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Some Remarks on the Definition of the Basic Building Blocks of Modern Life Insurance Mathematics

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1. Notation

For all real r, the floor (greatest integer) and the ceiling (least integer) function are defined respectively as

$\lfloor r \rfloor$ = the greatest integer less than or equal to r
$\lceil r \rceil$ = the least integer greater than or equal to r

The sign = will be used to denote an approximation that follows from a given assumption.

2. The remaining lifetime and related random variables

Let (x) denote a life aged x. The basic building blocks of modern life insurance mathematics are the following random variables

T (x) : the remaining lifetime of (x), thus x + T (x) is the age at death of (x).

K (x) : the number of completed future years lived by (x), or the curtate future lifetime of (x).

S (x) : the fraction of a year during which (x) is alive in the year of death.

$S^{(m)}(x)$: the time between the end of the last completed year and the end of the m-th part of the year in which death occurs.

To simplify the notation T (x) will be abbreviated as T, K (x) as K , and so on.

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3. The classical set up of the stochastic life insurance model

The standard textbooks of BOWERS et al. (1986) and GERBER (1990) develop the life insurance theory starting from the following assumption (A1) and definitions (A2 - A4)

(A1) T : a continuous random variable in $[0, \infty[$

(A2) $K = \lfloor T \rfloor$: a discrete random variable with range $0, 1, 2, \ldots$

(A3) $S = T - K$: a continuous random variable in $[0, 1[$

(A4) $S^{(m)} = \lfloor mS + 1 \rfloor / m$: a discrete random variable with range $\frac{1}{m}, \frac{2}{m}, \ldots, 1$.

4. Discussion of the classical setting

Several objections can be made against the assumption (A1) that states that T has to be a continuous type random variable.

a) From a mathematical point of view it is desirable to build up the theory of life insurance in a general probabilistic framework, independent of the type of distribution of T. For the definition of the force of mortality and the related density function of T a restriction to the special case of continuous distributed T can always be made.

b) To evaluate net single premiums like $\bar{A}_x$ and A_{x+u}, with x an integer and $0 < u < 1$, directly from a life table it is necessary to make an assumption about the distribution of S. In the cited standard textbooks it is assumed that S and K are independent random variables and that S is uniformly distributed between 0 and 1. However the resulting formulae, see GERBER (1990 (3.3.5) p. 26 and (4.8.9) p. 47), do not reflect insurance practice in a realistic way.

   To calculate the present value of a payment of 1 payable immediately on death practioners generally make the assumption that death occurs in the middle of a year, so that

$$\bar{A}_x = (1 + i)^{1/2} A_x \tag{1}$$

   Actual determination of the net single premium of a whole life policy issued at fractional age $x + u$, will mostly be done by rounding the age to the nearest integer, that is

$$A_{x+u} = \begin{cases} A_x & u < {}^1/_2 \\ A_{x+1} & u \geq {}^1/_2 \end{cases} \tag{2}$$

It is embarrasing that the practical approximations (1) and (2) can not be obtained from the general theory if one has to assume that T is always continuous. They can only be derived in a discrete model where, for each value of K , S has a one point distribution in $^1/_2$. The proof of (1) is trivial, the proof of (2) is left as an exercise to the reader.

c) It is common to define A_x as the net single premium of a whole life insurance with benefits payable at the end of the year of death. In doing so it is not necessary to make an assumption about the distribution of S.
For didactical purposes it would be instructive to relate the definition of A_x to that of $\bar{A}_x$, by showing that A_x can also be obtained as a special case of $\bar{A}_x$

$$\bar{A}_x = A_x \qquad (3)$$

if the assumption is made that death can only occur at the end of a year.
Similarly, an alternative way of defining $A_x^{(m)}$ is to consider

$$\bar{A}_x = A_x^{(m)} \qquad (4)$$

under the assumption that death can only occur at the end of intervals of length 1/m. This means that S has a discrete uniform distribution.

The interpretations (3) and (4) permit a better understanding of the actuarial symbols, but require that the theory allows that S , and thus T , can be discrete.

d) The use of the remaining lifetime concept is not restricted to human lives only. For the purpose of profit testing e.g. one can consider the remaining life time of a yearly renewable policy, which is of course a discrete random variable.
Life tables have applications not only in insurance, but in many fields where "death" can only occur at discrete points of time.

Now, some problems will be discussed which arise if the continuity assumption (A1) is dropped and the definitions (A2) - (A4) are applied as such.
As before, a whole life insurance will be considered for illustration purposes. To make the notation clear it is specified that $_{k|}q_x$ and A_x have their usual meaning.

$$_{k|}q_x = \Pr(k < T \leq k+1) \quad \text{with} \quad _{0|}q_x = q_x , \qquad (5)$$

$$A_x = \sum_{k=0}^{\infty} {}_{k|}q_x \, v^{k+1} . \qquad (6)$$

In the stochastic life insurance theory the net single premium of a given life insurance contract is defined as the expectation of the present value, at policy issue, of the benefits insured. Following BOWERS et al. (1986) and GERBER (1990), the net single premium of a whole life insurance is defined as

$$A_x^* = E\,(v^{K+1}) \tag{7}$$

where a star is added to the symbol to make a distinction with the traditional expression (6).
If T is continuous, (7) with the definition (A2) of K leads to the usual expression for the net single premium of a whole life insurance, that is

$$A_x^* = A_x \tag{8}$$

However, if the continuity assumption (A1) is dropped, it follows from (5) and (A2) that the probability function of K is given by

$$\Pr\,(K = k) = {}_{k|}q_x + \Pr\,(T = k) - \Pr\,(T = k + 1) \qquad k = 0, 1, \ldots \tag{9}$$

so that (7) leads to

$$A_x^* = A_x + v \Pr\,(T = 0) - d \sum_{k=1}^{\infty} v^k \Pr\,(T = k) \tag{10}$$

with $d = 1 - v$. Without (A1), application of (7) and (A2) thus give rise to correction terms which make the formula for the net single premium very cumbersome.
Further insight may be gained by considering the special case in which T is defined on the positive integers, that is $\Pr\,(T = k + 1) = {}_{k|}q_x$, $k = 0, 1, 2, \ldots$.
Then, (10) reduces to

$$A_x^* = v\, A_x \tag{11}$$

which clearly indicates the difference between the traditional A_x and A_x^*, as defined by (7) and (A2).
Another way of viewing the discrepancy in definition results if one considers

$$\bar{A}_x^* = E\,(v^T) \tag{12}$$

Under the assumption that T is defined on the positive integers, one has according to (A2) that $T = K$, so that

$$\bar{A}_x^* = (1 + i)\;A_x^* \tag{13}$$

The expected result (3) is thus not obtained by applying (7) and (12), if K is defined by (A2).

The special case with death occurring at integral points of time was given for simplicity. More elaborated examples can be given by considering e.g. $A_x^{(m)}$ and a discrete uniform distribution of deaths within a year. The problem caused by the definitions (A2) - (A4) can of course also be illustrated by considering endownments or life annuities.

5. The proposed set up for the stochastic life insurance model

From the discussion in the previous section it is clear that the set up for the stochastic life insurance model should fulfill the following two criteria :

(i) the theory must be applicable for both continuous and discrete remaining life times T

(ii) the random variables K, S and $S^{(m)}$ must be defined so that the expected value of the relevant present value random variables - such as E (V^{K+1}), ... that are considered in BOWERS et al. (1986) or GERBER (1990) - equals the corresponding traditional deterministic formula for the net single premium.

These requirements are met if (A1) - (A4) are replaced by the following assumption and definitions.

(B1) T	: a random variable, discrete or continuous, in $]0, \infty[$
(B2) $K = \lceil T \rceil - 1$	: a discrete random variable with range 0, 1, 2, ...
(B3) $S = T - K$	: a random variable, discrete or continuous, in $]0, 1]$
(B4) $S^{(m)} = \lceil mS \rceil / m$	: a discrete random variable with range 1/m , 2/m , ..., 1.

It is easily verified that with these definitions one has for arbitrary T

$\Pr[K = k] = {}_{k|}q_x$

and $A_x = \overset{*}{A}_x = E\,(v^{\lceil T \rceil})$

It is left as an exercise to the reader to verify that also the other inconvenienies mentioned in section 4 are resolved by building up the life insurance model starting from (B1) - (B4).

6. References

BOWERS, N.L.; GERBER, H.U.; HICKMAN, J.C.; JONES, D.A. and NESBITT, C.J. (1986) : Actuarial Mathematics, Society of Actuaries, Itacsa, IL.

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Summary

The definitions of the key random variables of the stochastic theory of life insurance, as introduced in the standard textbooks of BOWERS et al. (1986) and GERBER (1990), are discussed.

First some evidence is given for removing the assumption that time-until-death has always to be considered as a continuous type random variable. Then it is shown that ceiling brackets should be used instead of floor brackets in the definition of the curtate future lifetime and related random variables.

Resumé

Les définitions des variables aléatoires fondamentales de la théorie stochastique de l'assurance vie, comme elles ont été introduites dans les ouvrages classiques de BOWERS et al. (1986) et GERBER (1990), sont discutées.

D'abord quelques raisons sont données pour lesquelles ont devrait laisser tomber l'hypothèse de continuité de la durée de vie résiduelle. Puis il est démontré qu'on doit utiliser l'approximation entière vers le haut dans la définition de la durée de vie tronquée et des variables associées, au lieu de l'approximation entière vers le bas.

Zusammenfassung

Die Definitionen der wichtigsten Zufallsvariablen in der modernen Lebensversicherungsmathematik - wie in Standardwerken von BOWERS et al. (1986) und GERBER (1990) eingeführt - werden diskutiert. Erst werden einige Gründe genannt, warum die Annahme, die zukünftige Lebenszeit sei immer eine stetige Zufallsvariable, vernachlässigt werden kann. Dann wird gezeigt, daβ zur definition der gestutzten zukünftigen Lebenszeit und verbundenes Zufallsvariablen statt der Abgerundeten der Zufallsvariable, die um eins verminderte aufgerundete Zufallsvariable herangezogen werden sollte.