ON THE DEPENDENCY OF RISKS IN THE INDIVIDUAL LIFE MODEL *

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Abstract

The paper considers several types of dependencies between the different risks of a life insurance portfolio. Each policy is assumed to have a positive face amount (or an amount at risk) during a certain reference period. The amount is due if the policy holder dies during the reference period.
First, we will look for the type of dependency between the individuals that gives rise to the riskiest aggregate claims in the sense that it leads to the largest stop-loss premiums. Further, this result is used to derive results for weaker forms of dependency, where the only non-independent risks of the portfolio are the risks of couples.

Keywords

Individual life model, (in)dependent risks, stop-loss premiums

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1 Introduction

Consider a portfolio consisting of n life insurance policies, with each policy having a positive face amount (or an amount at risk) during a certain reference period, e.g. one year. The amount is due if the policyholder dies during the reference period. The aggregate claims of the portfolio is the sum of all amounts payable during the reference period. To find the distribution of the aggregate claims and related quantities such as stop-loss premiums is one of the main topics of the individual risk theory.
In order to solve this problem in its most general form, not only the marginal distributions of the claims on each separate contract have to be known, but also knowledge of the dependency relationships is required.
In practice and also in theory the problem is almost always simplified by assuming that the different contracts are mutually independent, so that the knowledge of the marginal distributions suffices to tackle the problem.
However it is obvious that the independence assumption does not always reflect reality:

- There may be duplicates in the portfolio, i.e. several policies may concern the same life. In this case the number of policies is not equal to the number of insured lives. See e.g. Beard and Perks (1949) and Seal (1947).
- A husband and his wife may both have a policy in the same portfolio. It is clear that there must be a dependency between their mortality. Both are more or less exposed to the same risks. Moreover there may be certain selectional mechanisms in the matching of couples (birds of a feather flock together). It is known that the mortality rate increases by the mortality of one's spouse (the "broken heart" syndrome). See e.g. Carrière et al. (1986), Norberg (1989) and Frees et al. (1995).
- A pension fund covers the pensions of persons that work for the same company, so their mortality will be dependent to a certain extent.
- If the density of insured people in a certain area or organisation is high enough then catastrophes such as storms, explosions, earthquakes, epidemics and so on can cause an accumulation of claims for the insurer. See e.g. Strickler (1960), Feilmeier et al. (1980) and Kremer (1983).

As pointed out by Kaas (1993) actuarial practioners are well aware of these phenomena but for convenience usually assume that their influence on the resulting stop-loss premiums is small enough to be negligible. The fact that dependencies may have disastrous effects on stop-loss premiums is illustrated numerically in Kaas (1993). He compares the stop-loss premiums of a portfolio consisting of independent risks to the stop-loss premiums of a portfolio that is identical to the basic portfolio except for the fact that a number of policies of it are based on the same life (duplicates). The stop-loss premiums can be seen to rise astronomically especially for large retentions.

In this paper we will look for the type of dependency between individuals that gives rise to the largest stop-loss premiums.
A similar non-life problem is treated in Heilmann (1986). First, this author considers some general results. Then he considers the special case of a portfolio consisting of two exponential

risks and derives the supremum of the stop-loss premiums for this portfolio, where the supremum is taken over the set of all probability measures in R^2 with given exponential marginals.

In the second part of the paper a life insurance portfolio is considered where the only dependencies that occur are the dependencies between the risks $(X_i, X_i^{'})$ of couples (e.g. wife and husband). We will examine the effect on the stop-loss premiums of changing the correlations between the individual risks of a couple.

2 Description of the model

Let $(X_1, X_2, \ldots, X_n)$ be a portfolio consisting of n risks $X_1, X_2, \ldots, X_n$ with X_i (i=1, 2,..., n) having a given two-point distribution in 0 and $\alpha_i > 0$:

$$\Pr(X_i = 0) = p_i \text{ and } \Pr(X_i = \alpha_i) = 1 - p_i = q_i \tag{1}$$

Usually it is assumed that the random variables $X_1, X_2, \ldots, X_n$ are mutually independent. In this case the distribution of the aggregate claims $X_1 + X_2 + \ldots + X_n$ of the portfolio is uniquely determined by the distributions (1) of the X_i.

In the sequel we will not assume independence. In this case the distribution of the aggregate claims is no longer uniquely determined by the survival probabilities p_i of the individual risks.

Therefore we will introduce the set $\Re(p_1, \ldots, p_n; \alpha_1 \ldots, \alpha_n) \equiv \Re_n$ consisting of all random variables S that can be written as

$$S = X_1 + X_2 + \ldots + X_n \tag{2}$$

with the distribution of the individual risks X_i determined by (1).

It follows immediately that for each $S \in \Re_n$ the mean is given by

$$E(S) = \sum_{i=1}^{n} q_i \alpha_i$$

Hence, the expected aggregate claims is not influenced by the type of dependence between the individual risks.

For convenience, we will assume that the risks $(X_1, X_2, \ldots, X_n)$ are arranged in such a way that

$$p_1 \leq p_2 \leq \ldots \leq p_n$$

which means that a risk with a lower index has a lower survival probability.

3 A particular type of dependency

In this section we will examine a special type of dependency between the risks of the life insurance portfolio. This is not only done for illustrative purposes, but we will need it in section 4 where we state our main result.

Let $S^* \in \Re_n$ with the dependencies between the individual risks given by the following relations

$$\Pr\left(X_{i+1} = 0 \middle| X_i = 0\right) = 1 \qquad (i = 1,2,\ldots,n-1) \tag{3}$$

From (3) we derive the following relations

$$\Pr(X_{i+1} = 0 | X_i = \alpha_i) = \frac{p_{i+1} - p_i}{1 - p_i} \tag{4}$$

$$\Pr(X_{i+1} = \alpha_{i+1} | X_i = 0) = 0 \tag{5}$$

$$\Pr(X_{i+1} = \alpha_{i+1} | X_i = \alpha_i) = \frac{1 - p_{i+1}}{1 - p_i} \tag{6}$$

From (3) it follows that if person (i) stays alive then person (i+1) stays alive, but if person (i+1) stays alive then person (i+2) stays alive,... . So we can conclude

$$\Pr(X_{i+j} = 0 | X_i = 0) = 1 \qquad (i = 1,2,\ldots,n-1; j = 1,\ldots,n-i) \tag{7}$$

This means that if a person will survive the exposure period, then all persons with greater survival probabilities will also survive.

From (6) we deduce

$$\Pr(X_{i-1} = \alpha_{i-1} | X_i = \alpha_i) = 1 \qquad (i = 2,\ldots,n) \tag{8}$$

and

$$\Pr(X_{i-j} = \alpha_{i-j} | X_i = \alpha_i) = 1 \qquad (i = 2,\ldots,n; j = 1,\ldots,i-1) \tag{9}$$

Hence, if a person dies then all persons with lower survival probabilities will die too.

From the reasoning above it follows that the possible outcomes for S^* are

$$0, \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \ldots + \alpha_n,$$

and we have

$$\Pr(S^* = 0) = \Pr(X_1 = 0; X_2 = 0; \ldots; X_n = 0) = \Pr(X_1 = 0) = p_1$$

$$\Pr(S^* = \alpha_1 + \alpha_2 + \ldots + \alpha_i) = \Pr(X_1 = \alpha_1; X_2 = \alpha_2; \ldots; X_i = \alpha_i; X_{i+1} = 0; \ldots; X_n = 0)$$
$$= \Pr(X_i = \alpha_i; X_{i+1} = 0) = \Pr(X_i = \alpha_i).\Pr(X_{i+1} = 0 | X_i = \alpha_i) = p_{i+1} - p_i (i = 1,2,\ldots,n-1)$$

$$\Pr(S^* = \alpha_1 + \ldots + \alpha_n) = \Pr(X_1 = \alpha_1; \ldots; X_n = \alpha_n) = \Pr(X_n = \alpha_n) = 1 - p_n$$

Denoting the distribution of S^* by F^* we can conclude

$$F^*(s) = \begin{cases} p_1 & : 0 \le s < \alpha_1 \\ p_{i+1} & : \alpha_1 + \ldots + \alpha_i \le s < \alpha_1 + \ldots + \alpha_{i+1} \\ 1 & : s \ge \alpha_1 + \ldots + \alpha_n \end{cases} \qquad (i = 1,2,\ldots,n-1) \qquad (10)$$

4 The riskiest aggregate claims

If X and Y are two risks then we say that X precedes Y in stop-loss order (written X $\le_{sl}$ Y), or also X is less risky than Y, if their stop-loss premiums are ordered uniformly:

$$E(X-d)_+ \le E(Y-d)_+$$

for all retentions d ≥ 0.

Y is said to stochastically dominate X (written X $\le_{st}$ Y) if the following order exists between their distribution functions:

$$F_X(x) \ge F_Y(x)$$

for all x.

In the following theorem we will show that in the class of aggregate claims $S = X_1 + \ldots + X_n$ with given marginal distributions of the risks X_i, the aggregate claims S^* with dependencies given by (3) will give rise to the maximal stop-loss premiums.

Theorem 1

Let S^* be the random variable contained in $\Re_n$ with dependencies between the individual risks given by (3). Then we have for any $S \in \Re_n$ that

$$S \le_{sl} S^* \qquad (11)$$

Proof:

The following expressions for the stop-loss premium with retention d of a random variable S having a distribution F(s) will be used:

$$E(S-d)_+ = \int_d^\infty (1-F(s))ds = E(S) - d + \int_0^d F(s)ds$$

In order to prove (11) we define

$$S_j = X_1 + \ldots + X_j \qquad (j = 1,2,\ldots,n)$$

and denote their respective distribution functions by F_j. The random variables S_j^* (j=1,2,...,n) are defined by their distribution functions F_j^*:

$$F_j^*(s) = \begin{cases} p_1 & :0 \le s < \alpha_1 \\ p_{i+1} & :\alpha_1 + \ldots + \alpha_i \le s < \alpha_1 + \ldots + \alpha_{i+1} \\ 1 & :s \ge \alpha_1 + \ldots + \alpha_j \end{cases} \qquad (i = 1,2,\ldots,j-1)$$

For j=1 we immediately have that $S_1 \le_{sl} S_1^*$.

Now assume that $S_j \le_{sl} S_j^*$ or equivalently, because $E(S_j) = E(S_j^*)$,

$$\int_0^d F_j(s)ds \le \int_0^d F_j^*(s)ds \qquad (d \ge 0)$$

Then we find for $d < \alpha_1 + \ldots + \alpha_j$

$$\int_0^d F_{j+1}(s)ds \le \int_0^d F_j(s)ds \le \int_0^d F_j^*(s)ds = \int_0^d F_{j+1}^*(s)ds$$

such that

$$E\left(S_{j+1} - d\right)_+ \le E\left(S_{j+1}^* - d\right)_+ \qquad (0 \le d < \alpha_1 + \ldots + \alpha_j)$$

In order to prove that the inequality above also holds for $d \ge \alpha_1 + \ldots + \alpha_j$ note that

$$F_{j+1}(\alpha_1 + \ldots + \alpha_j) = \Pr(X_1 + \ldots + X_{j+1} \le \alpha_1 + \ldots + \alpha_j) \ge \Pr(X_1 + \ldots + X_{j+1} \le \alpha_1 + \ldots + \alpha_j ; X_{j+1} = 0)$$
$$= p_{j+1} = F_{j+1}^*(\alpha_1 + \ldots + \alpha_j)$$

and hence

$$F_{j+1}(s) \geq F^*_{j+1}(s) \qquad\qquad (s \geq \alpha_1 + \ldots + \alpha_j)$$

such that for $d \geq \alpha_1 + \ldots + \alpha_j$

$$E\left(S_{j+1} - d\right)_+ = \int_d^\infty (1 - F_{j+1}(s))ds \leq \int_d^\infty (1 - F^*_{j+1}(s))ds = E\left(S^*_{j+1} - d\right)_+$$

Q.E.D.

We have proven that the dependency between the risks X_i as expressed by (3) gives rise to the riskiest aggregate claims random variable in the sense that it has the largest stop-loss premiums.

As

$$F_n(0) = \Pr(S = 0) = \Pr(X_1 = 0; \ldots; X_n = 0) \leq p_1 = F^*_n(0)$$

and

$$F_n(\alpha_1 + \ldots + \alpha_{n-1}) = \Pr(X_1 + \ldots + X_n \leq \alpha_1 + \ldots + \alpha_{n-1}) \geq p_n = F^*_n(\alpha_1 + \ldots + \alpha_{n-1})$$

we have that neither S stochastically dominates S^* nor S^* stochastically dominates S.
More generally, we can say that there are no non-trivial stochastic dominance relations between random variables in $\Re_n$. This follows from the fact that all elements of $\Re_n$ have the same mean.

For the more general class of risks S defined by its range $\left[0; \alpha_1 + \ldots + \alpha_n\right]$ and its mean $E(S) = \sum_{i=1}^n \alpha_i q_i$ we have that the riskiest risk is Z with

$$\Pr(Z = \alpha_1 + \ldots + \alpha_n) = \frac{\sum_{i=1}^n q_i \alpha_i}{\sum_{i=1}^n \alpha_i}$$

$$\Pr(Z = 0) = 1 - \Pr(Z = \alpha_1 + \ldots + \alpha_n)$$

see Goovaerts et al. (1990).

As any risk $S \in \Re_n$ is contained in this class, we have

$$S \leq_{sl} S^* \leq_{sl} Z \qquad (12)$$

As $E(S) = E(S^*) = E(Z)$ we find from Goovaerts et al. (1990) that

$$Var(S) \le Var(S^*) \le Var(Z) \tag{13}$$

Note that a dependency of the form "if one person dies, all persons die" is in general not possible for the portfolio $(X_1, X_2, \ldots, X_n)$ with given survival probabilities. The reason is that this latter dependency requires that $p_1 = p_2 = \ldots = p_n$.

If the portfolio is such that $p_1 = p_2 = \ldots = p_n$, the distribution of S^* equals the distribution of Z and the riskiest dependency can be expressed as "if one person dies, all persons die".

5 Applications

5A. In this subsection we will illustrate Theorem 1 numerically. We will use Gerber's (1979) portfolio which is represented in Table 1.

q_j \ amount at risk	1	2	3	4	5
0.03	2	3	1	2	-
0.04	-	1	2	2	1
0.05	-	2	4	2	2
0.06	-	2	2	2	1

Table 1 Gerber's portfolio: number of policies with given amount at risk and claim probability.

In Table 2 we give the stop-loss premiums for a number of retentions in the case of independent risks and in the case of the dependencies described by (3).

d	independent risks	dependencies described by (3)
0	4,490	4,490
4	1,776	4,250
6	1,001	4,130
9	0,361	3,950
14	0,048	3,650
19	0,004	3,350

Table 2 Stop-loss premiums for Gerber's portfolio

From these figures one sees that the riskiest form of dependencies leads indeed to "astronomical" increase of the stop-loss premiums, especially for large retentions.

5B. Let X be the random present value of a n-year temporary life annuity of 1 at the end of year 1,2,...,n provided that a certain person of age x, denoted by (x), survives. Further, let $(x_1),(x_2),\ldots,(x_n)$ be n persons of age x with identically distributed remaining life times as (x). We do not assume independence between the remaining life times. Y_i (i=1,2,...,n) is the random present value of 1 due at the end of i years provided that (x_i) survives. Then we have that

$$E(X) = \sum_{i=1}^{n} E(Y_i)$$

Now we will show that X will always be riskier (in terms of stop-loss premiums) than $\sum_{i=1}^{n} Y_i$.

Let v be the deterministic one year discount factor, then we see that X and $\sum_{i=1}^{n} Y_i$ both are elements of $\Re_n(p_1,\ldots,p_n;v,v^2,\ldots,v^n)$ with p_i $(i=1,2,\ldots,n)$ being the probability that a person of age x dies within i years.

Now we have that $p_1 \le p_2 \le \ldots \le p_n$ so that application of Theorem 1 gives that the most risky element of $\Re_n$ $(p_1,\ldots,p_n;v,v^2,\ldots,v^n)$ is S^* with

$$\Pr(S^* = 0) = p_1$$
$$\Pr(S^* = v+\ldots+v^i) = p_{i+1} - p_i \qquad (i = 1,2,\ldots,n-1)$$
$$\Pr(S^* = v+\ldots+v^n) = 1 - p_n$$

As X has the same distribution as S^* we can conclude that

$$\sum_{i=1}^{n} Y_i \quad \le_{sl} \quad X$$

and from Goovaerts et al. (1990) it follows that this implies

$$E\left(\left(\sum_{i=1}^{n} Y_i\right)^{\alpha}\right) \le E(X^{\alpha})$$

for all $\alpha \ge 1$. As the expectations of both random variables are equal we also have that

$$\mathrm{var}(\sum_{i=1}^{n} Y_i) \le \mathrm{var}(X)$$

6 Stop-loss order relations for sums of two dependent random variables

6A. The results of Theorem 1 can also be used for deriving upper bounds for stop-loss premiums of portfolios with weaker forms of dependency. In the remainder of this paper we will consider a portfolio consisting of couples whereby it is assumed that the claims produced by the different couples are mutually independent, but the claims of a husband and his wife are dependent. In this section we will consider one such couple (X_1, X_2) and derive some results which we will need in Section 7. We assume that each risk X_i (i=1,2) has a two-point distribution:

$$\Pr(X_i = 0) = p_i \qquad ; \Pr(X_i = \alpha_i) = q_i = 1 - p_i \tag{14}$$

with $\alpha_i > 0$.

Let $\Re_2(p_1, p_2; \alpha_1, \alpha_2) \equiv \Re_2$ be the class of all random variables S that can be written as

$$S = X_1 + X_2$$

with the distribution of the X_i given by (14).

In the following lemma an expression is derived which holds for the distribution function F_S of any $S \in \Re_2$. We will only consider the cases $\alpha_1 < \alpha_2$ and $\alpha_1 = \alpha_2$.
The case $\alpha_1 > \alpha_2$ follows from a symmetry argument.

Lemma 1

The distribution F_S of $S \in \Re_2$ is given by

$$F_S(s) = \begin{cases} p_2 - q_1 + \Pr(S = \alpha_1 + \alpha_2) & : 0 \le s < \alpha_1 \\ p_2 & : \alpha_1 \le s < \alpha_2 \\ 1 - \Pr(S = \alpha_1 + \alpha_2) & : \alpha_2 \le s < \alpha_1 + \alpha_2 \\ 1 & : s \ge \alpha_1 + \alpha_2 \end{cases}$$

if $\alpha_1 < \alpha_2$; and by

$$F_S(s) = \begin{cases} p_2 - q_1 + \Pr(S = \alpha_1 + \alpha_2) & : 0 \le s < \alpha_1 \\ 1 - \Pr(S = \alpha_1 + \alpha_2) & : \alpha_1 \le s < \alpha_1 + \alpha_2 \\ 1 & : s \ge \alpha_1 + \alpha_2 \end{cases}$$

if $\alpha_1 = \alpha_2$.

Proof:

Consider the case that $\alpha_1 < \alpha_2$.
Then we find that

$$\Pr(S = \alpha_1) = q_1 - \Pr(S = \alpha_1 + \alpha_2)$$

and

$$\Pr(S = \alpha_2) = q_2 - \Pr(S = \alpha_1 + \alpha_2)$$

so that

$$\Pr(S = 0) = 1 - \Pr(S = \alpha_1) - \Pr(S = \alpha_2) - \Pr(S = \alpha_1 + \alpha_2)$$
$$= p_2 - q_1 + \Pr(S = \alpha_1 + \alpha_2)$$

From these expressions we find $F_S(s)$.

The case $\alpha_1 = \alpha_2$ follows from a similar reasoning.

Q.E.D.

6B. Let $S = X_1 + X_2 \in \Re_2$ then we have

$$\operatorname{var}(S) = q_1 p_1 \alpha_1{}^2 + q_2 p_2 \alpha_2{}^2 + 2\alpha_1\alpha_2(\Pr(S = \alpha_1 + \alpha_2) - q_1 q_2) \tag{15}$$

and

$$\operatorname{cov}(X_1, X_2) = \alpha_1\ \alpha_2(\Pr(S = \alpha_1 + \alpha_2) - q_1 q_2) \tag{16}$$

From (15), (16) and Lemma 1 we conclude that the distribution of any $S \in \Re_2$ is uniquely determined by one of the following quantities: $\Pr(S = \alpha_1 + \alpha_2), \operatorname{var}(S), \operatorname{cov}(X_1, X_2)$.

Now we are able to state the following result concerning the relation between the correlations of X_1 and X_2 for different elements of $\Re_2$.

Lemma 2

Let S_i (i=1,2) be random variables contained in $\Re_2$ with the correlation coefficient between X_1 and X_2 given by $corr_i(X_1, X_2)$. Then the following statements are equivalent:

$(a)\ \Pr(S_1 = \alpha_1 + \alpha_2) \le \Pr(S_2 = \alpha_1 + \alpha_2)$
$(b)\ \operatorname{var}(S_1) \le \operatorname{var}(S_2)$
$(c)\ corr_1(X_1, X_2) \le corr_2(X_1, X_2)$
$(d)\ S_1 \le_{sl} S_2$

Proof:

From (15) and (16) we find immediately that (a), (b) and (c) are equivalent.
Now suppose that (a) holds, then it follows from Lemma 1 that the distribution functions of S_1 and S_2 cross once, with S_2 having the heavier tailed distribution. Hence, from Goovaerts et al. (1990) it follows that (d) holds.
Finally suppose that (d) holds. As $E(S_1) = E(S_2)$, we find from Goovaerts et al. (1990) that (b) holds so that the theorem is proven.

Q.E.D.

6C. From Lemma 2 it follows that the most risky element S^* in $\Re_2$ is the one which maximizes $\Pr(S = \alpha_1 + \alpha_2)$. As we have

$$\Pr(S = \alpha_1 + \alpha_2) \le \min(q_1, q_2)$$

we find

$$\Pr(S^* = \alpha_1 + \alpha_2) = \min(q_1, q_2).$$

Let us now assume that $p_1 \le p_2$ then we find that for the most risky random variable S^* in $\Re_2$ the following type of dependency exists between X_1 and X_2

$$\Pr(X_1 = \alpha_1 \,|\, X_2 = \alpha_2) = \frac{\Pr(S^* = \alpha_1 + \alpha_2)}{\Pr(X_2 = \alpha_2)} = 1$$

which means that the death of the younger one (the one with the higher survival probability) implies the death of the older one. This result could also be found from Theorem 1.

7 A life insurance portfolio with pairwise dependencies

7A. Let $\Im(p_1, p_1^{'}, \ldots, p_m, p_m^{'}, p_{m+1}, \ldots, p_n, \alpha_1, \alpha_1^{'}, \ldots \alpha_m, \alpha_m^{'}, \alpha_{m+1} \ldots, \alpha_n) \equiv \Im$ be the class of all random variables S of the following form:

$$S = \sum_{i=1}^{m} (X_i + X_i^{'}) + \sum_{i=m+1}^{n} X_i$$

where each X_i $(i = 1,2,\ldots,n)$ has a given two-point distribution in 0 and $\alpha_i > 0$, and each $X_i^{'}$ $(i = 1,2,\ldots,m)$ has a given two-point distribution in 0 and $\alpha_i^{'} > 0$.
Further, we assume that for any $S \in \Im$ all risks are mutually independent, except for the "coupled risks". This means that the only dependencies that occur are the dependencies between the two risks $(X_i, X_i^{'})$ of the couples $(i = 1,2,\ldots,m)$. We will also assume that the survival probabilities p_i and $p_i^{'}$ in each couple are ordered such that $p_i \le p_i^{'}$.

Theorem 2

Let $S_j \quad (j=1,2) \quad \in \Im$ *with the correlation coefficients between the risks of the couples given by* $corr_j(X_i, X_i^{'})$, $(i=1,2,\ldots,m)$. *Then we have that*

$$corr_1(X_i, X_i^{'}) \leq corr_2(X_i, X_i^{'}) \quad (i=1,2,\ldots,m)$$

implies

$$S_1 \leq_{sl} S_2$$

Proof:

The proof follows immediately from the equivalence of the statements (c) and (d) in Lemma 2 and from the preservation of stop-loss ordering under convolution for independent risks, see e.g. Goovaerts et al. (1990).

Q.E.D.

From Section 6C we find the following result concerning the most risky random variable in $\Im$.

Theorem 3

Let S^{**} *be the random variable in* $\Im$ *with the dependencies between the risks of the couples given by*

$$\Pr(X_i = \alpha_i \mid X_i^{'} = \alpha_i^{'}) = 1 \qquad (i=1,2,\ldots,m)$$

Then we have for any $S \in \Im$

$$S \leq_{sl} S^{**}.$$

In practice the risks $(X_i, X_i^{'})$ of a couple (wife and husband) will be positively correlated. Theorem 4 considers this case.

Theorem 4

Let S^{indep} *be the random variable in* $\Im$ *with all risks mutually independent and S be a random variable in* $\Im$ *with positively correlated couples* $(X_i, X_i^{'})$. *Then we have*

$$S^{indep} \leq_{sl} S$$

Proof:

The proof follows immediately from Theorem 2.

Q.E.D.

From Theorem 4 we conclude that the assumption of mutual independence will underestimate the stop-loss premiums, at least if the couples $(X_i, X_i^{'})$ are positively correlated.

7B. The result of Theorem 4 is only valid for portfolios with individual risks having a two-point distribution. This will be shown by the following example where we consider a portfolio consisting of only one couple with each individual risk having a three-point distribution.

Let the probability function of $X_i \ (i = 1,2)$ be given by

$$\Pr(X_i = x) = 1/3 \quad (x=0,1,2)$$

Further let S_1 be defined by $S_1 = X_1 + X_2$ with X_1 and X_2 independent.

Then we find

$$\text{cov}_1(X_1, X_2) = 0$$

and

$$E(S_1 - 3)_+ = \Pr(S_1 = 4) = 1/9$$

The random variable S_2 is defined by $S_2 = X_1 + X_2$ with

$$\Pr(X_2 = 0 \mid X_1 = 0) = 1$$
$$\Pr(X_2 = 1 \mid X_1 = 2) = 1$$
$$\Pr(X_2 = 2 \mid X_1 = 1) = 1$$

In this case we have

$$\text{cov}_2(X_1, X_2) = \Pr(X_1 = 1, X_2 = 1) + 2\Pr(X_1 = 2, X_2 = 1)$$

$$+2\Pr(X_1 = 1, X_2 = 2) + 4\Pr(X_1 = 2, X_2 = 2) - 1 = 1/3 > 0$$

and

$$E(S_2 - 3)_+ = \Pr(S_2 = 4) = 0$$

So we find from this example that in general a positive correlation between the individual risks of the couple does not necessarily imply larger stop-loss premiums than in the independence case.

8 Conclusion

In this paper we considered the effect of the dependency assumption on the stop-loss premiums of portfolios consisting of a fixed number of two-point distributed risks. Given the distribution functions of the individual risks, we found that the most risky dependency is the one that can be described as "if a risk leads to a claim, all risks with higher positive claim probability also lead to a claim".

For portfolios where only pairwise dependencies occur, it was shown that the covariances between the non-independent risks of the couples turn out to be the key quantities that contain the information concerning the riskiness of the portfolio, measured in terms of stop-loss premiums.

The generalization of the results of this paper from two-point distributed risks to the case of general non-negative risks is a topic for future research.

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