ONDERZOEKSRAPPORT NR 9024

APPROXIMATING THE COMPOUND NEGATIVE BINOMIAL DISTRIBUTION BY THE COMPOUND POISSON DISTRIBUTION

BY

J. Dhaene

D/1990/2376/29

Approximating the compound negative binomial distribution by the compound Poisson distribution.

Jan Dhaene

Catholic University of Leuven, Belgium.

Abstract

Improved bounds are derived for the difference between the net stop-loss premiums resulting from the compound negative binomial and the corresponding compound Poisson distribution.

Keywords

Stop-loss premium, error bound, compound negative binomial, compound Poisson.

1. Introduction

Assume that the aggregate claims of a portfolio of positive risks have a compound negative binomial distribution. Let S^{CNB} be the random variable representing the aggregate claims then

$$S^{CNB} = X_1 + X_2 + \ldots + X_{K(1)} \tag{1}$$

where K(1) is the number of claims and X_i is the size of claim i. By convention, $S^{CNB} = 0$ if $K(1) = 0$. Further, the X_i are positive mutually independent identically distributed random variables that are independent of K(1). Let the mean of the X_i be denoted by μ.
The number of claims K(1) has a negative binomial distribution,

$$\text{Prob } (K(1) = k) = \binom{k+r-1}{k} p^r q^k \qquad , k = 0, 1, 2, \ldots \tag{2}$$

with $r > 0$, $0 < p < 1$ and $q = 1-p$.

Remark that the Poisson distribution can be obtained as a limit from negative binomial distributions. Thus the negative binomial distribution with parameters r and p can be approximated by the Poisson distribution with parameter $\lambda = rq/p$, provided that p is "sufficiently" close to 1 and r is "sufficiently" large . As a consequence of this, GERBER (1984) remarks that the compound negative binomial distribution can be approximated by the corresponding compound Poisson distribution, i.e. S^{CNB} can be approximated by S^{CP} with

$$S^{CP} = Y_1 + Y_2 + \ldots + Y_{K(2)} \tag{3}$$

where K(2) has a Poisson distribution

$$\text{Prob } (K(2) = k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad , k = 0, 1, 2, \cdots \tag{4}$$

with the Poisson parameter given by

$$\lambda = \frac{rq}{p} \tag{5}$$

The Y_i are positive mutually independent random variables, independent of K(2) and have the same distribution as the X_i.
Again, $S^{CP} = 0$ if $K(2) = 0$.

Remark that

$$E[K(1)] = E[K(2)] = \frac{rq}{p} = \lambda \tag{6}$$

In section 3 bounds will be derived for the difference between the compound negative binomial distribution and the corresponding compound Poisson approximation. The difference between the two distributions will be expressed in terms of stop-loss premiums.
First two lemmas are given that will be used in the proof of our results.

2. Inequalities for stop-loss premiums

The stop-loss premium with retention t corresponding to a random variable X is denoted by $\pi(X, t)$:

$$\pi(X, t) = E[(X-t)_+] \tag{7}$$

Lemma 1

Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ be independent random variables satisfying for all t

$$a_i \le \pi(X_i, t) - \pi(Y_i, t) \le b_i \qquad i = 1, \ldots, k \tag{8}$$

Then one has for all t

$$\sum_{i=1}^{k} a_i \le \pi(\sum_{i=1}^{k} X_i, t) - \pi(\sum_{i=1}^{k} Y_i, t) \le \sum_{i=1}^{k} b_i \tag{9}$$

Lemma 2

Let $X_1, \ldots, X_k$ be independent and identically distributed positive random variables.
Then the following inequalities hold for all t

$$0 \le \pi(\sum_{i=1}^{k} X_i, t) - \sum_{i=1}^{k} \pi(X_i, t) \le (k-1)\, E[X_1] \tag{10}$$

For a proof of Lemma 1 and 2 see DE PRIL and DHAENE (1990).

3. Error bounds

Theorem

Under the assumptions given in section 1, the following inequalities hold for all t

$$0 \le \pi(S^{CNB}, t) - \pi(S^{CP}, t) \le \mu r \left(\ln p + \frac{q}{p}\right) \tag{11}$$

Proof :

Since only positive claims can occur, one find for $t \le 0$

$$\begin{aligned}\pi(S^{CNB},t) - \pi(S^{CP}, t) &= E[(S^{CNB} - t)_+] - E[(S^{CP} - t)_+] \\ &= E[S^{CNB}] - E[S^{CP}] \\ &= E[K(1)]\, E[X_1] - E[K(2)]\, E[Y_1] \\ &= 0\end{aligned}$$

so that (11) is satisfied.

Consider now the case $t > 0$.
The compound negative binomial distribution with parameters r and p and claim size distribution G is the convolution of n compound negative binomial distributions with parameters $\frac{r}{n}$ and p and claim size distribution G.
Thus, from Lemma 1 it follows that it is enough to give the proof for the case that r is sufficiently small.
By taking conditional expectations, one finds

$$\begin{aligned}\pi(S^{CNB},t) - \pi(S^{CP},t) &= E[(\sum_{i=1}^{K(1)} X_i - t)_+] - E[(\sum_{i=1}^{K(2)} Y_i - t)_+] \\ &= E[E[(\sum_{i=1}^{K(1)} X_i - t)_+ | K(1)]] - E[E[(\sum_{i=1}^{K(2)} Y_i - t)_+ | K(2)]]\end{aligned}$$

Using (6) and the assumption that the X_i and Y_i are identically distributed this expression can be written as

$$\begin{aligned}\pi(S^{CNB},t) - \pi(S^{CP},t) &= E[E[(\sum_{i=1}^{K(1)} X_i - t)_+ | K(1)] - K(1)\, E[(X_1 - t)_+]] \\ &\quad - E[E[(\sum_{i=1}^{K(2)} Y_i - t)_+ | K(2)] - K(2)\, E[(Y_1 - t)_+]]\end{aligned}$$

$$= \sum_{k=1}^{\infty} \{ \binom{k+r-1}{k} p^r q^k - \frac{\lambda^k}{k!} e^{-\lambda} \} \{ E[(\sum_{i=1}^{k} X_i - t)_+] - k\, E[(X_1 - t)_+] \}$$

If r is sufficiently small then the inequality

$$\binom{k+r-1}{k} p^r q^k - \frac{\lambda^k}{k!} e^{-\lambda} < o$$

only holds for k = 1. See also Gerber (1984).

So using Lemma 2 one finds that for r sufficiently small

$$o \le \pi(S^{CNB},t) - \pi(S^{CP},t) \le \mu \sum_{k=1}^{\infty} \{ \binom{k+r-1}{k} p^r q^k - \frac{\lambda^k}{k!} e^{-\lambda} \}(k-1)$$

$$= \mu\ (p^r - e^{-\lambda})$$

$$= \mu\ p^r\ (1 - \exp\ (-r\ (q/p + \ln p)))$$

$$\le \mu\ (1 - \exp\ (-r(q/p + \ln p)))$$

$$\le \mu r\ (q/p + \ln p)$$

which proves (11).

■

In GERBER (1984) the following bounds were derived

$$o \le \pi(S^{CNB},t) - \pi(S^{CP},t) \le \mu\ r\ q^2/p \qquad (12)$$

Now,

$$\mu r(\ln p + q/p) < \mu r\ (-q + q/p) = \mu r(q^2/p)$$

so that the upper bound derived in (11) is smaller than the one derived by GERBER (1984).

Further,

$$\mu\ r\ (\ln p + q/p) = \mu\ r\ (-\ q - q^2/2 - \ \dots + q\ (1+q+q^2 + \dots))$$

$$= \mu\ r\ (\frac{1}{2}\ q^2 + \frac{2}{3}\ q^3 + \frac{3}{4}\ q^4 + \dots)$$

and $\mu\ r\ q^2/p = \mu\ r\ q^2\ (1 + q + q^2 + \dots)$

$$= \mu\ r\ (q^2 + q^3 + \dots)$$

It follows that for q sufficiently small the upper bound derived here is approximately half of Gerber's upper bound.

4. References

GERBER, H.U. (1984). Error bounds for the compound Poisson approximation. Insurance : Mathematics and Economics. 3, 191-194.

DE PRIL, N. ; DHAENE, J. (1990). Error bounds for collective approximations of stop-loss premiums. Paper presented at the Oberwolfach meeting on risk theory, 1990.

Jan Dhaene
Catholic University of Leuven
Dekenstraat 2
B - 3000 Leuven
Belgium