Strategic voting in large elections under proportional representation: Why vote for center parties?*

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1 Introduction

Most of the countries in the democratic world use proportional representation to allocate seats in parliaments. There are typically more than two serious parties in those countries. Consequently, the two-party competition models do not help us much to understand the political economy of those countries.

When there are more than two parties, it is no longer rational for voters to vote for the party they prefer. For instance, if one’s preferred party has no chance of being part of the future government, it may be utility maximizing to vote for another party and help it join the government. Consequently, a complete picture of proportional representation requires to take account of strategic behaviour both by parties and by voters.

Parties are strategic in their choice of the proposed platform, as well as in their choice of which other parties they will form a coalition with in case they have their say in the forming of the winning government. Voters are strategic at the voting stage, and their choice of which party to vote for will

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typically depend on their expectations about how the government will form as a function of the outcome of the elections.

Few papers have tried to provide a complete picture of the strategic election process. In their seminal paper, Austen-Smith and Banks? model elections as a three step mechanism: 1) how competing parties determine their political platform, 2) how voters choose the party they vote for, and 3) how parties bargain to form the running coalition. They assume that information is complete and agents follow subgame perfect rationality. A key feature of their result is that the center party, that is, the one which proposes a policy platform that lies between the policy platforms of the other two parties, receives no more votes than the minimal requirement to be represented in parliament.

In other theoretical papers, center parties do even not succeed in surviving. In Gerber and Ortuno-Ortín?, and De Sinopoli and Iannantuoni??, only extreme parties get vote at equilibrium, should the policy space be one or multidimensional. The key assumption is that the policy that voters expect, as a function of the outcome of the election, is a kind of average of platforms announced or represented by existing parties. Rational voting consists in trying to affect the average as much as possible to move it closer to one’s preferred policy.

All these papers share the following centrifugal mechanism: center voters, that is, voters who prefer the policy proposed by the center party, may have an interest not to vote for the center party, as voting for a more extreme party may influence the expected implemented policy or the expected winning coalition in a way that is closer to their interest. No paper identifies a counterbalancing centripetal motivation.

In EU15 countries using proportional representation, even if Norway, Sweden and Portugal may qualify as countries without center party (in the sense that no party is used to belong to coalitions with sometimes leftist and sometimes rightists parties), the majority of countries have center parties. Moreover, even if center parties may be small (like the regionalist parties CiU and PNV in Spain, or FDP in Germany1), they may also be among the largest parties (like OVP in Austria, CD&V in Belgium, CE in Finland, CSV in Luxembourg and KVP in the Netherlands), or the largest one (like DC in Italy

\[1\] Even if FDP is most often called to join the government by either of the main two parties, there is no real consensus about which party qualifies as a center party in Germany. Here, we follow the measure used by Schofield, Martin, Quinn, and Whitford? leading to the conclusion that FDP proposes median political platforms.
before the electoral reform of 1993) or majoritarian (like PSC in Belgium after the 1954 elections). It is therefore clear that the centrifugal mechanism highlighted in the theory is not the only working mechanism.

In this paper, we develop a model where the center party may be small, medium or large at equilibrium. Two ingredients are key to obtain that result. The first ingredient is that voters think that the largest party is also the one who has the power to choose the winning coalition. The second ingredient is that parties may strategically commit, before the election, to the coalition they will form in case they arrive first after the election.

Pre-election commitments have been underestimated in the literature, whereas there are clear examples of such strategic moves. Before the 2007 Belgian legislative elections, for instance, polls documented a switch of many voters from the socialist to the green party. The leader of the French-speaking socialist party made then clear that his party would not participate in any coalition containing the conservative party. As a consequence of that commitment, all plausible coalitions had the property that, with or without the socialist party, they contained the green party. As a consequence, a rational voter hesitating between voting for green or voting socialist had to vote for the latter. This is indeed what happened, as the socialist party did unexpectedly well (and ended up in the winning coalition).

The consequence of our modelling of pre-election commitments can be described as follows. Rational agents vote as a function of how the outcome of the election game influences the winning coalition. In practice, voters build expectations on the possible coalitions, and, in spite of possible (strategic) announcements by parties, there typically remains some uncertainty. In our model, pre-election commitments play a double role. First, it reduces the uncertainty about what will take place after the election. Note that in models where the bargaining process is deterministic, backward induction reasoning also removes uncertainty. Second, it allows parties to strategically use that uncertainty. Indeed, if the legislative bargaining process is fixed, then voters know perfectly what will happen after the election and that process may not be the one that is preferred by the proposing party. Our pre-election commitment stage is equivalent to any pre or post-election mechanism that removes uncertainty about the relationship between the election outcome and the government coalition and that do so in line with the best interests of the winning party(ies).

The key lesson of this formal result is that there are centripetal voting forces. Here are the three main such forces. The main one works as follows.
If, say, a leftist voter expects that the left party will be small (the probability of it forming the government is negligible), she may be ready to vote for the party that commits to form a coalition with it. Consequently, a sufficiently large center party may commit to form the government with the leftist party to obtain the votes of leftist voters. This can indeed be an equilibrium strategy. There are other, less important, forces.

If the center party is expected to be small, to the extent that it may disappear from the parliament, then voters preferring the second largest party, fearing to see their opponent alone in government, may vote for center to help it keep representatives in parliament.

If the center party is very large, to the extent that there is some non-negligible probability that it ends up alone in government, then extreme voters may still prefer to vote for the center and have it alone in government than vote for their preferred party and see the center form a coalition with a party they even like less. Again, center can manipulate these voters by announcing coalitions that induce extreme voters to prefer the outcome where center is governing alone.

There are other, out of equilibrium, centripetal forces. If an extreme party is very large, then it may be tempting for its voters to vote for the center party to try to enforce the coalition they prefer. If this is the case, though, it is rational from the extreme party to commit to form a coalition with the center, thereby keeping its voters. Observe that the center party benefits from this rational use of the voters’ preferences by the extreme party as center ends up in the winning coalition.

Centrifugal forces still exist in the model. As soon as the center party has committed to form a coalition with an extreme party, the center voters preferring the opposite coalition may have an incentive to vote for the other extreme party. Our result implies that these forces never overturn the other forces, as long as the center party rationally commits to the coalition that maximizes its probability of being in the winning coalition.

We then discuss the possible variants of the model that would make the result vanish. It turns out that as soon as the role of coalition formation is allocated in some proportion of the election outcomes, then the centrifugal forces dominate. This comes from the fact that center voters maximize the probability that the center party is in the coalition they like by voting for the extreme party they prefer. This may come either from the fact that the role of government formateur is randomly allocated as a function of the outcome of the election, or that parties are not allowed to commit to a given coalition
before the election and voters are uncertain about the outcome of the future legislative bargaining process. We also show that, facing this uncertainty, there always exists a party which has an incentive to break the uncertainty and commit, before the election, to a particular coalition. We then come back to a similar result to the one of our benchmark model.

Of course, we take our model as a step in the direction of understanding why center parties are often in winning coalitions, and why they are not bound to remain small. On the other hand, our model does not explain why center parties exist in the first place.

The remaining of the paper is organised as follows. In Section 2, we define the benchmark model. In Section 3, we state and prove the main result. That results state that the center party always ends up being part of the winning coalition and it is not bound to be small. In Section 4, we discuss several possible extensions of the benchmark model, and we show that modifying some key assumptions leads to the result that the center party barely survives. In Section 5, we give some concluding comments.

2 The model

Our model has the following characteristics. There are three parties, $L, C$ and $R$. Voters vote for one party, and, as a function of the outcome of the election, a government is formed. There are six possible governments, $L, LC, C, LR, CR$ and $R$. Voters have preferences over governments, that is, coalitions of parties and not policy platforms.

That allows us to define the center party: if a voter prefers a party that is not the center party, then she prefers her preferred party to be in a coalition with the center party than with the other party. Also, if a voter prefers the center party, then the coalition containing the other two parties cannot be her preferred two party coalition.

More precisely, we have four types of voters, $L, CL, CR$ and $R$, which we name leftist, center-leftist, center-rightist and rightist respectively. A voter population is parameterised by a vector $\theta = (\theta_L, \theta_{CL}, \theta_{CR}, \theta_R)$, describing the proportion of voter types in the population. Voters preferences can be represented in the following table.
We assume that voters maximize their expected utility, and voter of type \( v \)’ utility is measured by a function \( u_v \).

At the time of the election, there is some uncertainty about who participates in the election. We model this uncertainty using the large Poisson games introduced by Myerson\(^?\). The actual number of voters \( N \) is a random variable that is Poisson distributed with parameter \( n \), that is, \( N \sim \mathcal{P}(n) \Leftrightarrow \text{Prob}(N = k) = \frac{e^{-n} n^k}{k!} \).

As a consequence of this assumption, all major random variables are also Poisson distributed, which makes the model technically simple. For instance, the number of actual voters of type \( i \) is also Poisson distributed with parameter \( \theta_v n \), \( v \in \{L, C, R\} \). Another consequence is that strategies are defined for types and not for agents. For voters of type \( v \), \( v \in \{L, C, R\} \) and party \( p \in \{L, C, P\} \), we let \( \sigma^p_v \) denote the probability that a voter of type \( v \) votes for party \( p \). Let \( \Delta^2 \) denote the two-dimensional simplex. The strategy of voters of type \( v \) is therefore the vector \( \sigma_v = (\sigma^L_v, \sigma^C_v, \sigma^R_v) \in \Delta^2 \). A strategy profile is denoted \( \sigma = (\sigma_L, \sigma_C, \sigma_R) \). Consequently, the actual number of voters for party \( p \in \{L, C, P\} \), \( N^p \), as a function of strategy profile \( \sigma \), is itself Poisson distributed with parameter \( \lambda^p n \), where

\[
\lambda^p = \sum_{v \in \{L, C, R\}} \theta_v^p \sigma^p_v. \tag{1}
\]

As a consequence, for each strategy profile, each voter has a strictly positive probability to be pivotal. Maximizing expected utility requires that voters vote as a function of the most likely tie event in which their vote can be decisive.

Before the election, parties commit to the party with which they will form the governing coalition in case they win the elections. Formally, each
party \( p \in \{L, C, R\} \) announces a future partner \( P^p \in \{L, C, R\} \setminus \{p\} \). Let \( P = (P^L, P^C, P^R) \) denote a profile of strategies for the parties. Once parties have announced their commitments, voters vote, assuming that the winning party, that is, the party obtaining the largest number of votes, will form the coalition it committed to form. We assume that parties’ preferences consist of lexicographically applying the following criteria:

1. first, they maximize the probability of being in the winning coalition,

2. second, they maximize the probability of being alone in the winning coalition, and

3. third, they maximize the probability of being in the winning coalition with an ideologically close partner.

The third criterion is restrictive for parties \( L \) and \( R \) only. The criterion requires that when these parties have no influence on the probability of being in the winning coalition or being alone in the coalition, they prefer to try to be in a coalition with party \( C \).

After the elections, the parliament is formed. We assumed that parties having obtained more than a fraction \( a \) of votes are represented, \( 0 < a < \frac{1}{3} \), and that seats are allocated proportionally to votes among represented parties. Then the government forms.

To sum up, at stage 1, each party announces its partner. At stage 2, voters who are selected by nature to vote choose the party they vote for, knowing the parties’ commitments, but facing uncertainty about voters’ participation in the election. A complete set of actions is a pair \( (P, \sigma) \). After the election, the government is formed by the party having obtained the largest number of votes according to its pre-electoral commitment. We look for perfect Bayesian equilibria of that game in which parties play a pure strategy. As it will become clear in the next section, for a given population preference profile, there can be several equilibria at stage 2, following the same profile of announcements at stage 1. Some of these equilibria involve a huge change in voting behavior by voters following a change in the announcement by parties. We find it reasonable to first focus on the equilibria in which a change in parties’ announcements does not lead to a major change in voters’ behavior, that is, in which the equilibria in subgames out of the equilibrium path satisfy the following consistency property:
Assumption 1  The consistency property: a perfect Bayesian equilibrium of the electoral game satisfies the consistency property if the equilibrium of a subgame that is not reached at equilibrium is, among the possible equilibria of that subgame, the one that has the largest set of voters voting in the same way as along the equilibrium path.

Finally, to express our focus on large elections, we look at the limit of sequences of equilibria as the population parameter, $n$, tends to infinity. We call them equilibria of our electoral game.

3  The result

Our main result is that, under these assumptions, the center party always belongs to the winning coalition. Moreover, it can be small, medium or large.

Theorem 1 Let us assume that $a \leq \theta^L, \theta^C, \theta^R \leq 0.5$. Let $(P, \sigma)$ be an equilibrium satisfying the consistency property, and such that all three parties are represented in parliament and no party is majoritarian.

1. Party $C$ is in the government.

2. Party $C$ is either the largest party, the second largest party or the smallest party.

Let us begin with the proof of the second claim. It is sufficient to give examples, but we need to introduce the following terminology. There are four sets of events at which a vote can be decisive. Let us define them formally. In the first set, a vote can be pivotal in allowing a party to obtain strictly more than fifty per cent of the seats, that is, to become majoritarian. For parties $p, q, r \in \{L, C, P\}$, $p \neq q \neq r \neq p$:

$$Piv_1^p : N^p = N^q + N^r.$$

In the second set, a vote can be pivotal in modifying which party wins the election:

$$Piv_2^{pq} : N^p = N^q - 1 \leq N^r \text{ or } N^p = N^q > N^r.$$

In the third set, a vote can be pivotal in modifying which party arrives second in the election:

$$Piv_3^{pq} : N^r - 1 > N^p = N^q - 1 \text{ or } N^r - 1 > N^p = N^q.$$
In the fourth set, a vote can be pivotal in allowing a party to obtain the number of votes that is necessary to be represented in parliament, preventing another party to become majoritarian:

\[ Piv^pq_4 : N^p = a(N^p + N^q + N^r + 1) - 1 \text{, and } N^q > N^r. \]

These pivotal events allow us to write voters’ utility. For instance, in the subgame where both parties \(L\) and \(R\) have committed to form a government with \(C\) and party \(C\) has committed to form a government with \(R\), a voter of type \(v\) will vote for \(L\) rather than \(C\) if and only if

\[ Piv^L_1(u_v(L) - u_v(LC)) + Piv^{LC}_2(u_v(LC) - u_v(CR)) + Piv^{LC}_4(u_v(CR) - u_v(C)) + Piv^{LR}_4(u_v(CR) - u_v(R)). \]

As \(n\) becomes large, the probability of all \(Piv\) events converges to zero, but at different speed, so that the probability of the event that converges at the slowest rate tends to be infinitely larger than all the others. Voters can then focus on those events. For instance, in the above example, if \(Piv^{LC}_2\) converges at the slowest rate, then only the sign of \((u_v(LC) - u_v(CR))\) matters, and a voter of type \(v\) will prefer voting for \(L\) than for \(C\) if it is positive.

The most useful measure of the convergence rate is given by the following formula, which, following Myerson, we call magnitude and denote by \(\mu\). For a strategy profile \(\sigma\), let \(\text{Prob}(Piv(\sigma)|n)\) denote the probability of an event \(Piv\) when the expected number of voters is \(n\). The magnitude of event \(Piv\) is computed as

\[ \mu(Piv(\sigma)) = \lim_{n \to \infty} \frac{1}{n} \ln(\text{Prob}(Piv(\sigma)|n)). \]

Magnitudes are non-positive. Events of which the probability does not converge to zero, or does converge at a low rate (say, of the order of \(n^{-t}\) for some positive \(t\)) have a magnitude of zero. Events that converge extremely quickly (say, of the order of \(n^{-n}\)) to zero have a magnitude of minus infinity. Pivotal events we are interested in typically have a negative but finite magnitude.

**Lemma 1** For a given strategy profile \(\sigma\) and the corresponding expected fractions of votes \(\lambda^L, \lambda^C\) and \(\lambda^R\) determined by Eq. 1, magnitudes are computed by the following formulas: For parties \(p, q, r \in \{L, C, P\}\), \(p \neq q \neq r \neq p\), such that \(\lambda^p \geq \lambda^q \geq \lambda^r\):

- \(\mu(Piv^p_1) = 2\sqrt{\lambda^p(\lambda^q + \lambda^r)} - 1\),
\[\mu(Piv_{pq}^2) = 2\sqrt{\lambda_p\lambda_q} - (\lambda_p + \lambda_q),\]
\[\mu(Piv_{qr}^3) = 2\sqrt{\lambda_q\lambda_r} - (\lambda_q + \lambda_r),\]
\[\mu(Piv_{rp}^4) = (\lambda_p + \lambda_q)^{1-a} \left(\frac{\lambda_r}{a}\right)^a - 1.\]

The lemma is proven in the appendix.

Two events having the same magnitude do not necessarily have the same probability. On the contrary, the probability ratio of the more likely over the less likely event may converge to any positive number, or to infinity.

We can compute that if
\[\lambda_p = 0.46, \lambda_q = 0.43, \lambda_r = 0.11,\] (2) then
\[\mu(Piv_{pq}^2) = -0.0005, \mu(Piv_{rp}^4) = -0.0005, \mu(Piv_{qr}^3) = -0.0032.\] (3)

We use these numbers in the following examples.

**Example 1:** party C is the largest. Let us consider the following voter population:
\[\theta_L = 0.40, \theta_{CL} = 0.03, \theta_{C}= 0.40, \theta_R = 0.17.\]

Let us assume that parties have made the following announcements:
\[P_L = C, P_C = R, P_R = C.\]

The strategy profile is defined as follows:
\[\sigma_L^L = 1, \sigma_{CL}^L = 1, \sigma_{C}^R = 1, \sigma_{C}^C = \frac{6}{17}, \sigma_{R}^R = \frac{11}{17},\]
that is, all L and C\(^L\)-voters vote for L, all C\(^R\)-voters vote for R, and R-voters share their votes between C and R, to maximize the probability that C wins the election, as the leader of party C has announced that it would form a coalition with R, under the constraint that R obtains sufficiently many votes so that it is represented in parliament.

This strategy profile yields \(\lambda_L = 0.43, \lambda_C = 0.46, \lambda_R = 0.11\). We claim that it is an equilibrium at stage 2.

The most likely ties are \(\mu(Piv_{pq}^{LC})\) and \(\mu(Piv_{qr}^{RC})\). Observe that a vote for either L or C has the same effect on \(Piv_{qr}^{RC}\). Consequently, those who
hesitate between L and C only look at $Piv_2^{LC}$. Given the announcements by party leaders, the choice is between voting for L and favouring a coalition LC or voting for C and favouring a coalition CR. By voting for L, voters of type L and $C_L$ play a best reply. By voting for C, voters of type $C^R$ also play a best reply.

Voters of type R, on the other hand, prefer R to be represented in parliament rather than having C becoming majoritarian. Moreover, they prefer a coalition CR over a coalition LC. Consequently, their best reply consists in playing a mixed strategy between voting for C and voting for R, so that the magnitudes of $\mu(Piv_2^{LC})$ and $\mu(Piv_4^{RC})$ are equalized. Indeed, if, say, the former is strictly larger than the latter, than they have a strict incentive to vote for C (thereby decreasing $\mu(Piv_2^{LC})$).

Let us note that the equilibrium values of $\sigma_C^R$ and $\sigma_R^R$ must depend on n, as they need to equalize the utility of voting for C and promoting a CR coalition and voting for R to guarantee that R is represented in parliament. Our analysis, here, is restricted to computing the limit value of the strategy as n becomes large.

Now, we need to prove that the above party announcements form a Nash equilibrium. More precisely, we need to prove that there exist Nash equilibria at stage 2, following deviations by parties, making these deviations unprofitable.

First, party L cannot do better by deviating. Indeed, if it announces $P^L = \emptyset$, then the same equilibrium as the one we are studying may arise at stage 2. If it announces $P^L = R$, then voters change their expectations: by voting for L they now favor a LR coalition. Again, that may lead to the same equilibrium as the current one in stage 2, as a voter prefers LC over CR if and only if she prefers LR over CR.

Second, party C cannot gain either. Indeed, by announcing $P^C = \emptyset$ or $P^C = L$, it makes the choice between voting for L or voting for C irrelevant: both ballots lead to a LC coalition. Therefore, the key choice is between voting for L or C and getting a LC coalition, or voting for R and getting a CR coalition. Given that a fraction 0.57 of the population (that is, all R and $C^R$ voters) prefer a coalition CR over a coalition CL, it is easy to see that C cannot gain by deviating and pushing its $C^R$ voters to vote for R.

Third, party R cannot gain by deviating. This is straightforward as its announcement does not influence the way voters vote. The proof is thus complete.
The above example exhibits the main mechanism of the model. It explains why the center party is not bound to disappear, that is, why there may be centripetal forces among strategic voters. In that example, voters are divided between those who prefer coalition $LC$, that is, $L$ and $C^L$-voters, and those who prefer coalition $CR$, that is, $C^R$ and $R$-voters. The reason why $C$ succeeds in being the largest party is because it gathers the votes of all voters preferring coalition $CR$. Conversely, it is rational for $R$-voters to vote for $C$ as this is the best strategic way to obtain the coalition they prefer.

This is not the whole story, though. The argument is only complete after one observes that the other parties do not have the means to change the result of the elections by changing their announcement. This comes from the fact that the alternative coalition $LR$ is never the most preferred coalition of a voter. Consequently, shifting towards $LR$ does not allow $L$, in our example, to increase its number of votes. This is why $C$ succeeds in becoming the largest party, in spite of the fact that $C^L$-voters vote for $L$, trying to induce a $LC$ coalition, a typical centrifugal force.

In the example below, the mechanism is similar, but given the relative fractions of voters in the population, $C$ only succeeds in being the second largest party. Observe that it still belongs to the winning coalition, as $L$ announces $P^L = C$ to be sure to attract the $C^L$ voters.

Example 2: party $C$ is the second largest. Let us consider the following voter population:

$\theta^L = 0.43, \theta^{C^L} = 0.03, \theta^{C^R} = 0.37, \theta^R = 0.17$.

Let us assume that parties have made the following announcements, identical to the ones above:

$P^L = C, P^C = R, P^R = C$.

The strategy profile, also identical to the one above, is defined as follows:

$\sigma^L_L = 1, \sigma^{C^L}_C = 1, \sigma^{C^R}_C = 1, \sigma^R_C = \frac{6}{17}, \sigma^R_R = \frac{11}{17}$.

As a result, the expected vote shares are as in Eq. 4. Observe that $C$ is now the second largest party. Magnitudes are given by Eq. 5. The proof that it is an equilibrium is similar to the one above.

Example 3: party $C$ is the smallest. Let us consider the following voter population:

$\theta^L = 0.43, \theta^{C^L} = 0.06, \theta^{C^R} = 0.11, \theta^R = 0.40$. 
Let us assume that parties have made the following announcements:

\[ P_L = C, P^C = R, P^R = C. \]

The strategy profile is defined as follows:

\[ \sigma^L_L = 1, \sigma^C_L = 0.5, \sigma^C_R = 0.5, \sigma^R_C = \frac{8}{11}, \sigma^R_R = \frac{3}{11}, \sigma^R_R = 1. \]

Again, expected vote shares and magnitudes are given by Eqs. 4 and 5. Voters of type \( C^L \) and \( C^R \) play a mixed strategy between voting for \( C \) to guarantee that it is represented, and voting for their preferred partner, \( L \) or \( R \). In the sequence of equilibria we are interested in, the relative probability of \( Piv^{LR}_2 \) and \( Piv^{CL}_4 \) must be such that expected utilities of voting for \( L \) or \( C \) for \( C^L \)-voters, and voting for \( C \) or \( R \) for \( C^R \)-voters are equalized.

Voters of type \( L \) (resp., \( R \)) vote for their preferred party, thereby trying to enforce their preferred coalition. That completes the proof that these strategies are equilibrium strategies at stage 2.

Parties’ announcements are optimal, too. If either party \( L \) or \( R \) announces \( \emptyset \), then the expected partner does not change, and the same equilibrium as the one we just described applies, so that this deviation is not profitable. If party \( L \) (resp., \( R \)) announces \( P^R = R \) (resp., \( P^R = L \)), then again the same equilibrium holds, as voters prefer \( LR \) over \( CR \) (resp., \( LR \) over \( LC \)) if and only if they prefer \( LC \) over \( CR \) (resp., \( CR \) over \( LC \)). Such a deviation is not, therefore, profitable. That completes the proof.

Example 3 is consistent with the typical mechanism one finds in the literature. It is in the interest of all \( C^L \) and \( C^R \) voters not to vote for \( C \) but for their second preferred party, in order to try to induce the coalition they prefer, or, more precisely, in order to maximize the probability that the coalition proposer is the second party they prefer, given that that party has committed to form a coalition with \( C \). Party \( C \) only gets the number of votes that guarantees that it is represented in parliament, or, more precisely, it gets the expected share of votes that equalizes the magnitude of the two events of interest: the event that the proposer becomes the second preferred party of the voters, and the event that \( C \) is represented in parliament.

The proof of the first part of the theorem is more technical. We sketch it here.

If party \( C \) is the largest party, then, by assumption, it belongs to the government. Let us assume, then, that \( C \) is the second largest party. Let us
say, w.l.o.g., that $L$ is the largest. If $L$ plans to form a coalition with $C$, then, again, we are done. Let us assume, then, that $L$ has announced a coalition with $R$.

If $C$ announces a coalition with $L$, then $L$-voters and $C^L$-voters prefer that $C$ wins the election, and all the others prefer that $L$ wins the election. If $C$ announces a coalition with $R$, then voters’ preferences are reversed. Consequently, $C$ announces $L$ or $R$ as a function of whether $\theta^L + \theta^{CL} > \theta^R + \theta^{CR}$ or the reverse. In any case, by choosing the best response, $C$ can attract more voters than $L$, contradicting the premise of the argument.

Finally, let us assume that $C$ is the smallest party. This excludes the possibility that both $L$ and $R$ announce that they will form a coalition with each other. Indeed, if it were the case, then voters are indifferent between voting for $L$ and for $R$, and, by announcing $L$ or $R$, party $C$ can attract all $L$- and $C^L$-voters or $R$- and $C^R$-voters, and become at least the second largest party, thereby contradicting the premise of the argument.

Therefore, at least one of $L$ and $R$ announces a future coalition with $C$. Let us assume that $L$ announces such a coalition. Then, necessarily, $L$- and $C^L$-voters prefer to vote for $L$ (as long as $C$ is sure to be represented in parliament) and $R$- and $C^R$-voters prefer to vote for $R$ (under the same constraint that $C$ is represented). Independently of whether $R$ announces $L$ or $C$, it attracts the same set of voters (given the consistency property which equilibria are assumed to satisfy). Consequently, given that it prefers to be in a coalition with $C$, it announces $C$. As a result, both $L$ and $R$ announce that they plan to form a coalition with $C$, so that $C$ ends up in the winning coalition.

To conclude this section, let us discuss the equilibria that are excluded by our consistency property and the requirement that no party is majoritarian. First, independently of the voters’ population, there are three equilibria where only two parties are represented in parliament. Indeed, if we take any two parties, voters voting for the one they prefer among the two is an equilibrium at stage 2, independently of the parties’ announcements at stage 1. Given that we are interested in equilibria with three parties, we have removed these equilibria. But we have removed more. Indeed, any equilibrium at stage 2 can be enforced as an equilibrium of the whole electoral game, independently of the announcements of the parties, if out-of-equilibrium-subgames are solved by selecting the equilibrium where the deviating party is “punished” by the equilibrium where it is not represented in parliament. We clearly don’t think these equilibria are interesting.
Even if we restrict ourselves to equilibrium paths where out-of-equilibrium subgames have equilibria where all three parties are represented, there is a large possibility to enforce particular equilibrium paths by “punishing” the deviating party. The only aim of our consistency property is to prevent these kind of punishments. By restricting the selection of out-of-equilibrium-subgame equilibria to the ones that involve the minimal change in voters’ behavior, we focus on the most realistic ones. They appear to also be the ones where C always belongs to the winning coalition.

Finally, there are equilibria satisfying the consistency property, where all three parties are represented in parliament, but where one party is majoritarian. As a matter of fact, that party cannot but be C.

Example 4: party C is majoritarian. Let us consider the following voter population:

\[ \theta^L = 0.40, \theta^{CL} = 0.10, \theta^{CR} = 0.10, \theta^R = 0.40. \]

Let us assume that parties have made the following announcements:

\[ P^L = C, P^C = R, P^R = C. \]

The strategy profile is defined as follows:

\[ \sigma^C_L = 1, \sigma^C_{CL} = 1, \sigma^C_{CR} = 1, \sigma^L_R = 0.5, \sigma^R_R = 0.5, \]

that is, all L and C-voters vote for C and R-voters mix between voting for L and voting for R in such a way that the expected vote fractions of L and R are the same. That is, R-voters vote in such a way as to minimize the probability that either L or R fails to be represented, which corresponds to minimizing the probability that C is majoritarian.

We can compute that

\[ \lambda^L = 0.20, \lambda^C = 0.60, \lambda^R = 0.20, \]

so that the most likely ties have magnitudes

\[ \mu(Piv^C_1) = -0.0204, \mu(Piv^{LC}_4) = \mu(Piv^{RC}_4) = -0.0360. \]

Let us prove that this is an equilibrium strategy profile at stage 2. The most likely tie is between C being majoritarian, and, therefore, alone in government, or C being force to form a coalition, in which case, given its
announcement, a CR coalition will emerge. All L and C-voters strictly prefer the former government to the latter. They play a best reply. The R-voters, on the contrary, prefer CR to C. Their objective is to minimize the probability that C ends up alone in government. That, of course, forces them not to vote for C. That does not tell them, however, how to chose between L and R. The only impact of the choice between L and R is on the fourth type of pivotal event. If one of the two parties ends up being excluded from the government, then the threshold C needs to reach to be majoritarian decreases. To minimize the probability of that event, R-voters vote for R if and only if \( \mu(Piv_4^{RC}) > \mu(Piv_4^{LC}) \) (other wise they vote for L). Equilibrium is indeed reached only if these magnitudes are equalized.

Let us now prove that it is an equilibrium at stage 1. Given the symmetry in the parameters of the example, if C announces \( P^C = L \), the symmetric equilibrium in which all R-voters vote for C and L-voters split their vote between L and R may prevail, in which case C does not have any interest to announce \( P^C = L \). By announcing \( P^C = \emptyset \), C cannot gain either. Indeed, there is no stage 2 equilibrium following that announcement in which C obtains a larger victory margin. Indeed, a larger margin could only be obtained if both L and R voters vote for C, which is impossible as voters of the party that is expected to be the second largest always have a strict interest not to vote for C.

The interest in example 4 does not lie in the precise shape of the equilibrium. It lies in the fact that it exhibits an asymmetry among parties in favor of C. In the example, indeed, C is expected to be majoritarian, and yet it obtains votes from voters who prefer another party. Indeed, by strategically announcing \( P^C = R \), C give incentive to L-voters to vote for C, as they prefer government C to government CR. Parties L and R do not have that power. As soon as, say, the most likely tie event is \( Piv_1^L \), voters vote as if their vote were decisive between a government of L alone and the coalition of L and its announced partner. Here is the asymmetry: all \( C^L \), \( C^R \) and R-voters strictly prefer any coalition containing L to L alone.
4 Extensions

4.1 Models of strategic voting

An informal criticism to the use of Poisson games to describe the strategic behavior of voters is that voters don’t compute pivotal probabilities, and, more generally, don’t behave as a function of the event in which their vote may be decisive. Before discussing this criticism a little bit, let us summarize the role played by our modelling of strategic voting in this model.

In this model, there are four reasons for the voters not to vote sincerely.

1. A $L$- (resp., a $R$-) voter may choose to vote for $C$ rather than $L$ (resp., $R$) with some probability when $C$ has announced that it would form a coalition with $L$ (resp., $R$) and $C$ has much more probability to win the elections than $L$ (resp., $R$).

2. A $C_L$- (resp., a $C_R$-) voter may choose to vote for $L$ (resp., $R$) rather than $C$ with some probability when $L$ (resp., $R$) has announced that it would form a coalition with $C$ and $L$ (resp., $R$) has much more probability to win the elections than $C$.

3. A $C_L$- (resp., a $C_R$-) voter may choose to vote for $L$ (resp., $R$) rather than $C$ with some probability when they are sure that $C$ will be in government and she tries to enforce the coalition she prefers.

4. A $L$- (resp., a $R$-) voter may choose to vote for $C$ rather than $L$ (resp., $R$) with some probability when $L$ (resp., $R$) is expected to be the second largest party whereas $C$ is not even sure to be represented in parliament and this voter is afraid that $R$- (resp., a $L$-) is alone in the government.

Our modelling of strategic voting and our use of the Poisson games is no more than a rigorous foundation to the assumptions that voters follow the above strategies. Any model of strategic voting that would be consistent with these strategies would give us the same result. Note that the fourth one, which, to our opinion, is the least plausible, is not necessary for the result.

4.2 Introducing the possibility for parties to not commit

In the model of the previous section, parties have to announce with whom they plan to form a coalition. As we explained, this stage stands for all
possible devices that parties may choose and that result in voters perfectly foreseeing the relationship between the outcome of the election and the winning coalition. Let us now add a third possible action, that is, let us allow parties not to commit. Either that leads to uncertainty, with the consequence that voters will vote in the same way as in other cases of uncertainty reviewed below. Or that leads to another, certain, outcome. Let us assume that voters believe that a winning party which did not commit will form the government with the second largest party.\footnote{Assuming that a party which did not commit plans to form a coalition with the smallest party creates the incentive for voters to try to decrease the size of their preferred party, which does not look very interesting.} Formally, each party \( p \in \{L, C, P\} \) announces a future partner \( P^p \in \{L, C, P\} \setminus \{p\} \cup \{\emptyset\} \), where \( P^p = \emptyset \) means that party \( p \) does not commit to a specific partner and, in case of victory, will choose the second largest party to form a government.

The result above does no hold any more. Indeed, by not committing, a party may give an incentive to voters to try to make their party the second largest, which creates a new kind of possible equilibrium. Let us give an example.

Example 5: An equilibrium with a non-committing party. Let us consider the following voter population:

\[
\theta_L = 0.46, \theta_{CL} = 0.22, \theta_{CR} = 0, \theta_R = 0.32.
\]

Let us assume that parties have made the following announcements:

\[
P_L = \emptyset, P_C = L, P_R = C.
\]

The strategy profile is defined as follows:

\[
\sigma_L^L = 1, \sigma_C^C = 1, \sigma_C^{CR} = 1, \sigma_R^R = 1,
\]

that is, all voters vote sincerely. We compute that the most likely pivotal event is the one in which \( L \) is likely to be majoritarian: \( \mu_1 = -0.0032 \). The second most likely pivotal event is a tie between \( C \) and \( R \) for being second in the election: \( \mu_3 = -0.0093 \).

Let us prove that this is an equilibrium strategy profile at stage 2. Given the most likely tie event, all \( L \)-voters vote for \( L \). None other voter vote for \( L \), as any coalition containing \( L \) is better for all of them compared to \( L \) alone. That tells them not to vote for \( L \), but that does not tell them how to
vote. The second largest pivotal event is a tie between $C$ and $R$. Given that $C$-voters, who are all $c^L$-voters, prefer a $LC$ coalition to a $LR$ coalition, they vote for $C$. On the contrary, all $R$-voters vote for $R$.

4.3 Minority and surplus coalitions

Here we stick to the notion of minimal winning coalition, justified by Riker\textsuperscript{3}. It was soon noticed (see, for instance, Herman and Pope\textsuperscript{4}) that coalition governments could be minority or surplus coalitions. We could easily introduce the possibility of minority government by requiring a threshold lower than 0.50 to be allowed to form the government alone. Conversely, by requiring that the majority contain parties representing more than 0.50 of the seats, we could also let surplus government form. In either case, none of the current result would be affected. The nature of the government (majority, minority, surplus), though, would stay essentially exogenous, and the paper can in no way be seen as explanatory of the nature of governments.

4.4 Policy space dimensions

Our assumptions on preferences are incompatible with a one-dimensional policy space. It is consistent with a policy space defined in terms of several public goods, or in terms of the combination of one public (the typical left-right economic policy space) and one private good (benefits voters expect from “their” representatives). If we were to restrict our attention to a one-dimensional space, then we would need to change the preferences in the following way: at least some center voters prefer the Left-Right coalition over any coalition containing the center party.

4.5 Microfoundations of the assumptions

Voters’ preferences can be explained in a two-dimensional setting where one dimension is religion, where the center party is at the extreme of the space. The second dimension can also be seen as private benefits which voters expect from their party.

We assume that the policy which the winning coalition will implement does not depend on the relative number of votes obtained by the parties in the coalitions. That can be founded easily using the Baron and Diermeier\textsuperscript{5} type of bargaining process.
4.6 More on the legislative bargaining stage

Testable prediction: our model predicts that the center party can be large (largest or second largest party) only if voters believe that the party obtaining the largest score has the power to choose its partner(s) in the government formation and if parties have the power to remove uncertainty about which party they would choose in case they form the government.

Our result show that in equilibrium it is always in the interest of a large party to announce the party with which it will form a coalition once in power. In the equilibrium profile associated to some of the parameters, however, the commitments of the largest two parties may be incompatible. Left may have committed to rule the country with Center, whereas Center has committed to rule the country with Right. Consequently, we completely disregard the possible strategy of a party to not commit to form a coalition with another one in order to avoid being ostracised by the winning party. If pre-election commitments are synonymous of loosing likely friends, it may be rational to avoid doing so, especially if post-election bargaining is uncertain.

5 Concluding remarks

We have introduced an equilibrium refinement that we find natural for Poisson games. Without this refinement, there would be more equilibria. First, there would be equilibria where only two parties are present in parliament and voters vote for the party they prefer between these two. That may be interesting to focus on two party equilibria in PR systems (think of Spain). These are not the equilibria we are interested in, though. We could have simply restricted ourselves to three party equilibria. Even under this restriction, we would have had a lot of additional equilibria. For instance, if $R$ is the largest party, followed by $L$, with $C$ being small, still we have the equilibrium where $L$ and some $C^L$ voters vote for $L$, the other $C^L$, the $C^R$ and a lot of $R$ voters vote for $C$, so that they are the largest two parties and $Piv_1^L = Piv_2^{LC}$, supported by announcements that $L$ would form a coalition with $C$ and $C$ would form a coalition with $R$. That shows that we can sort of replicate the equilibria we have obtained under our refinement almost independently of the true fractions of voters in the society. On the other hand, we don’t know whether other equilibria would emerge. That is, we have not been able to characterise the set of all equilibria, and that is why we have had to impose

20
our refinement.