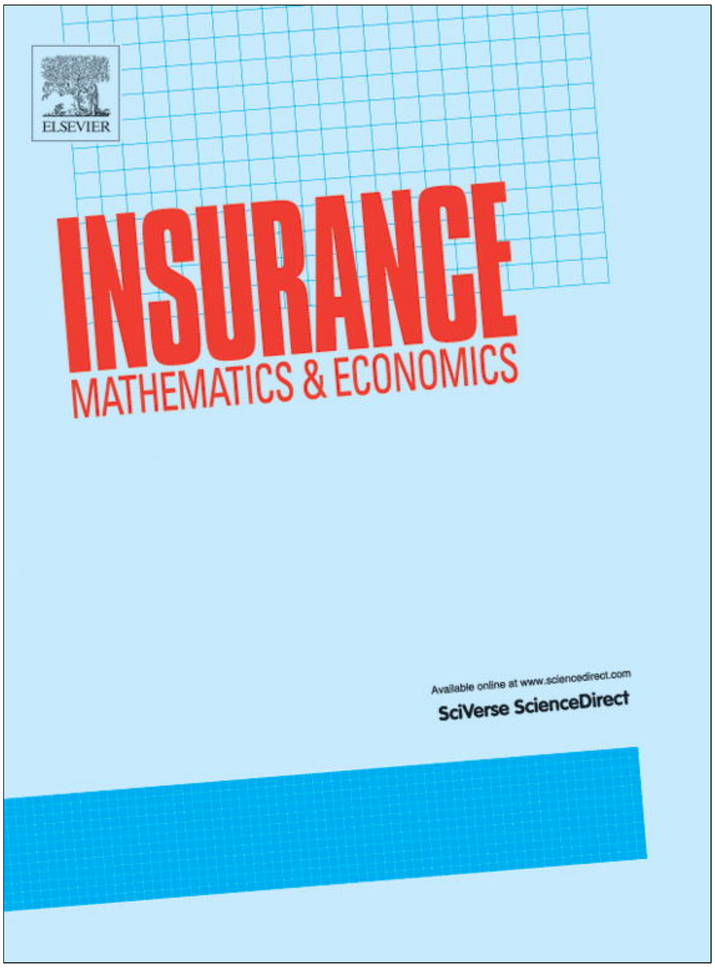
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Convex order and comonotonic conditional mean risk sharing

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ABSTRACT

Using a standard reduction argument based on conditional expectations, this paper argues that risk sharing is always beneficial (with respect to convex order or second degree stochastic dominance) provided the risk-averse agents share the total losses appropriately (whatever the distribution of the losses, their correlation structure and individual degrees of risk aversion). Specifically, all agents hand their individual losses over to a pool and each of them is liable for the conditional expectation of his own loss given the total loss of the pool. We call this risk sharing mechanism the conditional mean risk sharing. If all the conditional expectations involved are non-decreasing functions of the total loss then the conditional mean risk sharing is shown to be Pareto-optimal. Explicit expressions for the individual contributions to the pool are derived in some special cases of interest: independent and identically distributed losses, comonotonic losses, and mutually exclusive losses. In particular, conditions under which this payment rule leads to a comonotonic risk sharing are examined.

1. Introduction and motivation

Loss sharing mechanisms have been studied for decades in the economics and actuarial literatures. The pioneering work by Borch (1960, 1962) considered equilibrium in a reinsurance market. Under appropriate conditions (including that agents are expected utility maximizers and have the same probability on the state space), this author established that any Pareto-optimal loss sharing mechanism is equivalent to a pool arrangement, i.e. all the agents hand their individual losses over to a pool and agree on some rule as to how the total pooled loss has to be divided amongst agents. This fundamental result explains why comonotonicity plays a central role in the study of Pareto-optimality of risk sharing mechanisms, as each component of a comonotonic random vector is (almost surely) equal to a non-decreasing function of the sum of all of its components.

After Borch (1962) established that agents' optimal risk sharing depends only on aggregate loss, Landsberger and Meilijson (1994) have shown that Pareto-optima are comonotonic if agents' preferences agree with second degree stochastic dominance. Specifically, Landsberger and Meilijson (1994) provided an algorithm to construct an improvement of any non-comonotonic risk allocation in the discrete case. This result has been extended to the general case by Dana and Meilijson (2003) and Ludkovski and Rüschendorf (2008). In this paper, we consider the particular conditional mean risk sharing rule and we investigate its comonotonicity and Pareto-optimality. More precisely, we show that whatever the risks faced by decision-makers, there is always a mutually beneficial risk pooling mechanism with respect to second degree stochastic dominance. A noteworthy feature of the analysis conducted in this paper is that risk sharing remains mutually beneficial even if the loss random variables are (positively) correlated. This result is obtained by a standard reduction argument involving conditional expectations, that can be found, e.g., in Dana and Meilijson (2003). In some special cases, explicit expressions for the individual contributions to the pool are derived. We study several particular cases where the risk sharing based on conditional expectations leads to a comonotonic allocation. We also further stress the importance of comonotonicity in the context of Pareto-optimal risk sharing schemes.

Let us briefly describe the contents of this paper. In Section 2, the definition of the convex order is recalled, and some of its basic properties are presented. Section 3 introduces risk sharing and related notions. In Section 4, we define the conditional mean risk allocation and stress the importance of comonotonicity for establishing Pareto-optimality. It is shown that risk-averse decision-makers can always reduce their respective risks by pooling them together. The result guarantees the existence of a mutually beneficial risk exchange. When comonotonic, that risk exchange turns out to be Pareto-optimal. We study the respective contributions of each participant to the pool and establish conditions under which those participants bringing larger losses have to contribute more to the pool, as should hold for any reasonable risk sharing mechanism. In general, the conditional mean risk sharing rule can only be

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applied if we know the conditional distributions of the individual risks, given the total pooled loss. This requires knowing the joint distribution of the individual risks to be pooled. However, there are situations where a weaker form of knowledge is sufficient to apply our conditional mean risk allocation rule. Examples of such situations are given where conditions under which the proposed risk sharing rule produces comonotonic individual payments are also studied. Some particular cases are examined in Section 4: independent and identically distributed losses, comonotonic losses, mutually exclusive losses, and independent losses with log-concave densities.

Henceforth, all the equalities between random variables and random vectors are assumed to hold almost surely, unless stated otherwise.

2. Convex order

Let X and Y be two random variables such that

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)] \quad \text{for all convex functions } g : \mathbb{R} \to \mathbb{R}, \tag{2.1}$$

provided the expectations exist. Then X is said to be smaller than Y in the convex order (denoted as $X \preceq_{CX} Y$). Now, X is said to be strictly smaller than Y in convex order, which is denoted as $X \prec_{CX} Y$, if $X \preceq_{CX} Y$ holds true and X and Y are not identically distributed.

The stochastic inequality $X \preceq_{CX} Y$ intuitively means that X and Y have the same magnitude (as $\mathbb{E}[X] = \mathbb{E}[Y]$ holds) but that Y is more variable than X. For instance, the variance of Y is larger than the variance of X. For a thorough description of the convex order and its applications in an actuarial context, we refer the reader, e.g., to Denuit et al. (2005).

An important characterization of $\preceq_{CX}$ is as follows. The random variables X and Y satisfy $X \preceq_{CX} Y$ if, and only if, there exist two random variables $\widetilde{X}$ and $\widetilde{Y}$, defined on the same probability space, such that $\widetilde{X}$ and X (resp. $\widetilde{Y}$ and Y) are identically distributed, and

$$\mathbb{E}[\widetilde{Y}|\widetilde{X}] = \widetilde{X}. \tag{2.2}$$

More generally, whatever the random variable (or random vector) Z,

$$\mathbb{E}[X|Z] \preceq_{CX} X. \tag{2.3}$$

The economic intuition behind (2.3) is that averaging a loss (i.e., taking a conditional expectation of it) decreases the risk involved (in the sense of convex order). Applications of (2.2)–(2.3) to actuarial science are described in Denuit and Vermandele (1998, 1999). See also Leitner (2004, 2005) for a use of (2.2) in connection with risk measures and Dhaene et al. (2002a,b) for an application of (2.3) in connection with (comonotonic) approximations for sums of non-independent random variables.

The convex order can also be characterized by means of Tail-VaR risk measures. Recall that the Value-at-Risk (or VaR) for a risk X with distribution functions F_X is defined as

$$\text{VaR}[X; p] = F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \quad 0 < p < 1.$$

The Tail-VaR at probability level p is then defined as

$$\text{TVaR}[X; p] = \frac{1}{1-p} \int_p^1 \text{VaR}[X; \epsilon] \, d\epsilon.$$

Then, $X \preceq_{CX} Y$ if, and only if, $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\text{TVaR}[X; p] \leq \text{TVaR}[Y; p]$ holds for all p. See, e.g., Denuit et al. (2005). We will use this characterization of convex order in the proof of our main result. Notice that $X \prec_{CX} Y$ implies that there exists a probability level $p_0 \in (0, 1)$ such that $\text{TVaR}[X; p_0] < \text{TVaR}[Y; p_0]$.

3. Risk sharing

3.1. Definitions

Consider n decision-makers (economic agents), numbered $i = 1, 2, \ldots, n$. Each of them faces a possible risk (or loss), denoted by X_i. No particular assumption is made about the distribution of the random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$.

Definition 3.1 (*Risk Sharing Scheme*)**.** Consider a portfolio of risks represented by the random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. A risk sharing (or risk allocation) scheme for $\boldsymbol{X}$ is a random vector $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X}))$ where the (measurable) functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are such that

$$\sum_{i=1}^n h_i(\boldsymbol{X}) = \sum_{i=1}^n X_i. \tag{3.1}$$

In the end, each agent will pay $(h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \ldots, h_n(\boldsymbol{x}))$ where $\boldsymbol{x}$ is the observed realization of $\boldsymbol{X}$. The condition (3.1) is called the full risk allocation condition. Consider n economic agents facing total risk

$$S = \sum_{i=1}^n X_i. \tag{3.2}$$

In the sequel we will exclusively use the notation S for the total risk (3.2) of the portfolio $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. The risk sharing scheme characterized by $(h_1, h_2, \ldots, h_n)$ allocates the total risk S to the different agents. The i-th agent bears the risk $h_i(\boldsymbol{X})$, $i = 1, 2, \ldots, n$. Notice that we allow the h_i to be depending on (the distribution of) $\boldsymbol{X}$, as it will be the case for the conditional mean risk allocation discussed in the next section.

An important subclass of risk allocations consists of

$$(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X})) = (g_1(S), g_2(S), \ldots, g_n(S))$$

for some functions $g_1, g_2, \ldots, g_n : \mathbb{R} \to \mathbb{R}$. We will call a risk allocation scheme fulfilling this property a risk pooling scheme.

3.2. Pareto-optimality

In this paper, we study Pareto optimal risk sharing schemes. The following definition is in line with Dana and Meilijson (2003).

Definition 3.2 (*Pareto Optimal Risk Sharing Schemes*)**.** A risk sharing scheme $(h_1^\star(\boldsymbol{X}), h_2^\star(\boldsymbol{X}), \ldots, h_n^\star(\boldsymbol{X}))$ for $\boldsymbol{X}$ is Pareto-optimal if there exists no risk sharing scheme $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X}))$ for $\boldsymbol{X}$ such that the stochastic inequalities

$$h_i(\boldsymbol{X}) \preceq_{CX} h_i^\star(\boldsymbol{X})$$

hold for $i = 1, 2, \ldots, n$, with at least one of these convex order inequalities being strict.

Hence, we have that a risk sharing scheme is Pareto-optimal if no agent can be made strictly better off (in the sense of convex order) without worsening the situation of another agent. Notice that we define here better in terms of convex order. In the expected utility paradigm, one has that a risk sharing scheme is Pareto-optimal if there exists no risk sharing scheme that increases the expected utility of all (risk-averse assumed) agents, with a strict increase for at least one of them.

Remark 3.3. Note that the convex order naturally appears in the context of Pareto-optimality, because of the condition (3.1) which rules out many other stochastic order relations. For instance, replacing the convex order with stochastic dominance $\preceq_{ST}$ does not lead to a useful concept, as explained next. Recall the $X \preceq_{ST} Y$ holds if $\Pr[X > t] \leq \Pr[Y > t]$ is valid for all real t. The inequality $X \preceq_{ST} Y$ is strict if X and Y are not identically distributed, that is, if there is at least one value t_0 such that $\Pr[X > t_0] < \Pr[Y > t_0]$. Then, requiring that $h_i(\boldsymbol{X}) \preceq_{ST} h_i^\star(\boldsymbol{X})$ holds for $i = 1, 2, \ldots, n$, with at least one of the stochastic dominance inequalities being strict implies that $\mathbb{E}[h_i(\boldsymbol{X})] \leq \mathbb{E}[h_i^\star(\boldsymbol{X})]$ holds for $i = 1, 2, \ldots, n$, with at least one strict inequality. But this contradicts the full risk allocation condition (3.1) which ensures that $\sum_{i=1}^n \mathbb{E}[h_i(\boldsymbol{X})] = \sum_{i=1}^n \mathbb{E}[h_i^\star(\boldsymbol{X})]$ must hold whatever the risk sharing scheme for $\boldsymbol{X}$.

Remark 3.4. Notice that the definition of Pareto-optimality given in this paper is not the only one possible. For instance, Ludkovski and Rüschendorf (2008) require that there is no risk sharing scheme $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X}))$ for $\boldsymbol{X}$ such that

$$\rho_i\left[h_i(\boldsymbol{X})\right] \leq \rho_i\left[h_i^\star(\boldsymbol{X})\right], \quad i = 1, 2, \ldots, n$$

holds, with at least one strict inequality, and this for a given choice of risk measures $\rho_i[\cdot]$ consistent with $\preceq_{CX}$. See also Goovaerts et al. (2010) for a careful treatment of the difference existing between risk measures and decision principles.

3.3. Comonotonicity of the risk sharing rule

Comonotonicity of the risk sharing rule is closely related to Pareto-optimality, as explained in the Introduction. Recall that a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is said to be comonotonic if there exists a random variable Z and non-decreasing functions f_i such that $\boldsymbol{X}$ is distributed as $(f_1(Z), f_2(Z), \ldots, f_n(Z))$. See, e.g., Dhaene et al. (2002a,b) for a review of comonotonicity and of its applications in actuarial science and finance. Comonotonicity of $\boldsymbol{X}$ is equivalent to stating that there exist continuous and non-decreasing functions $g_i : \mathbb{R} \to \mathbb{R}$ such that

$$g_1(z) + \cdots + g_n(z) = z \quad \text{for any } z \in \mathbb{R}$$

and

$$\boldsymbol{X} = (g_1(S), \ldots, g_n(S)).$$

Broadly speaking, this characterization of comonotonicity implies that a comonotonic risk allocation scheme is always a risk pooling scheme. This explains why comonotonicity is so intimately connected to Pareto-optimality since Borch (1962) established that under mild assumptions, agents' optimal risk sharing depends only on aggregate loss S given in (3.2). Landsberger and Meilijson (1994), Dana and Meilijson (2003), and Ludkovski and Rüschendorf (2008) have shown under various sets of assumptions that Pareto-optima are comonotonic if agents' preferences agree with $\preceq_{CX}$.

4. Conditional mean risk sharing

4.1. Definition

In this paper, we study the risk sharing rule $g_i^\star$, $i = 1, 2, \ldots, n$, defined as

$$g_i^\star(S) = \mathbb{E}[X_i|S]. \tag{4.1}$$

We call (4.1) the conditional mean risk sharing (or allocation) of $X_1, \ldots, X_n$. Clearly, the conditional mean risk allocation satisfies the full allocation condition (3.1) as

$$\sum_{i=1}^n \mathbb{E}[X_i \mid S] = S.$$

In particular, we consider the situation where the functions $s \mapsto g_i^\star(s) = \mathbb{E}[X_i|S = s]$ are non-decreasing for every $i = 1, 2, \ldots, n$, making $\mathbb{E}[X_1|S], \ldots, \mathbb{E}[X_n|S]$ comonotonic.

The next result shows that the conditional mean risk sharing always results in a larger covariance between the individual contributions to the pool, compared to the initial risks.

Property 4.1. *Whatever the individual risks X_1 and X_2 with sum $S = X_1 + X_2$, the covariance between $g_1^\star(S)$ and $g_2^\star(S)$ is always larger than the covariance between X_1 and X_2.*

Proof. Clearly, $X_1 + X_2 = g_1^\star(S) + g_2^\star(S)$ gives

$$\begin{aligned}\mathbb{V}[S] &= \mathbb{V}[X_1] + \mathbb{V}[X_2] + 2\mathbb{C}[X_1, X_2] \\ &= \mathbb{V}[g_1^\star(S)] + \mathbb{V}[g_2^\star(S)] + 2\mathbb{C}[g_1^\star(S), g_2^\star(S)].\end{aligned}$$

Now, as $g_i^\star(S) \preceq_{CX} X_i$ holds, we have $\mathbb{V}[X_i] \geq \mathbb{V}[g_i^\star(S)]$ so that

$$\mathbb{C}\left[g_1^\star(S), g_2^\star(S)\right] \geq \mathbb{C}[X_1, X_2]$$

must indeed hold. □

Note that a pooling arrangement cannot be improved using the reduction technique underlying the conditional mean risk sharing as $\mathbb{E}[g_i(S)|S] = g_i(S)$. We discuss below several important cases where the $g_i^\star$ in (4.1) are non-decreasing, making the conditional mean risk sharing $(g_1^\star(S), \ldots, g_n^\star(S))$ comonotonic.

4.2. Pareto-optimality of the comonotonic conditional mean risk sharing

Either the individuals remain with their own loss X_i, or they start to bargain with each other to find a sharing solution. The next result indicates that there always exists a mutually beneficial risk sharing mechanism (with respect to $\preceq_{CX}$). If economic agents resort to the conditional mean risk allocation then they improve their situation in the $\preceq_{CX}$-sense, meaning that they increase their respective expected utilities (assuming that they are risk-averse). Furthermore, if $\mathbb{E}[X_1|S], \ldots, \mathbb{E}[X_n|S]$ are comonotonic then the conditional mean risk allocation (4.1) is optimal.

Proposition 4.2. *Whatever the X_i's, the conditional mean risk allocation (4.1) is mutually beneficial, that is,*

$$g_i^\star(S) \preceq_{CX} X_i \quad \text{for } i = 1, 2, \ldots, n.$$

If $g_1^\star(S), \ldots, g_n^\star(S)$ are comonotonic, that is, if $s \mapsto g_i^\star(s) = \mathbb{E}[X_i|S = s]$ is non-decreasing for every $i = 1, 2, \ldots, n$, then the conditional mean risk allocation is Pareto-optimal.

Proof. The first part of the result is obvious since a direct application of (2.3) gives

$$g_i^\star(S) = \mathbb{E}[X_i|S] \preceq_{CX} X_i \quad \text{for } i = 1, \ldots, n.$$

The proof of the Pareto-optimality is by contradiction. Assume that there is a risk allocation $g_1(S), \ldots, g_n(S)$ such that

$$g_i(S) \preceq_{CX} \mathbb{E}[X_i|S] \quad \text{for } i = 1, \ldots, n, \quad \text{and} \quad \sum_{i=1}^n g_i(S) = S$$

with a least one strict improvement, in that for some $i_0 \in \{1, \ldots, n\}$ there exists a probability level p_0 such that

$$\text{TVaR}[g_{i_0}(S); p_0] < \text{TVaR}[g_{i_0}^\star(S); p_0].$$

Recall from Dhaene et al. (2006) that TVaR is a subadditive risk measure, which is additive for comonotonic risks. Then, we have

$$\begin{aligned}\text{TVaR}[S; p_0] &\leq \sum_{i=1}^n \text{TVaR}[g_i(S); p_0] \\ &< \sum_{i=1}^n \text{TVaR}[g_i^\star(S); p_0] \\ &= \text{TVaR}[S; p_0]\end{aligned}$$

which is a contradiction. We can conclude that the conditional mean risk sharing is optimal when it results in comonotonic contributions to the pool. □

Note that the conditional mean risk allocation also applies to risks with infinite expectation (like Pareto losses) which are not insurable. Also risk sharing, due to its nature of sharing risks, whatever will be their outcome, avoids the need for keeping capital to ensure solvency. Recall from Kalashnikov and Norberg (2002) that the financial risk often dominates in the insurance industry and greatly increases the insolvency probability for the insurer. Specifically, these authors show that the probability of ultimate ruin decreases slowly (not faster than a power function) if the premiums and reserve are currently invested in a risky asset, that is, an asset that may bear negative interest. The main message of Kalashnikov and Norberg (2002) is that risky investments may impair the insurer's solvency just as severely as do large claims, roughly speaking. In these respects (effectiveness towards large losses and no need for capital), risk sharing is superior to conventional fixed premium insurance.

Notice that Proposition 4.2 can be generalized as follows.

Proposition 4.3. *Any comonotonic risk sharing scheme* $(h_1(\boldsymbol{X}), h_2(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X}))$ *of a portfolio of risks* $\boldsymbol{X}$ *is Pareto-optimal.*

The proof follows the same lines as for Proposition 4.2.

Pareto-optimal risk pooling arrangements are not always appealing in practice. The next result gives an interesting property of the conditional mean risk sharing rule, which is shown to minimize the sum of the expected squared difference between the individual risks to be shared and the pooling arrangement. Hence, the risk share $g_i^\star(S)$ is "as close as possible" to the original risk X_i, taking into account the full risk allocation condition, in the sense of the quadratic distance.

Property 4.4. *The conditional mean risk allocation* $g_1^\star(S), \ldots, g_n^\star(S)$ *for* $\boldsymbol{X}$ *minimizes*

$$\sum_{i=1}^{n} \mathbb{E}\left[(X_i - g_i(S))^2\right]$$

over all risk pooling arrangements $g_1(S), \ldots, g_n(S)$.

Proof. The result immediately follows from the properties of conditional expectations, ensuring that

$$\mathbb{E}\left[\left(X_i - g_i^\star(S)\right)^2\right] \leq \mathbb{E}\left[(X_i - g_i(S))^2\right]$$

must hold for every $i = 1, 2, \ldots, n$, whatever g_i. □

4.3. Respective contributions to the pool

The classical way to compare the respective sizes of two random variables X and Y is by using stochastic dominance $\preceq_{\text{ST}}$. If, given that they belong to some interval, X and Y can still be compared by means of $\preceq_{\text{ST}}$ then the likelihood ratio order $\preceq_{\text{LR}}$ is obtained. Specifically, $X \preceq_{\text{LR}} Y$ if $[X|a \leq X \leq b] \preceq_{\text{ST}} [Y|a \leq Y \leq b]$ holds for all $a < b \in \mathbb{R}$. Let X and Y be continuous (or discrete) random variables with respective probability (or mass) density functions f_X and f_Y. Then,

$$X \preceq_{\text{LR}} Y \Leftrightarrow f_X(u) f_Y(v) \geq f_X(v) f_Y(u) \quad \text{for all } u \leq v. \tag{4.2}$$

For a description of $\preceq_{\text{LR}}$ and its applications in an actuarial context, we refer the reader, e.g., to Denuit et al. (2005).

The next result shows that with the conditional mean risk allocation, an agent bringing a smaller loss to the pool (in the $\preceq_{\text{LR}}$-sense) will indeed contribute a smaller amount in the total loss provided individual losses are independent.

Property 4.5. *Consider a pool with two participants. Their independent losses are denoted as* X_1 *and* X_2, $S = X_1 + X_2$ *being the total loss. Assume that the loss* X_1 *brought to the pool is smaller than* X_2 *in that the stochastic inequality* $X_1 \preceq_{\text{LR}} X_2$ *holds true. Then,*

$$\mathbb{E}[X_1|S = s] \leq \mathbb{E}[X_2|S = s] \quad \text{for all } s$$

so that the contribution paid for X_1 *is always smaller than the contribution paid for* X_2.

Proof. This result can be deduced from Theorem 1.C.26 in Shaked and Shanthikumar (2007) which gives

$$[X_1|S = s] \preceq_{\text{LR}} [X_2|S = s] \quad \text{for all } s.$$

This ensures that the means are ordered accordingly, that is, the announced result holds. □

Note that independence is a crucial assumption in this last result. In general, an agent bringing a larger loss to the pool may nevertheless be required a smaller contribution due to the fact that his loss is negatively related with the total pooled loss.

4.4. Particular cases

Recall that, in general, the conditional mean risk sharing rule can only be applied if we know the conditional distributions of the X_i, given the aggregate claims S. Broadly speaking, this requirement comes down to knowing the joint distribution of the random vector $\boldsymbol{X}$. This section discusses situations where a weaker form of knowledge is sufficient to apply the conditional mean allocation rule.

4.4.1. Independent and identically distributed risks

If the X_i's are independent and identically distributed then it is well-known that

$$g_i^\star(S) = \frac{S}{n}.$$

In such a case, the individuals share the total losses equally. Since the $g_i^\star$ are indeed non-decreasing, we know from Proposition 4.2 that the conditional mean risk allocation (4.1) is Pareto-optimal for independent and identically distributed losses, which complies with intuition.

This result remains true for exchangeable risks $X_1, X_2, \ldots, X_n$.

4.4.2. Comonotonic risks

If the X_i's are comonotonic then they can be represented as non-decreasing functions of their sum S, that is, there are non-decreasing functions $f_1, \ldots, f_n$ such that $X_i = f_i(S)$ holds almost surely. Then,

$$g_i^\star(S) = f_i(S) = X_i$$

and the Pareto-optimal conditional mean allocation leaves each agent with his own loss.

4.4.3. Mutually exclusive risks

Recall from Dhaene and Denuit (1999) that the (non-negative) risks $X_1, X_2, \ldots, X_n$ are said to be mutually exclusive when

$$\Pr\left[X_i > 0, X_j > 0\right] = 0 \quad \text{for all } i \neq j.$$

Clearly, mutual exclusivity of $(X_1, X_2, \ldots, X_n)$ means that the probability mass of this random vector is concentrated on the axes.

Let us now determine the conditional mean risk allocation scheme $(g_1^\star(S), g_2^\star(S), \ldots, g_n^\star(S))$ for the mutually exclusive risk portfolio $(X_1, X_2, \ldots, X_n)$. Hereafter, we assume that all X_i have a discrete distribution. An extension to more general distributions is straightforward. Furthermore, we introduce the following notation:

$$q_i = \Pr[X_i > 0], \quad i = 1, 2, \ldots, n.$$

Clearly, it holds that $\mathbb{E}[X_i \mid S = 0] = 0$ for every i. On the other hand, for $s > 0$ with $\Pr[S = s] > 0$, we have that

$$\begin{aligned} g_i^\star(s) &= \mathbb{E}[X_i \mid S = s] \\ &= \sum_{j=1}^n \mathbb{E}\left[X_i \mid S = s, X_j > 0\right] \Pr\left[X_j > 0 \mid S = s\right] \\ &= s \Pr[X_i > 0 \mid S = s]. \end{aligned}$$

Hence, for any $s \geq 0$ with $\Pr[S = s] > 0$, it holds that

$$g_i^\star(s) = \frac{\Pr[X_i > 0 \text{ and } S = s]}{\sum_{j=1}^n \Pr\left[X_j > 0 \text{ and } S = s\right]} s.$$

Introducing the notations $p_j(s) = \Pr[X_j > 0 \text{ and } S = s]$, we find that

$$g_i^\star(S) = \frac{p_i(S)}{\sum_{j=1}^n p_j(S)} S.$$

If $p_i(s) = p_j(s)$ for all i and j then it is easily seen that $g_i^\star(S) = \frac{S}{n}$ and the conditional mean risk allocation is Pareto-optimal. In general, the non-decreasingness of the $g_i^\star$ has to be checked before we can conclude that the conditional mean risk sharing scheme is Pareto-optimal.

Example 4.6. In the special case where all the mutually exclusive risks X_i are two-point distributions with probability mass in 0 and $a > 0$,

$$\Pr[X_i = a] = q_i$$

and

$$p_j(a) = \Pr\left[X_j > 0 \text{ and } S = a\right] = q_j.$$

The allocation rule $g_i^\star$ reduces to

$$g_i^\star(s) = \frac{q_i}{\sum_{j=1}^n q_j} s, \quad s \in \{0, a\}.$$

Hence,

$$g_i^\star(S) = \frac{q_i}{\sum_{j=1}^n q_j} S,$$

which means that the conditional mean risk allocation is comononotic and Pareto-optimal in this case.

4.5. Pareto-optimality of the conditional mean risk sharing rule for independent losses with log-concave densities

Assume that $X_1, X_2, \ldots, X_n$ are independent, each of them having a log-concave probability density function (that is, the logarithm of their probability density function is concave). For instance, Gamma and Weibull distributions are log-concave for appropriate values of their parameters. Log-concave densities enjoy numerous attractive properties. For instance, log-concave densities are unimodal, that is, they are non-decreasing up to some point and non-increasing beyond that point, and convolutions of log-concave densities remain log-concave.

We know from Efron (1965) that each such X_i increases in the sum S in the $\preceq_{\text{LR}}$-sense, that is,

$$[X_i|S = s] \preceq_{\text{LR}} [X_i|S = s'] \quad \text{for } s \leq s'.$$

This ensures that $s \mapsto g_i^\star(s) = \mathbb{E}[X_i|S = s]$ is non-decreasing so that the conditional mean risk sharing is comonotonic. Thus, the conditional mean risk sharing is Pareto-optimal for independent log-concave risks.

Example 4.7. Let us assume that X_i is Normally distributed with parameters μ_i and σ_i^2 for $i = 1, 2$. Furthermore, assume that (X_1, X_2) is bivariate Normal with Pearson linear correlation coefficient

$$\rho = r[X_1, X_2].$$

Then we find that

$$\mathbb{E}[X_i \mid S = s] = \mu_i + \frac{\mathbb{C}[X_i, S]}{\mathbb{V}[S]}(s - \mathbb{E}[S]).$$

This leads to

$$g_i^\star(S) = \mathbb{E}[X_i \mid S] = \mu_i + \frac{\mathbb{C}[X_i, S]}{\mathbb{V}[S]}(S - \mathbb{E}[S]).$$

If $\rho \geq 0$ then this is a comonotonic risk sharing scheme.

We have that $(g_1^\star(S), g_2^\star(S))$ is bivariate Normal with

$$\mathbb{E}\left[g_i^\star(S)\right] = \mu_i,$$

$$\mathbb{V}\left[g_i^\star(S)\right] = \frac{(\mathbb{C}[X_i, S])^2}{\mathbb{V}[S]} = (r[X_i, S])^2 \sigma_i^2 \leq \sigma_i^2.$$

Furthermore, we know from Property 4.1 that

$$\mathbb{C}\left[g_1^\star(S), g_2^\star(S)\right] = \frac{\mathbb{C}[X_1, S]\,\mathbb{C}[X_2, S]}{\mathbb{V}[S]} \geq \mathbb{C}[X_1, X_2].$$

From this expression for the covariance between $g_1^\star(S)$ and $g_2^\star(S)$, we find that

$$r\left[g_1^\star(S), g_2^\star(S)\right] = 1$$

so that $g_1^\star(S)$ and $g_2^\star(S)$ are perfectly correlated, and hence comonotonic.

Now let us consider the special case that the X_i are mutually independent, that is, $\rho = 0$. Since X_1 and X_2 have logconcave densities, we know that the conditional mean risk sharing is Pareto-optimal. We find that

$$g_i^\star(S) = \mathbb{E}[X_i \mid S] = \mu_i + \frac{\sigma_i^2}{\sigma_1^2 + \sigma_2^2}(S - \mathbb{E}[S]).$$

Obviously, the risk sharing scheme $(g_1^\star(S), g_2^\star(S))$ is also comonotonic in this case.

We also find

$$\mathbb{V}\left[g_i^\star(S)\right] = \frac{\sigma_i^2}{\sigma_1^2 + \sigma_2^2}\sigma_i^2 \leq \sigma_i^2$$

and

$$\mathbb{C}\left[g_1^\star(S), g_2^\star(S)\right] = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \geq 0 = \mathbb{C}[X_1, X_2].$$

Example 4.7 can be generalized to any dimension n if we assume that $r[X_i, X_j] \geq 0$ for all i and j. This is because comonotonicity is equivalent to $r[X_i, X_j] = 1$ for all i and j in the multivariate Normal case (see Dhaene et al. (2002a)). Also, we refer the interested reader to Liggett (2000) for a discrete analog of logconcavity, and for further examples.

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