The application of an accurate approximation in the risk management of investment guarantees in life insurance

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Abstract

Investment guarantees in life insurance business have generated a lot of research recently due to the earlier mispricing of such products. These guarantees generally take the form of exotic options and are therefore difficult to price analytically, even in a simplified setting. A possible solution to the risk management problem of investment guarantees contingent on death and survival is proposed through the use of a conditional lower bound approximation of the corresponding embedded option value. The derivation of the conditional lower bound approximation is outlined in the case of regular premiums with asset-based charges and the implementation is illustrated in a Black-Scholes-Merton setting. The derived conditional lower bound approximation also facilitates verifying economic scenario generator based pricing and valuation, as well as sensitivity measures for hedging solutions.

Keywords: Investment Guarantees, Comonotonicity, Life Insurance

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1 Introduction

Life insurers have traditionally concerned themselves with mainly mortality risk, which is a diversifiable risk if the insurer is able to aggregate a large number of independent insured lives. The risks inherent to investment guarantees, however, are largely dependent and require an approach to fair pricing of these guarantees and a hedging programme to effectively transfer investment risk to third parties or market. According to Hardy (2003), the three main aspects of risk management relates to the price, the amount of capital to hold and how to invest the capital.

Initial attempts at pricing investment guarantees were mainly of a statistical real-world approach. Significant improvements in the field of finance with the papers by Black and Scholes (1973) and Merton (1973) led to attempts to combine the fields of actuarial science and finance. Of the first substantial attempts were the publications by Boyle and Schwartz (1977), Brennan and Schwartz (1977) and Brennan and Schwartz (1979). The last mentioned authors not only considered fair pricing of both single and recurring premium structures in a Black-Scholes-Merton framework, but also looked at a possible delta hedging strategy and the sensitivities of the hedging strategy to model parameters.

The martingale approach to risk-neutral pricing by Harrison and Kreps (1979) and Harrison and Pliska (1981) further facilitated the adoption of modern finance techniques by the actuarial profession. The standard Black-Scholes-Merton framework under the martingale approach was applied to minimum guarantees at death and maturity by, among others, Delbaen (1986) and Aase and Persson (1994). The complexity in benefits as well as in the assumptions underlying the approaches considered in literature have grown substantially. Bacinello and Ortu (1993a) considered endogenous minimium guarantees, i.e. minimum guarantees that depend functionally on the premium of the policies. Bacinello and Ortu (1993b), Nielsen and Sandmann (1995) and Nielsen and Sandmann (1996) extended existing results to include stochastic interest rate risk. Boyle and Hardy (1997) considered a Value at Risk (VaR) methodology and a dynamic replicating portfolio approach. Møller (1998) considered risk-minimising strategies set in a Black-Scholes-Merton framework and Møller (2001) proposed a more pragmatic risk-minimising strategy in a discrete Cox-Ross-Rubinstein framework.

The policyholder contributions are usually solved for under fair value principles. We assume that the policyholder contributions are exogenously given, which is largely the case in practice. The payment of regular premiums results in the payoff being dependent on the underlying asset price throughout the duration of the contract and leads to an analogy with path-dependent Asian options. Upper and lower bounds in terms of double integrals for approximately valuing Asian options have been developed by Rogers and Shi (1995) and Thompson (1999). The application of stochastic bounds to financial products in actuarial science was introduced mainly by Simon et al. (2000), Dhaene et al. (2002a,b), Nielsen and Sandmann (2003) and Shrager and Pelsser (2004). The last mentioned authors use a change of numeraire technique to derive a general pricing formula in the case of stochastic interest rates in the lognormal case for rate of return guarantees in regular premium business. Hürlimann (2008) considers GMDB and GMAB guarantees in regular premium unit-linked (UL) business in the lognormal case with a two-factor fund diffusion process, consisting of a one-factor fund price and a one-factor stochastic interest rate model with deterministic bond price volatilities. The author considers the call-type option representation of the guaranteed benefit and determined bounds for the premium payable by the policyholder. In variable annuity (VA) and UL business, the contribution is typically specified by the policyholder, although rider risk benefits are mostly charged for explicitly through risk premiums before investment of the contribution. The investment guarantee charges are levied from the policyholder's underlying fund in the form of asset-based charges, i.e. the charges are expressed as a percentage of the value of the underlying fund or sub-account.

In the following, we derive the conditional lower bound approximation for the value of different types of embedded option and the asset-based investment guarantee charge for these options. We consider regular premium VA business and investment guarantee rider benefits that are contingent on death and survival, i.e. the GMDB and GMAB investment guarantee types. We further show how the conditional lower bound approximation can be used to validate sensitivity measures, the so-called greeks, for hedging. We assume a Black-Scholes-Merton setting throughout, although it is important to note that the conditional lower bound is a versatile approach and can be determined in a model dependent or model independent case. An example of the model independent case is the recent paper by Chen et al. (2008) in which the authors investigate static super-replicating strategies for European-type call options written on a weighted sum of asset prices. For complex models or in some model independent cases, the conditional lower bound might result in an approximation that is not analytically tractable. In such cases and where a point estimate suffices, numeric solutions can be used.

Investment guarantee offerings pose several challenges to insurance firms. The guarantees of individual policies are largely dependent, although the diversification benefit arising from the various underlying fund choices offsets the dependency to some extent. Insurers will fund the initial cost of setting up a hedging portfolio or reserving requirement at inception by recouping the cost over the policy term through asset-based charges. This income at risk poses an additional cost in the price of the guarantee. Finally, the embedded options of investment guarantees take the form of exotic path-dependent derivatives and require multiple nested simulations when valuing across stochastic economic scenarios.

The conditional lower bound is derived in section 2 for a simplified financial product, i.e. the benefit considered is not contingent on death or survival. In section 3, we show that the conditional lower bound theory easily extends to more sophisticated benefits and costing structures by adding typical costs as well as mortality to the simplied financial product considered in section 2. The popular GMDB and GMAB / GMMB structures are covered in section 3 and in section 4, we propose a possible dynamic hedging solution for the aforementioned structures.

2 A simple product

We first consider a simplified financial contract for ease of exposition. Consider an investor who invests a regular contribution π_k, $k = 0, 1, ..., n$,with a firm. In exchange for these contributions, the firm guarantees the investor the greater of a guaranteed minimum benefit of b_n and the fund value S_n at maturity of the contract, i.e. the benefit payout at time n is $B_n = \max(b_n, S_n)$. We assume that the investor, age x at inception, survives the contract with probability 1, i.e. ${}_n p_x = 1$, and we do not allow for other decrements such as surrenders. Let F_k denote the value at time k of one unit of the fund in which the contributions are invested by the firm. The value of the investor's portfolio V_{k+1}^- at the end of the period $(k, k+1)$, before the next period's contribution, is given by the recursive formula:

$$V_{k+1}^- = [V_k + \pi_k] \frac{F_{k+1}}{F_k} \tag{1}$$

Equation (1) states that the value at the end of a period during the contract term is equal to the regular contribution added to the existing value of the fund and then accumulated to the end of the period by the fund growth factor $\frac{F_{k+1}}{F_k}$. We assume $V_0^- = 0$ by construction. Under the assumption that the investor survives each period and that all contributions have been paid, we can express equation (1) as:

$$V_{k+1}^- = \sum_{j=0}^{k} \pi_j \frac{F_{k+1}}{F_j} \ , \quad k = 0, ..., n-1 \tag{2}$$

The minimum guaranteed benefit at maturity can be written as follows:

$$B_n = b_n + \max\left(0, V_n^- - b_n\right) \tag{3}$$

or:

$$B_n = V_n^- + \max\left(b_n - V_n^-, 0\right) \tag{4}$$

Equation (3) describes the guaranteed benefit as the value of the guarantee plus a call option on the fund value at maturity with a strike price equal to the value of the guarantee. Equation (4), on the other hand, describes the guaranteed benefit as the fund value at maturity plus a put option on the fund value at maturity with a strike price equal to the value of the guarantee. The expressions in equations (3) and (4) naturally allow for the use of financial economics in the pricing of the guaranteed benefit.

2.1 The concept of comonotonicity

The concept of comonotonicity, or "common monotonicity", provides the theoretical framework for the conditional lower bound approximation. A recent and largely self-contained overview of comonotonicity is given by Deelstra et al. (2010). For the purposes of this paper, the following definition of comonotonicity suffices.

Definition 1 *For any n-dimensional random vector* $\underline{X} = (X_1, X_2, ..., X_n)$ *with multivariate cumulative distribution function (cdf)* $F_{\underline{X}}(\underline{x})$ *and marginal univariate cdf's* $F_{X_1}, F_{X_2}, ..., F_{X_n}$ *and for any* $\underline{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$*, there exists a random variable* Z *and non-decreasing functions* f_i*,* $i = 1, 2, ..., n$*,* *such that:*

$$\underline{X} \stackrel{d}{=} \left(f_1(Z), f_2(Z), ..., f_n(Z)\right)$$

The concept of comonotonicity allows the use of certain properties to arrive at an accurate approximation of the true distribution of the underlying fund and the contingent benefits on the underlying fund. The following definitions will be used in the following:

Definition 2 *For a comonotonic sum S^c of the vector $\underline{X}^c = (X_1^c, X_2^c, ..., X_n^c)$, the inverse distribution function of a sum of comonotonic random variables is equal to the sum of the inverse distribution functions of the marginal distributions:*

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \; p \in [0,1] \tag{5}$$

Definition 3 *For a comonotonic sum S^c of the vector $\underline{X}^c = (X_1^c, X_2^c, ..., X_n^c)$, the distribution of the comonotonic sum S^c can be given in terms of the stop-loss premiums of the marginal components:*

$$E\left[(S^c - d)_+\right] = \sum_{i=1}^{n} E\left[(X_i - d_i)_+\right] \tag{6}$$

where $F_{S^c}^{-1}(0) \leq d \leq F_{S^c}^{-1}(1)$. The d_i, for $i = 1, 2, ..., n$, are given by:

$$d_i = F_{X_i}^{-1(\alpha_d)}\left(F_{S^c}(d)\right) \tag{7}$$

where $\alpha_d \in [0,1]$ is determined by:

$$F_{S^c}^{-1(\alpha_d)}\left(F_{S^c}(d)\right) = d \tag{8}$$

For the lognormal case, the inverse distribution function is left-continuous and strictly monotonic and the value of α_d in equation (8) is therefore equal to one. The decomposition formula in equation (6) can be interpreted financially such that the value of an European call option on a stochastic sum S^c of share prices with strike price d is equal to the sum of the values of European call options on the constituent shares with strike prices d_i as determined in equation (7). In fact, Jamshidian (1989) proved that in the case of one-factor mean-reverting Gaussian interest rate models, a European option on a portfolio of discount coupon-bearing bond decomposes into a portfolio of European options on the individual discount bonds in the portfolio.

2.2 Value of the embedded option

For ease of exposition, assume that the parameters of the contract $(b_n,\ \pi_k)$ are given at inception of the contract, and that the investment fund price process $\{F(t),\ t \geq 0\}$ evolve according to a geometric Brownian motion process with constant drift μ and constant volatility σ, i.e. assume a Black-Scholes-Merton setting:

$$\frac{dF(t)}{F(t)} = \mu dt + \sigma dW(t), \quad t \geq 0 \tag{9}$$

with initial value $F(0) > 0$, and where $\{W(t),\ t > 0\}$ is a standard Brownian motion.

Under a unique equivalent martingale measure Q, see e.g Harrison and

Pliska (1981), we have:

$$F(t) = F(0)\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}, \quad t \geq 0 \tag{10}$$

By combining the results in equation (2) and in equation (10), we find the following expression for the fund value at maturity V_n^-:

$$V_n^- = \sum_{k=0}^{n-1} \pi_k \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right)(n-k) + \sigma\left(W(n) - W(k)\right)\right\} \tag{11}$$

The contingent claims on the fund value in equation (11) are path-dependent and take the mathematical form of an arithmetic Asian option. To show this in the Black-Scholes-Merton setting, we use the time reversal property of Brownian motion. This property allows us to state a new process that maintains the probabilistic structure as the old process, i.e. $\tilde{W}(\tau) = W(n) - W(k)$,where $\tau = n - k$. By substituting the time reversed Brownian motion in equation (11), we find:

$$V_n^- = \sum_{\tau=0}^{n-1} \pi_\tau \exp\left\{\left(\delta - \frac{\sigma^2}{2}\right)\tau + \sigma\tilde{W}(\tau)\right\} = \sum_{\tau=0}^{n-1} \pi_\tau F_\tau \tag{12}$$

Assume now that the contributions are level across the policy term, so that we have:

$$\pi_k = \frac{1}{n}\sum_{j=0}^{n-1} \pi_j = \frac{1}{n}\Pi \tag{13}$$

where $k = 0, 1, ..., n-1$ and Π denotes the total premium paid over the policy term. Combining the results in equation (12) and equation (13), we have:

$$V_n^- = \Pi \sum_{\tau=0}^{n-1} \frac{1}{n} F_\tau \tag{14}$$

The result in equation (14) implies that any contingent claim on the policyholder's fund will take the form of a contingent claim on the arithmetic average of the underlying fund prices across the policy term, i.e. take the form of an arithmetic Asian option on the underlying fund. The value of Asian options do not have analytical solutions and various approximations are available in the form of numerical methods or double integral type bounds. Rogers and Shi (1995) suggest a conditional lower bound in the continuous averaging case. Dhaene et al. (2002a) derived comonotonic bounds for an arithmetic Asian option and illustrated the efficacy of these bounds in a Black-Scholes-Merton setting. In particular, the authors demonstrated the incredible accuracy of the conditional lower bound as an approximation to the exact value of an arithmetic Asian option.

2.3 Conditional lower bound (CLB) approximation

In the following, we approximate the distribution of a sum of partially dependent random variables such that true fund value V_n^- stochastically dominates the approximation of the fund $V_n^{-(l)}$ in a convex order sense, but in an optimal way. The conditional lower bound is defined to be the expection of V_n^- conditioned on some information variable Λ, i.e.:

$$V_n^{-(l)} = E\left[V_n^- \mid \Lambda\right] \tag{15}$$

Ideally, the choice of Λ is such that V_n^- and Λ are as alike as possible.

This choice reduces to opting for a significant level of dependence between V_n^- and Λ. In the following, we initially choose Λ such that it is a linear combination of the variability of V_n^-, i.e.:

$$\Lambda = \sum_{k=0}^{n-1} \gamma_k \left[W(n) - W(k) \right] \tag{16}$$

where γ_k is a deterministic constant. The CLB approximation is optimised by selecting the appropriate value of the contant γ_k. The CLB approximation can also be optimised with respect to the tail of the distribution, e.g. Vanduffel et al. (2008) propose locally optimal approximations in the sense that the relevant tail of the distribution of $E\left[S\,|\Lambda\right]$ is an accurate approximation for the corresponding tail of the distribution of S.

The variability of the fund value in equation (12) and the conditional random variable Λ in equation (16) stem from the Brownian motion differences, $Y_k = W(n) - W(k)$, for $k = 0, 1, ..., n-1$. The distribution of any Brownian difference Y_k given Λ follows a conditional bivariate normal distribution. The CLB approximation that is yet to be optimised therefore follows from equation (15):

$$\begin{aligned} V_n^{-(l)} &= \sum_{k=0}^{n-1} \pi_k E\left[\frac{F(n)}{F(k)} \mid \Lambda \right] = \sum_{k=0}^{n-1} \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2\right)(n-k)} E\left[e^{\sigma Y_k} \mid \Lambda \right] \\ &= \sum_{k=0}^{n-1} \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} \end{aligned} \tag{17}$$

where r_k is Pearson's correlation coefficient for the pair (Y_k, Λ) and can be found in the usual way:

$$r_k = \frac{Cov\left[Y_k, \Lambda\right]}{\sigma_{Y_k}\sigma_\Lambda}$$

where the covariance of the pair (Y_k, Λ) is given by:

$$Cov\left[Y_k, \Lambda\right] = \sum_{l=0}^{n-1} \gamma_l \min(n-k, n-l) \tag{18}$$

The expression in equation (17) is not an optimal bound and the variance of $V_n^{-(l)}$ needs to be maximised with respect to Λ in order to optimise the CLB approximation. The variance of $V_n^{-(l)}$ is given by:

$$Var\left[V_n^{-(l)}\right] = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \pi_k \pi_l e^{\delta(2n-k-l)} \left(e^{\sigma^2 r_k r_l \sqrt{n-k}\sqrt{n-l}} - 1\right) \tag{19}$$

By expanding the exponential term in brackets by a first order Taylor series, we can approximate the variance of $V_n^{-(l)}$ by:

$$Var\left[V_n^{-(l)}\right] \approx \sigma^2 r_S^2 \sigma_S^2 \tag{20}$$

where $S = \sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} Y_k$, and the correlation coefficient of the pair (S, Λ) is denoted by r_S. We can therefore maximise the expression for the variance of $V_n^{-(l)}$ in equation (20) by maximising the correlation coefficient r_S. This is only the case if the pair (S, Λ) is perfectly correlated, negatively or positively. The optimum choice of Λ is therefore given as:

$$\Lambda = S = \sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} Y_k \tag{21}$$

Comparing our optimal choice in equation (21) to our initial choice in equation (16), it is evident that the optimal choice remains a linear combination of the Brownian differences Y_k but with the constant γ_k defined as:

$$\gamma_k = \pi_k e^{\delta(n-k)} \tag{22}$$

The correlation coefficient r_k therefore becomes:

$$r_k = \frac{\sum_{l=0}^{n-1} \pi_l e^{\delta(n-l)} \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j \pi_l e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{23}$$

2.4 Value of the embedded options

Now that we have an approximate distribution for the value of the fund V_n^- under the equivalent martingale measure Q, we can use this distribution to approximate the discounted payoff or value of the embedded guarantee of our simple financial contract. From equation (17), it is evident that $V_n^{-(l)}$ is a function of a multiple of the standard normal random variable Z. By using the expression in Definition 3, we find:

$$E\left[\left(\sum_{k=0}^{n-1} \alpha_k e^{\sigma r_k \sqrt{n-k} Z} - b_n\right)_+\right] = \sum_{k=0}^{n-1} E\left[\left(X_k - F_{X_k}^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right)_+\right] \tag{24}$$

where $F_X(x)$ and $F_X^{-1}(p)$ denote the cumulative distribution function of the random variable X in the value x and the inverse of the function in the value p, respectively. We also defined the lognormal random variable $X_k = \alpha_k e^{\sigma r_k \sqrt{n-k} Z}$, where $\alpha_k = \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)}$.

By using the result of Definition 2, we find an expression for the inverse distribution function of the CLB approximation $V_n^{-(l)}$:

$$F^{-1}_{V_n^{-(l)}}(p) = \sum_{k=0}^{n-1} F^{-1}_{X_k}(p) = \sum_{k=0}^{n-1} \alpha_k e^{\sigma r_k \sqrt{n-k}\Phi^{-1}(p)} \tag{25}$$

We now have the tools needed in order to determine the value of the discounted payoffs as given in equations (3) and (4). First, consider the call-type payoff in equation (3). By using the results from equations (24) and (25), we obtain:

$$\begin{aligned} & E\left[\left(\sum_{k=0}^{n-1} \alpha_k e^{\sigma r_k \sqrt{n-k}Z} - b_n\right)_+\right] \\ = & \sum_{k=0}^{n-1} \alpha_k E\left[\left(e^{\sigma r_k \sqrt{n-k}Z} - e^{\sigma r_k \sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)}\right)_+\right] \end{aligned} \tag{26}$$

The lognormal random variable X leads to the final result for the value of the call-type payoff:

$$\begin{aligned} & E\left[\left(V_n^{-(l)} - b_n\right)_+\right] \\ & = \sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] - b_n \left[1 - F_{V_n^{-(l)}}(b_n)\right] \end{aligned} \tag{27}$$

The put-type payoff in equation (4) is found by using put-call parity, i.e.:

$$E\left[\left(b_n - V_n^{-(l)}\right)_+\right] = E\left[\left(V_n^{-(l)} - b_n\right)_+\right] + b_n - E\left[V_n^{-(l)}\right]. \tag{28}$$

The result for the value of the put-type payoff is given by:

$$E\left[\left(b_n - V_n^{-(l)}\right)_+\right]$$
$$= -\sum_{k=0}^{n-1} \pi_k e^{\delta(n-k)} \Phi\left[-\sigma r_k \sqrt{n-k} + \Phi^{-1}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right] + b_n F_{V_n^{-(l)}}\left(b_n\right) \quad (29)$$

The unknown quantity $F_{V_n^{-(l)}}\left(b_n\right)$ in equation (27) and equation (29) is solved by using the expression in equation (25), i.e.:

$$\sum_{k=0}^{n-1} \pi_k e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} e^{\sigma r_k \sqrt{n-k}\Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)} - b_n = 0 \quad (30)$$

2.5 Asset based price of the embedded option

In the previous section, the value of the embedded options at inception was determined. In this section, we determine the periodic price charged in respect of the embedded options. An annual management fee is charged on the value of the fund at the end of each respective year or contribution period. The charge can be stated annually but deducted monthly or quarterly. Note that at inception the embedded option has to be purchased by the insurer or a replicating portfolio has to be set up in order to effectively manage the risk of the guarantee. The management charges therefore recover the initial guarantee liability outgo. The annual management fee solution is a discrete case of the continuous pricing solution, which is a deduction in the yield of the underlying fund. Milevsky (2006) gives a structured approach to obtaining the actual price of the guarantee when recouping the cost of the guarantee continuously from the underlying fund.

Consider the asset based charge e deducted annually from the fund V_n^-. The fund value at time n is given as:

$$V_n^- = \sum_{k=0}^{n-1} \pi_k \left(1-e\right)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2\right)(n-k)+\sigma(W(n)-W(k))} \tag{31}$$

Comparing equation (31) above to equation (11) we see that only the premium vector changes, i.e. the guarantee charge reduces the contribution that participates in the fund growth. By following the same steps in deriving the CLB approximation as in section 2.3, we find the value of the optimising constant γ_k of the information variable Λ as:

$$\gamma_k = \pi_k \left(1-e\right)^{n-k} e^{\delta(n-k)} \tag{32}$$

and the optimal correlation coefficient r_k as:

$$r_k = \frac{\sum_{l=0}^{n-1} \pi_l \left(1-e\right)^{n-l} e^{\delta(n-l)} \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j \pi_l \left(1-e\right)^{2n-j-l} e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{33}$$

The value of the embedded call option at inception of the contract is given as:

$$\begin{aligned} C_0 &= \sum_{k=0}^{n-1} \pi_k \left(1-e\right)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}\left(b_n\right)\right)\right] \\ &\quad - e^{-\delta n} b_n \left[1 - F_{V_n^{-(l)}}\left(b_n\right)\right] \end{aligned} \tag{34}$$

and the value of the embedded put option at inception of the contract as:

$$
\begin{aligned}
P_0 \;\; = \;\; & -\sum_{k=0}^{n-1} \pi_k \left(1-e\right)^{n-k} e^{-\delta k} \Phi \left[-\sigma r_k \sqrt{n-k} + \Phi^{-1} \left(F_{V_n^{-(l)}} \left(b_n\right) \right) \right] \\
& + e^{-\delta n} b_n F_{V_n^{-(l)}} \left(b_n\right) \qquad (35)
\end{aligned}
$$

Again, the unknown quantity $F_{V_n^{-(l)}} \left(b_n\right)$ in the expressions of equations (34) and (35) is found by using the result in equation (25), i.e.:

$$
\sum_{k=0}^{n-1} \pi_k \left(1-e\right)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k)} e^{\sigma r_k \sqrt{n-k} \Phi^{-1} \left(F_{V_n^{-(l)}} (b_n) \right)} - b_n = 0 \qquad (36)
$$

The annual charge for the guarantee e can now be found by fair value principles, i.e. equating the expected present value of the premium with the expected present value of the benefits. For the embedded put option, we have:

$$
\sum_{k=0}^{n-1} \pi_k e^{-\delta k} = e^{-\delta n} E\left[V_n^{-(l)}\right] + P_0 \qquad (37)
$$

The unknown guarantee charge e is therefore found by numerically solving the following expression with respect to e:

$$
e^{-\delta n} b_n F_{V_n^{-(l)}} \left(b_n\right) - \sum_{k=0}^{n-1} \pi_k e^{-\delta k} \left[1 - \left(1-e\right)^{n-k} \Phi \left[\sigma r_k \sqrt{n-k} - \Phi^{-1} \left(F_{V_n^{-(l)}} \left(b_n\right) \right) \right] \right] = 0 \qquad (38)
$$

The derivation for the embedded call option follows in a similar way.

2.6 Illustrative Example

The accuracy of the conditional lower bound approximation is illustrated in the following by determining the value of the embedded guarantee using a Monte Carlo method as a proxy of the true value. In Table 1, we calculated the discounted values of the embedded put options for the simplified product. The accuracy of the call options follow by put-call parity.Variance reduction in the form of antithetic variates was used. Simulated results are based on 50 000 paths for each set of parameter values. We assume a continuous risk-free rate of 5% per annum with volatilities of 20%, 30% or 40% per annum. The contributions π_k, $k = 0, ..., n$, are assumed at 100 per annum and the term of the contract is taken as $n = 10$ years. We also vary the guaranteed amount b_n by considering the following five values $(500,\ 750,\ 1000,\ 1250,\ 1500)$. The *CLB* column gives the conditional lower bound approximation, while the *MC* and *s.e.* columns give the Monte Carlo estimate with associated standard error.

The annual charge was solved for by using equation (38) and is given for different parameter values in Table 2. The charges are given in basis points (bps) of the fund value charged annually. Charges in excess of or approaching 100% are omitted.

It is evident from Table 2 that the charges increase substantially as the volatility increases or as the risk-free rate decreases. The sensitivity of the option value to these parameters and therefore of the price of the option need to be considered in a comprehensive risk management strategy. In section 4, we consider and test a possible hedging strategy that aims to immunise the

Table 1: Conditional Lower Bound (CLB) estimates of the embedded put option values with $n = 10$, $\delta = 5\%$ and varying volatilities, compared to Monte Carlo (MC) estimates and their standard errors (s.e.).

σ	b_n	CLB	MC	s.e.
20%	500	0.2899	0.3191	0.00061
	750	7.6583	7.7911	0.00368
	1000	39.3632	39.5205	0.00924
	1250	104.2183	104.3376	0.01103
	1500	198.3930	198.5049	0.00816
30%	500	4.6067	4.9362	0.00299
	750	30.2476	30.7541	0.00824
	1000	84.6857	85.1418	0.01132
	1250	164.6151	164.9986	0.01243
	1500	264.0077	264.3668	0.00794
40%	500	15.6902	16.7220	0.00561
	750	60.3649	61.5619	0.01058
	1000	131.4565	132.5241	0.01172
	1250	222.2414	223.1759	0.01005
	1500	327.2443	328.0961	0.00796

Table 2: Charges in basis points (bps) for the embedded put options with $n = 10$ and varying volatilities and risk-free rates.

	b_n	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
$\delta = 1\%$	500	0.3664	2.8282	7.5429
	750	6.9304	18.7686	32.8233
	1000	51.1506	86.5640	120.8808
$\delta = 5\%$	500	0.06095	0.9881	3.4656
	750	1.6931	7.1870	15.1582
	1000	10.5377	25.1368	41.4655
	1250	48.1448	81.8928	114.5406
$\delta = 10\%$	500	0.00425	0.2254	1.2115
	750	0.2218	2.0356	5.7732
	1000	1.7803	7.3267	15.2321
	1250	6.9371	18.4198	31.8342
	1500	20.2496	40.7130	61.8769

important risk factors.

3 Life-contingent guaranteed minimum benefits

In this section, we generalise the simplified pure investment product of section 2 to allow for mortality risk by considering guaranteed death and guaranteed survival benefits. The premiums for ancillary risk benefits such as dread disease, disability and premium waiver benefits are usually deducted from the policyholder's contributions before the contribution is invested in the sub-account. We assume that the insurer has a risk-neutral position to mortality risk, i.e. the VA business portfolio consists of a large enough population of independent policyholders such that the aggregate mortality risk is diversified. In the case where mortality risk is fully diversifiable, the random lifetimes of policyholders can be replaced by their expected lifetimes. The Law of Large Numbers imply that the variability in the expected value of the independent random lifetimes of a cohort of identical policyholders tends to zero, i.e. the expected lifetime of a cohort of policyholders becomes deterministic. We also assume that mortality risk is independent of investment risk, thereby allowing us to express the real world probability of an integrated risk event as a the product of the individual investment risk and mortality risk events. Independence between financial and biometric risks are not necessarily maintained when replacing the physical probability measure with the equivalent martingale measure. Dhaene et al. (2010) provides an overview of the implications of modelling integrated risks in the combined market of financial and biometric risks. We assume in the following that the combined market is complete and that independence between the sub-markets holds for the equivalent martingale measure.

Consider a VA product that is underwritten on a life aged x at time 0 and who aims to annuitise the VA in n years. Annual premiums π_k are used to purchase units of the opted underlying investment fund. An allocation cost charge c is charged for when purchasing units, which implies that $(1-c)\,\pi_k$ worth of units are available of each contribution π_k. We assume for simplicity that any fractional unit of the underlying fund can be purchased. The allocation cost typically covers the intial expenses, commission or advisory fees and ongoing allocation expenses. The allocated premiums are used to purchase units of the investment fund at the fund's bid price F_k at the start of the year $(k, k+1)$. We assume that the bid price of one unit of the investment fund at time t is equal to the value of the fund F_k, i.e. the investment fund prices are given net of the bid-offer spread. The insurer charges the bid-offer spread by increasing the offer price at time k, i.e. setting the offer price at $\frac{F_k}{1-\alpha}$ for some value $0 < \alpha < 1$. This means that the policyholder has to buy the units at a higher price than the value of the fund. The premium for ancillary mortality benefits is charged as a deduction, the risk premium $\pi_k^{(r)}$, from the regular contribution. The savings premium or net contribution at time k is defined by:

$$\pi_k^{(s)} = (1-\alpha)\,(1-c)\,\pi_k - \pi_k^{(r)} \tag{39}$$

Assuming that the policyholder is still alive at time k and that no units have been withdrawn before time k, the fund value V_{k+1}^{-} at time $k+1$ is given by:

$$V_{k+1}^{-} = \sum_{j=0}^{k} \pi_j^{(s)} \frac{F_{k+1}}{F_j} (1-e)^{k+1-j}, k = 0, ..., n-1 \tag{40}$$

3.1 Guaranteed minimum maturity benefit

We assume that a sufficiently large book of independent insured lives are held by the insurer. This means that the probability that a policyholder aged x surviving to time t is replaced by the frequency of survival of the cohort of the population aged x:

$$P[T_x > t] = {}_tp_x \tag{41}$$

where T_x denotes the random lifetime of a policyholder aged x. The CLB approximation for the sub-account after allowing for deductions and charges is given as:

$$V_n^{-} = \sum_{k=0}^{n-1} \pi_k^{(s)} (1-e)^{n-k} e^{\left(\delta - \frac{1}{2}\sigma^2\right)(n-k) + \sigma(W(n) - W(k))} \tag{42}$$

By following the same steps in deriving the CLB approximation as in section 2.3, we find the value of the optimising constant γ_k of the information variable Λ as:

$$\begin{aligned} \gamma_k &= \left((1-\alpha)(1-c)\pi_k - \pi_k^{(r)}\right)(1-e)^{n-k} e^{\delta(n-k)} \\ &= \pi_k^{(s)} (1-e)^{n-k} e^{\delta(n-k)} \end{aligned} \tag{43}$$

and the optimal correlation coefficient r_k as:

$$r_k = \frac{\sum_{l=0}^{n-1} \pi_l^{(s)} (1-e)^{n-l} e^{\delta(n-l)} \min(n-k, n-l)}{\sqrt{n-k}\sqrt{\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} \pi_j^{(s)} \pi_l^{(s)} (1-e)^{2n-j-l} e^{\delta(2n-j-l)} \min(n-j, n-l)}} \tag{44}$$

The present value at time 0 of the embedded call option is given by:

$$\begin{aligned} C_0 &= \sum_{k=0}^{n-1} {}_n p_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad - {}_n p_x e^{-\delta n} b_n \left[1 - F_{V_n^{-(l)}}(b_n)\right] \end{aligned} \tag{45}$$

and the present value of the embedded put option is given by:

$$\begin{aligned} P_0 &= -\sum_{k=0}^{n-1} {}_n p_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[-\sigma r_k \sqrt{n-k} + \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad + {}_n p_x e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) \end{aligned} \tag{46}$$

In the case of the put-type option representation of the GMMB, the fair value principle leads to the asset-based charge e to be solved from:

$$\begin{aligned} \sum_{k=0}^{n-1} {}_k p_x \pi_k^{(s)} e^{-\delta k} &= {}_n p_x e^{-\delta n} E\left[V_n^{-(l)}\right] + {}_n p_x P_0 \\ &= \sum_{k=0}^{n-1} {}_n p_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi\left[\sigma r_k \sqrt{n-k} - \Phi^{-1}\left(F_{V_n^{-(l)}}(b_n)\right)\right] \\ &\quad + {}_n p_x e^{-\delta n} b_n F_{V_n^{-(l)}}(b_n) \end{aligned} \tag{47}$$

Note that the present value of contributions, left-hand side of equation (47), excludes the contribution attributable to ancillary risk benefits and allocation charges. For the call-type option representation of the GMMB, which equals a n-term bond and a call option, the application of the fair value principle results in:

$$\begin{aligned}\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} &= {}_np_x e^{-\delta n} b_n + {}_np_x C_0 \\ &= \sum_{k=0}^{n-1} {}_np_x \pi_k^{(s)} (1-e)^{n-k} e^{-\delta k} \Phi \left[\sigma r_k \sqrt{n-k} - \Phi^{-1} \left(F_{V_n^{-(l)}} (b_n) \right) \right] \\ &\quad + {}_np_x e^{-\delta n} b_n F_{V_n^{-(l)}} (b_n) \qquad (48)\end{aligned}$$

It is evident from equations (47) and (48) that the solution for the guarantee charge e is identical for the call-type option and put-type option representations.

3.2 Guaranteed minimum death benefit

The risk premium $\pi_k^{(r)}$ can be used to explicitly charge for the GMDB on a periodic basis, i.e. the risk premium $\pi_k^{(r)}$ is solved for from the expression:

$$\pi_k^{(r)} = A(x+k) e^{-\delta} E[S_R] \qquad (49)$$

where $A(x+k)$ is the cost-loaded actuarial premium for a one-year term insurance with death benefit equal to 1 and sold to the policyholder of age $x+k$, and where:

$$S_R = \max\left(b_{k+1} - V_{k+1}^-,\ 0\right) \qquad (50)$$

where V_{k+1}^- contains the risk premium $\pi_k^{(r)}$, cf. equation (39).

We assume in the following that the charge for the GMDB is recouped by an asset-based charge e_g that forms part of the regular management fee e, i.e. $e = e_a + e_g$ where e_a denote all other asset-based charges such as fund management charges. The call option and put option representations remain the same as in equations (45) and (46), but we now need to solve for the asset-based charge e_g by applying fair value principles. In the case of the put-type option representation of the GMDB, the asset-based charge e_g is solved from the expression:

$$\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} = \sum_{k=0}^{n-1} {}_kp_x \; q_{x+k} e^{-\delta(k+1)} E\left[V_{k+1}^{-(l)} + \left(b_{k+1} - V_{k+1}^{-(l)}\right)_+\right] \quad (51)$$

The asset-based charge for other fees besides the investment guarantee charge is assumed to be exogenously given. Therefore, the only unknown in equation (51) is the GMDB asset-based charge e_g.

3.3 Guaranteed minimum death and survival benefits

Endowment type VA products typically offer a guaranteed minimum benefit on both death and survival. Since these life-contingent events are mutually exclusive, the investment guarantee benefit simplifies to being the sum of the two life-contingent benefits, i.e.:

$$\begin{aligned} B_0 &= B_0^q + B_0^p \\ &= \sum_{k=0}^{n-1} {}_kp_x \; q_{x+k} e^{-\delta(k+1)} E\left[\max\left(b_{k+1}^q, V_{k+1}^{-(l)}\right)_+\right] \\ &\quad + {}_tp_x e^{-\delta n} E\left[\max\left(V_n^{-(l)}, b_n^p\right)\right] \end{aligned} \quad (52)$$

where b_{k+1}^q denotes the guaranteed minimum death benefit and b_n^p denotes the guaranteed minimum survival benefit. By using the split of the annual management charge, we solve for the unknown asset-based guarantee charge e_g from the expression:

$$\sum_{k=0}^{n-1} {}_kp_x \pi_k^{(s)} e^{-\delta k} = B_0^q + B_0^p \tag{53}$$

Note that we assume one asset-based guarantee charge for both the death and survival benefits. Note that should the insurer guarantee contributions inclusive of ancillary benefit charges and allocation charges, the savings premium $\pi_k^{(s)}$ is then replaced by the full contribution π_k on the discounted contributions side. The portion of the total guarantee charge applicable to the death benefit can be established by:

$$\frac{B_0^q}{B_0} e_g \tag{54}$$

and similarly for the portion of the guarantee charge applicable to the survival benefit:

$$\frac{B_0^p}{B_0} e_g \tag{55}$$

3.4 Illustrative example

In the following, we demonstrate the application of the results obtained for solving for the asset-based guarantee charge e_g. We consider the guarantee charge for the death benefit and for the survival benefit for varied economic assumptions. The contributions π_k, $k = 0, ..., n$, are assumed at 100 per annum with a contract term of $n = 10$ years. The GMDB, b_{k+1}, and GMMB,

b_n, are assumed to provide return of contributions . We also consider a benefit equal to 50%, 75%, 125% and 150% of contributions. The policyholder is assumed to be a male aged 30 at inception of the policy with mortality according to the PMA92 mortality tables as published by the Continuous Mortality Investigation Bureau (CMIB) of the Institute and Faculty of Actuaries, U.K. A deterministic mortality assumption can result in serious consequences to an insurer's solvency if its book of policies has few policyholders or if the book of policies consist of largely homogeneous policyholders that are susceptible to the same mortality shocks. In such cases, the Law of Large Numbers does not apply and the added mortality risk implies an additional cost, which must be charged to the policyholder for the insurer to accept the addtional risk. This cost stems from either more aggressive reinsurance treaties or more stringent reserving requirements.

An extreme upper bound is considered for the embedded option value in the case of the GMMB. If the underlying fund value were to fall to zero, the payoff becomes certain and the only randomness relates to the survival of the policyholder. The benefit therefore becomes a pure endowment contract with a sum assured equal to the certain payoff. It is evident from Table 3 that the value of the embedded option in the combined market case will approach the extreme upper bound for large values of the volatility parameter and small values of the risk-free rate parameter, i.e. implying no time-value of money.

In Table 4, the extreme upper bound to the value of the embedded option is also given. Similar to the GMMB example, the only randomness relates to the mortality of the policyholder. Therefore, in this case the benefit would

Table 3: Conditional Lower Bound (CLB) results for the discounted values of the GMDB embedded put options with $n = 10$ for a policyholder aged 30.

δ	b_n	σ 20%	30%	40%	Pure Endowment
1%	50%	1.9260	14.2503	36.3826	451.5138
	75%	31.1084	76.2113	125.1575	677.2707
	100%	120.4741	189.4874	255.4479	903.0276
	125%	266.2231	340.6167	413.5743	1128.7845
	150%	448.6732	516.8435	590.2473	1354.5413
5%	50%	0.2893	4.5975	15.6588	302.6587
	75%	7.6430	30.1871	60.2442	453.9881
	100%	39.2845	84.5163	131.1935	605.3175
	125%	104.0098	164.2858	221.7969	756.6468
	150%	197.9962	263.4797	326.5898	907.9762
10%	50%	0.0178	0.9375	4.9634	183.5718
	75%	0.9197	8.1576	22.2728	275.3577
	100%	7.0436	26.9571	53.0722	367.1436
	125%	24.3388	58.5864	95.3486	458.9295
	150%	55.9512	101.8676	146.7768	550.7154

be the term life death benefit with a sum assured equal to the certain payoff in the event of death. Again, the embedded option value in the combined market case will approach the extreme upper bound for high volatility values and low risk-free parameter values.

4 Hedging strategies

4.1 Introduction

In the following, we derive sensitivity measures, the so-called greeks, to implement a dynamic hedging strategy. A good review of dynamic hedging strategies and the various measures used can be found in Taleb (1997) and in Hull (2008). Life companies might opt for static hedging strategies, as opposed

Table 4: Conditional Lower Bound (CLB) results for the discounted values of the GMMB embedded put options with $n = 10$ for a policyholder aged 30.

δ	b_n	σ 20%	30%	40%	Term Life Benefit
1%	50%	0.0012	0.0106	0.0298	0.5309
	75%	0.0265	0.0696	0.1183	0.7964
	100%	0.1242	0.1943	0.2620	1.0618
	125%	0.2997	0.3720	0.4451	1.3273
	150%	0.5229	0.5840	0.6542	1.5927
5%	50%	0.0002	0.0038	0.0144	0.4014
	75%	0.0079	0.0320	0.0650	0.6021
	100%	0.0511	0.1024	0.1550	0.8028
	125%	0.1487	0.2135	0.2764	1.0035
	150%	0.2907	0.3547	0.4202	1.2043
10%	50%	0.00002	0.0010	0.0055	0.2871
	75%	0.0014	0.0113	0.0297	0.4306
	100%	0.0148	0.0437	0.0785	0.5741
	125%	0.0565	0.1025	0.1497	0.7177
	150%	0.1297	0.1843	0.2384	0.8612

to dynamic hedging strategies, should the investment guarantee structure allow it, e.g. recurring premium business with a GMMB requires a portfolio of forward-starting European put options to be hedged according to a static hedging strategy where these might not be available in the market. In this section, we consider approximate measures for the three greeks typically considered by life insurance firms in their dynamic hedging programmes, i.e. the delta, vega and rho of a portfolio.

It might be possible for an insurer to transfer its exposure to investment risk to a third party, such as a reinsurer or an investment bank, although these markets might prove infeasible during and following financial turmoil. Reinsurance firms typically have a set risk appetite for investment business and do not readily take on the investment risk, while Investment banks can offer a structured product to the insurer to transfer some or all of the investment risk or aid in the management of an existing hedging programme. The application of using existing market instruments, including over the counter instruments, to set up a static hedging portfolio is formally considered by, for example, Chen et al. (2008).

4.2 A Proposed Hedging Solution

In order to derive the delta at time t of the embedded options, we need to express the values of the embedded options in terms of the fund price at time t, where t is defined on the continuous interval $(0, n)$. In the previous sections, we assumed that time elapses in a discrete way, i.e. that the time variable k measures the discrete moments in time $k = 0, 1, ..., n-1$. In the following, we first derive an expression for the value of the embedded option at some instantaneous time t, where $0 \leq t \leq n$ and then find the delta measure by taking the derivative of the derived expression with respect to the underlying fund price at time t, F_t. In the follwing, we consider only the put-type embedded option since the derivation for the call-type option follows in a similar way.

In our dervivation of the CLB approximation, the initial fund value $F(0)$ cancelled out. Assume now the filtration $\mathcal{F}_t$ and $k < t$, we therefore have:

$$\frac{F(n)}{F(k)} = \frac{F(t)}{F(k)} \exp\left\{ \left(\delta - \frac{\sigma^2}{2}\right)(n-t) + \sigma\left(W(n) - W(t)\right) \right\} \tag{56}$$

The CLB approximation for the fund value V_n^- is given as:

$$V_n^{-(l)} = \sum_{k=0}^{n-1} \pi_k^{(s)} g_k(Z) \tag{57}$$

where $g_k(Z)$ is defined as follows:

$$g_k(Z) = \begin{cases} \frac{F(t)}{F(k)} e^{\left(\delta - \frac{1}{2}\sigma^2 r_t^2\right)(n-t) + \sigma r_t \sqrt{n-t} Z} & \text{for } k < t \\ e^{\left(\delta - \frac{1}{2}\sigma^2 r_k^2\right)(n-k) + \sigma r_k \sqrt{n-k} Z} & \text{for } k \geq t \end{cases} \tag{58}$$

By following the same steps in deriving the CLB approximation as in section 2.3, we find the value of the embedded put option at time t as:

$$\begin{aligned} P_t &= -F(t) \sum_{k=0}^{\lceil t \rceil - 1} \frac{\pi_k^{(s)}}{F(k)} \Phi \left[-\sigma r_t \sqrt{n-t} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] \\ &- \sum_{k=\lceil t \rceil}^{n-1} \pi_k^{(s)} e^{-\delta(k-t)} \Phi \left[-\sigma r_k \sqrt{n-k} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] + e^{-\delta(n-t)} b_n F_{V_n^{-(l)}}(b_n) \end{aligned} \tag{59}$$

where the value of $F_{V_n^{-(l)}}(b_n)$ can be found in the usual way, viz.:

$$\sum_{k=0}^{n-1} F^{-1}_{\pi_k g_k(Z)} \left(F_{V_n^{-(l)}}(b_n) \right) - b_n = 0 \tag{60}$$

The delta measure of the embedded put option of the GMMB in the combined market is given by:

$$\Delta_t = \frac{\partial P_t}{\partial F(t)} = -\, {}_t p_x \sum_{k=0}^{\lceil t \rceil - 1} \frac{\pi_k^{(s)}}{F(k)} \Phi \left[-\sigma r_t \sqrt{n-t} + \Phi^{-1} \left(F_{V_n^{-(l)}}(b_n) \right) \right] \tag{61}$$

The delta measure of the GMDB embedded put option, assuming that the policyholder is alive at time t, is given as:

$$\Delta_t = -\sum_{j=\lceil t\rceil-1}^{n-1} {}_jp_x\ q_{x+j} \sum_{k=0}^{\lceil t\rceil-1} \frac{\pi_k^{(s)}}{F(k)} \Phi\left[-\sigma r_t\sqrt{n-t} + \Phi^{-1}\left(F_{V_{j+1}^{-(l)}}(b_{j+1})\right)\right] \tag{62}$$

The delta measure of an embedded option in the combined market is therefore a weighted sum of the constituent delta measures, where the weights are determined by the probability of the event that drives the benefit, i.e. survival or death.

The vega and rho of the embedded option aim to immunise the portfolio with respect to the model parameters σ and r, respectively, thereby mitigating model risk. Although the delta measure was found analytically, a numeric finite differencing approach could have been used. A point estimate approximation of the embedded option value allows up and down shifts in the embedded option value to be calculated with speed and ease. The vega and rho of the embedded option measure changes in the embedded option value due to changes in the model parameters and are not analytically tractable. Therefore, the vega and rho are calculated using the finite differencing approach. A common approach is to consider both an up and a down shift in the option value, see e.g. Wilmott (2006). The vega for the put-type option is calculated as:

$$\begin{aligned}\mathcal{V} &= \frac{\partial P_t}{\partial \sigma} \\ &\approx \frac{P_t(\sigma+\Delta\sigma) - P_t(\sigma-\Delta\sigma)}{2\Delta\sigma}\end{aligned} \tag{63}$$

where $P_t(\sigma)$ denotes the put-type option value as a function of the given parameter value and $\Delta\sigma$ denotes the magnitude of the small shift in the parameter value. Likewise, we find the rho for the put-type option as:

$$\begin{aligned} \rho_t &= \frac{\partial P_t}{\partial \delta} \\ &\approx \frac{P_t(\delta + \Delta\delta) - P_t(\delta - \Delta\delta)}{2\Delta\delta} \end{aligned} \tag{64}$$

The magnitude of the shift in the parameter value depends on the firm's risk management framework and is usually a function of expected future volatility.

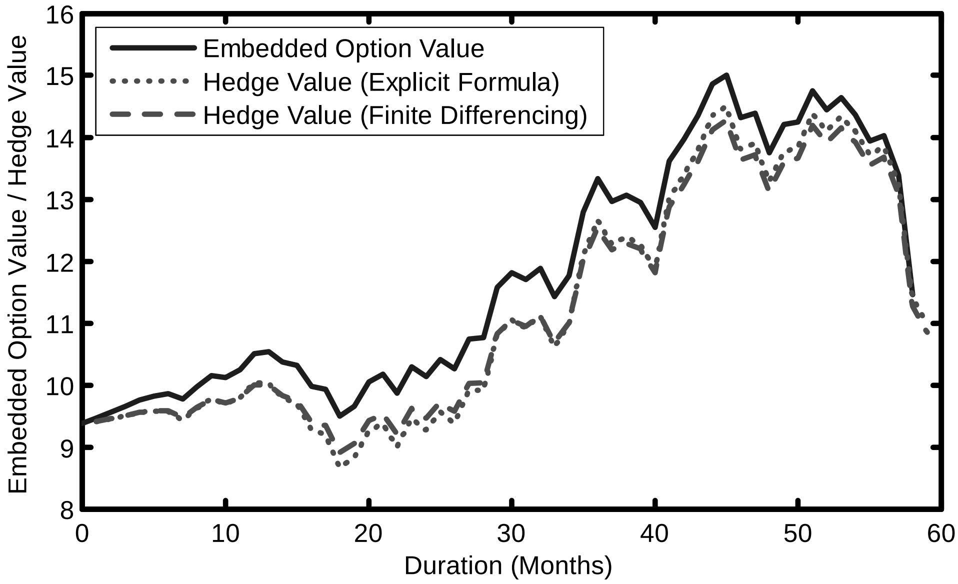
4.3 Illustrative Example

Consider an investor who contributes a monthly contribution of 1 at times, $k = 0, 1, ..., n$, for a term of 5 years, i.e. $n = 60$. For simplicity, we assume first that all payments are paid with certainty and that death is the only decrement, i.e. no lapses or surrenders. To assist the illustration, an in-the-money 200% return of contributions guarantee on the investment is assumed, i.e. a strike price of $b_n = 120$. A single market scenario was simulated from a Black-Scholes-Merton economy with parameters $\sigma = 20\%$ and $\delta = 5\%$. A delta hedging strategy is considered by using both the analytical delta measure of section 4.2 and the delta measure found by finite-differencing, i.e.:

$$\Delta_t \approx \frac{P_t[F(t) + \Delta F(t)] - P_t[F(t) - \Delta F(t)]}{2\Delta F(t)} \tag{65}$$

The GMMB is illustrated in Figure 1, while the GMDB is illustrated

Figure 1: Comparison of performance of delta measures for GMMB.

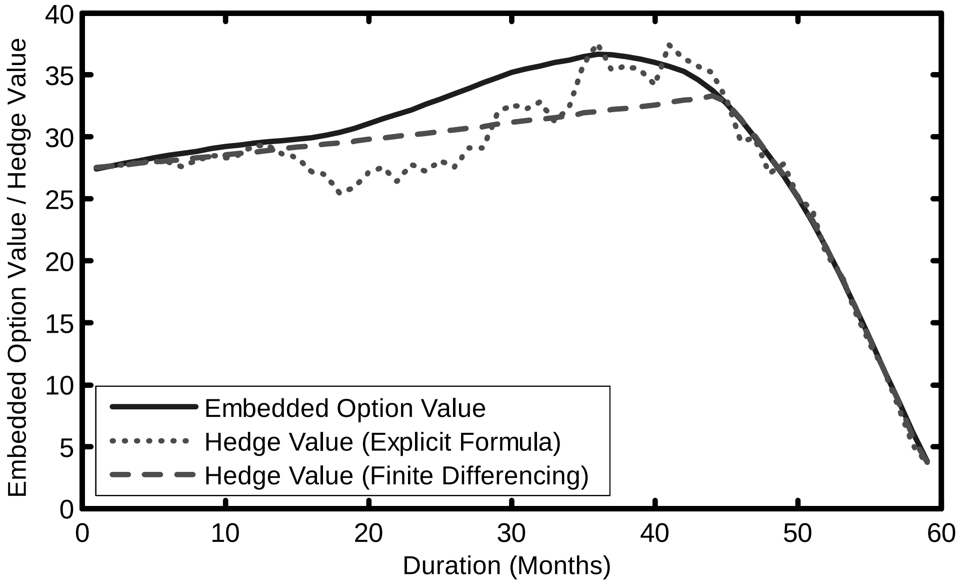


in Figure 2. In both figures, the embedded option value is illustrated by the solid line, while the dotted line represents the approach of equations (61) and (62) and the dashed line represents the approach of equation (65).

5 Conclusion

The complexity of insurance contracts set in the combined market of financial risk and mortality risk pose signficant challenges to the risk management frameworks of insurers. The conditional lower bound approximation aids in addressing the three questions posed by Hardy (2003). It allows quick feasiblity studies of product structures during product development, and estimates of the cost of capital in the case of reserving or the cost of a suitable hedging programme in the case of dynamic or static hedging. The use of the conditional lower bound approximation further allows insurers to verify the

Figure 2: Comparison of performance of delta measures for GMDB.



reasonableness of estimates arising from more complex and resource intensive pricing and reserving models. The conditional lower bound approximation can be extended to allow for more complex asset pricing models and mortality models.

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