Measuring the Impact of a Dependence Among Insured Lifelengths

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Résumé. Les actuaires calculent la plupart du temps le montant de la prime afférente à un contrat d'assurance-vie impliquant plusieurs têtes sur base de l'hypothèse peu réaliste d'indépendance des durées de vie restante des assurés. De nombreuses études cliniques ont cependant mis en évidence la corrélation entre les durées de vie des époux. Dans cette optique, le présent article tente de répondre à la question suivante: cette simplification de la réalité constitue-t-elle un réel danger pour l'assureur? A la lumière de la présente étude, il semble bien que la dépendance existant entre les durées de vie des époux puisse sensiblement influencer les primes relatives à des contrats d'assurance sur plusieurs têtes. Afin de quantifier cet impact sur les primes relatives à des annuités avec ou sans réversion, nous utilisons les bornes de Fréchet, les processus de Markov ainsi que certains modèles de "coupleur". Ces techniques sont appliquées à quelques contrats classiques reposant sur la tête d'un couple marié et illustrées sur des données récoltées dans la Région de Bruxelles Capitale.

Samenvatting. Gewoonlijk gaan actuarissen bij de berekening van premies op meer hoofden uit van de onrealistische assumptie van onafhankelijkheid tussen de resterende levensduren van de verzekerden. Nochtans hebben verscheidene studies de afhankelijkheid tussen de resterende levensduren van samenwonenden, zoal man en vrouw, aangetoond. In dit artikel pogen we een antwoord te vinden op de vraag of deze vereenvoudigende hypothese een financieel gevaar vormt voor de verzekeraar. Het antwoord op deze vraag is bevestigend: deze afhankelijkheid beïnvloedt wel degelijk de waarden van annuiteiten en verzekeringen die betrekking hebben op meerdere hoofden. Om de impact van een mogelijke afhankelijkheid op de premies voor verzekeringsprodukten waarin twee hoofden betrokken zijn te kunnen meten zullen we gebruik maken van de Fréchet-grenzen en van een aantal copula-modellen We zullen deze techniek toepassen op gebruikelijke verzekeringscontracten die afgesloten worden op de beide hoofden van een koppel. We zullen gebruik maken van een data-set met gegevens over personen die leven in Brussel.

Abstract. Actuaries usually compute multiple life premiums based on the unrealistic assumption of independence of the lifelengths of insured persons. Many clinical studies, however, have demonstrated dependence of the lifetimes of paired lives such as husband and wife. In this respect, the present article tries to give an answer to the following question: does this simplifying hypothesis constitute a real financial danger for the insurance company? The answer turns out to be affirmative: this dependence materially affects the values of annuities and insurances involving multiple lives. In order to quantify the impact of a possible dependence on the amount of premium charged for annuities, insurances and widow's pension, we resort here on the Fréchet bounds, Markov processes and some copula models. These techniques are applied to classical insurance contracts issued to married couples and illustrated on NIS data as well as on observations from Brussels city.

Mots-clés: Dépendance positive; primes relatives à des contrats d'assurance-vie reposant sur plusieurs têtes; durées de vie dépendantes; rentes viagères au premier et au dernier décès; rentes avec réversion; bornes de Fréchet; modèles de Markov; coupleurs

Sleutelwoorden: Positieve afhankelijkheid; premies voor verzekeringen op meer hoofden; afhankelijke resterende levensduren, joint-life en last-survivor statusen, overlevingsrenten, Fréchet-grenzen; Markov modellen, copula functies

Keywords: Positive dependence; multiple life insurance premiums; dependent lifetimes; joint-life and last-survivor annuities; reversionary annuities; Fréchet bounds; Markov models; copulas

1 Introduction-Motivation

Standard actuarial theory of multiple life insurance traditionally postulates independence for the remaining lifetimes in order to evaluate the amount of premium relating to an insurance contract involving multiple lives. Nevertheless, this hypothesis obviously relies on computational convenience rather

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than realism. A fine example of possible dependence among insured persons is certainly a contract issued to a married couple. Indeed, the husband and wife are more or less exposed to the same risks since they share a common way of life, go together away on holiday and, as the saying goes, “birds of a feather flock together". Moreover, from the medical point of view, several clinical studies put the “broken heart syndrome" in a prominent position; the latter may cause an increase of the mortality rate after the death of one’s spouse (using a data set consisting of 4,486 55-year-old widowers, Parkes, Benjamin and Fitzgerald (1969) showed that there is a 40% increase in mortality among the widowers during the first few months after the death of their wives; see also Jagger and Sutton (1991)). There is thus strong empirical evidence that supports the dependence of mortality of pairs of individuals. Investigations carried out by the Belgian National Institute of Statistics (NIS, in short) established that the marital status significantly modifies the mortality profile of individuals. Similar conclusions have been drawn in actuarial sciences, e.g. by Maeder (1995, Section 2.3). In such a case, the actuary has to wonder whether the independence assumption is reasonable and he has to build an appropriate price list taking into account the possible effects of a dependence among the time-until-death random variables involved in the contract.

To illustrate these remarks, we have Figure 1 based on the data collected by the Belgian NIS during 1991. The observed probabilities q_x (i.e. the probability that a life aged x will die within one year) are plotted as a function of the age x (for $x = 25$ to 90), separately for Belgian men and women, splitted according to the marital status. It can be seen that the mortality depends on the marital status, especially for men. The mortality experienced by the widows seems worse than the mortality experienced by the entire Belgian population. This speaks in favour of a model incorporating the marital status of the insured persons and taking into account the dependence among the involved lifelengths to calculate the amount of premium relating to policies issued to married couples such as the widow’s pension. Of course, one could convincingly argue that the society drastically changed during the last few decades and that the marriage is no more the obligatory prerequisite when two persons decide to start a life together. Consequently, many individuals counted as “single" by the Belgian NIS should in fact be considered as “married" from a sociological point of view. The marital status will not appear as the most relevant explanatory variable. However, the fiscal legislation often subordinates the tax incentives granted for some insurance contracts to the fact that the assured persons are indeed officially married. Therefore, the data collected by the governmental statistical services may be considered as relevant as far as contracts like the widow’s pension are concerned.

Recently, a number of articles have been devoted to the study of the impact of a possible dependence among insured risks in setting premium rates. Several authors based their analysis on multivariate stochastic orderings (see, e.g., Bäuerle and Müller (1998), Denuit, Lefèvre and Mesfioui (1999a,b), Dhaene and Goovaerts (1996,1997), Dhaene, Vanneste and

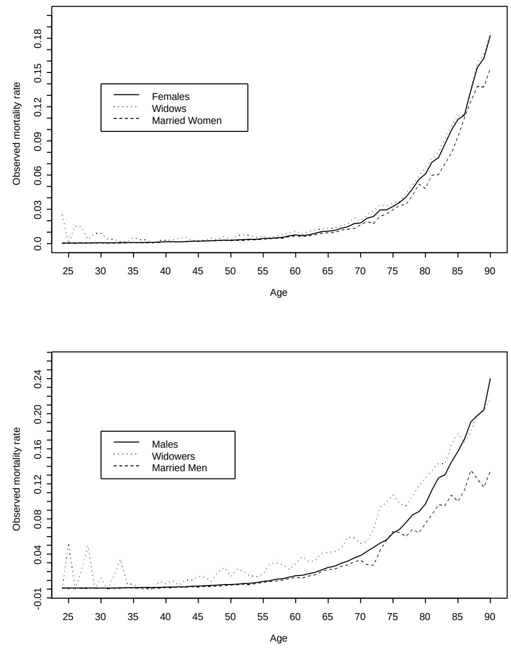

Figure 1. The observed probabilities q_x for the Belgian population, for married individuals and for widow individuals, as functions of the age x (from 25 to 90).

Wolthuis (2000) and Müller (1997)). Others used copula models to take this dependence into account (see, e.g., Carrière and Chan (1986), Carrière (1994,2000), Frees, Carrière and Valdez (1996) and Frees and Valdez (1998)). The situation where the dependence of lives arises from an exogeneous event that is common to each life, can be described by a “common shock" model. A reference to this kind of models is Marshal and Olkin (1988). To the best of our knowledge, the first actuarial textbook explicitly introducing multiple life models in which the future lifetime random variables are dependent is Bowers, Gerber, Hickmann, Jones and Nesbitt (1997). In Chapter 9 of this book, copula and common shock models are introduced to describe dependencies in joint-life and last-survivor statuses. Also other models can be used to incorporate dependencies between life times, e.g. frailty models or Markov models. For a more extensive overview of dependency models, we further refer to Frees, Carrière and Valdez (1996) and the references in that paper.

In this paper, we quantify the effect of a possible dependence of time-until-death random variables on the amount of premium relating to various insurance policies sold to married couples. For this purpose, we use the Fréchet bounds and a Markovian model inspired from Norberg (1989) and Wolthuis (1994). The paper is organized as follows. Section 2 gives the basic notations used throughout the paper. Section 3 recalls the main features of the Makeham model. In Section 4, we show that the Fréchet-Höffding bounds provide poor margins for widow's pensions while the margins obtained for most multiple life premiums are quite accurate. Moreover, the qualitative approach based on the positive quadrant dependence developed by Norberg (1989) is briefly exposed. Then, in Section 5, we present the Markovian model built to study the impact of a possible dependence among the husband's and wife's time-until-death random variables on the amount of insurance premiums. After having estimated the parameters of this model on the basis of data collected by the Belgian NIS during 1991[5], we compute the amount of premium relating to a widow's pension in the Markovian model. In Section 6, we present two copula models of ratemaking built to study the possible impact of dependence on amounts of premiums. This technique is illustrated on a data set from Brussels city. In the last Section 7, we propose the correlation order as a tool for describing and understanding dependencies in multiple life statuses. It is shown that this order is preserved (or reversed) when pricing multiple life and last-survivor insurance and annuity contracts. In particular, we establish conditions that provide information on phenomenon of over/underpricing when the usual assumption of mutual independency of the life times involved is made.

2 Notations

Let us now introduce the notation used throughout this paper. Henceforth, $\mathbb{R}$ is the real line $(-\infty,+\infty)$, $\mathbb{R}^+$ is the half-positive real line $[0,+\infty)$, $\mathbb{N}$ is the set $\{0,1,2,\dots\}$ of the non-negative integers. The symbol "$=_d$" means "is distributed as". In the remainder, T_x and T_y represent the remaining lifetimes (or time-until-death random variables) of a x-year-old man and of his y-year-old wife, respectively; T_x (resp. T_y) is assumed to be valued in $[0,\omega_x]$ (resp. $[0,\omega_y]$) where ω_x (resp. ω_y) denotes the difference between the ultimate age of the lifetable describing the probability distribution of T_x (resp. T_y) and x (resp. y). As usual, we denote by ${}_tq_x$ and ${}_tq_y$ (resp. by ${}_tp_x$ and ${}_tp_y$) the distribution functions (resp. survival functions) of T_x and T_y, respectively, i.e.

$${}_tp_x=\mathbb{P}[T_x>t]=1-{}_tq_x,\quad t\in\mathbb{R}^+,$$

and

$${}_tp_y=\mathbb{P}[T_y>t]=1-{}_tq_y,\quad t\in\mathbb{R}^+.$$

As usual, ${}_1p_x\equiv p_x$ and ${}_1q_x\equiv q_x$. The probability of the joint-life status "$\min(T_x,T_y)$" surviving to time t, denoted by ${}_tp_{xy}$, is given by

$$\begin{aligned}{}_tp_{xy} &= \mathbb{P}[\min(T_x,T_y)>t]\\ &= \mathbb{P}[T_x>t,T_y>t],\quad t\in\mathbb{R}^+.\end{aligned}$$

The probability ${}_tq_{xy}$ that the joint-life status fails before time t is then given by

$$\begin{aligned}{}_tq_{xy} &= \mathbb{P}[\min(T_x,T_y)\le t]\\ &= \mathbb{P}[T_x\le t \text{ or } T_y\le t]\\ &= 1-{}_tp_{xy},\quad t\in\mathbb{R}^+.\end{aligned}$$

[5] The details of these data can be found in a publication of the Belgian NIS (1992).

Similarly, the probability that the last-survivor status $\max(T_x,T_y)$ surviving to time t, denoted by ${}_tp_{\overline{xy}}$, is

$$\begin{aligned}{}_tp_{\overline{xy}} &= \mathbb{P}[\max(T_x,T_y)>t]\\ &= \mathbb{P}[T_x>t \text{ or } T_y>t]\\ &= {}_tp_x+{}_tp_y-{}_tp_{xy},\quad t\in\mathbb{R}^+.\end{aligned}$$

The probability ${}_tq_{\overline{xy}}$ that the last-survivor status fails before time t is given by

$$\begin{aligned}{}_tq_{\overline{xy}} &= \mathbb{P}[\max(T_x,T_y)\le t]\\ &= \mathbb{P}[T_x\le t \text{ and } T_y\le t]\\ &= 1-{}_tp_{\overline{xy}},\quad t\in\mathbb{R}^+.\end{aligned}$$

Finally, $v=(1+\xi)^{-1}$ stands for the discount factor associated with the constant annual effective rate ξ. Basic life insurance theory can be found in Gerber (1995).

We will consider in this chapter the standard net single premiums relating to annuities and widow's insurance. Of course, a similar study can be achieved for term life or whole life insurances payable on the first or on the last death. We restrict the present study to life benefits only to save space.

Annuities are contractual guarantees that promise to provide periodic income over the lifetime(s) of individuals. An important variation of the standard life annuity is the joint-life annuity and the last-survivor annuity. In the case of a married couple, the n-year last-survivor annuity pays \$ 1 at the end of the years 1, 2, ..., n, as long as either spouse survives; it is defined as

$$a_{\overline{xy};\overline{n}|}=\sum_{k=1}^{n}v^k{}_kp_{\overline{xy}}.$$

The corresponding perpetuity is $a_{\overline{xy}}\equiv a_{\overline{xy};\overline{\infty}|}$. The n-year joint-life annuity pays \$ 1 at the end of the years 1, 2, ..., n, as long as both spouses survive; it is defined

$$a_{xy;\overline{n}|}=\sum_{k=1}^{n}v^k{}_kp_{xy}.$$

The corresponding perpetuity is $a_{xy}\equiv a_{xy;\overline{\infty}|}$. Many variations are offered in the market place, including a joint and 50% annuity that pays a level amount while both annuitants survive with a 50% reduction of that amount upon the death of one annuitant.

The widow's pension is a reversionary annuity with payments starting with the husband's death and terminating with the death of his wife. Such an insurance is used as post-retirement benefit in some pension plans and is also widely used in the European social security systems. The corresponding net single life premium for a x-year-old husband and his y-year-old wife, denoted as $a_{x|y}$, is given by $a_{x|y}=a_y-a_{xy}$ where

$$a_y=\sum_{k=1}^{\omega_y}v^k{}_kp_y \text{ and } a_{xy}=\sum_{k=1}^{\min(\omega_x,\omega_y)}v^k{}_kp_{xy}.$$

Calculating the exact values of $a_{\overline{xy};\overline{n}|}$, $a_{xy;\overline{n}|}$ or $a_{x|y}$ requires the knowledge of the joint distribution of the lifetime random vector (T_x, T_y). In practice, the actuary is only able to approximate $a_{\overline{xy};\overline{n}|}$, $a_{xy;\overline{n}|}$ and $a_{x|y}$ with the help of various probabilistic models. The easiest approach certainly consists in considering T_x and T_y as independent. In the remainder of the work, the superscript "$\perp$" indicates that the corresponding amount of premium is calculated under the independence hypothesis; it is thus the premium from the tariff book. More precisely, $a^{\perp}_{\overline{xy};\overline{n}|}$, $a^{\perp}_{xy;\overline{n}|}$ and $a^{\perp}_{x|y}$ are given by

$$a^{\perp}_{\overline{xy};\overline{n}|} = \sum_{k=1}^{n} v^k \Big\{ {}_kp_x + {}_kp_y - {}_kp_x {}_kp_y \Big\},$$

$$a^{\perp}_{xy;\overline{n}|} = \sum_{k=1}^{n} v^k {}_kp_x {}_kp_y$$

and

$$a^{\perp}_{x|y} = \sum_{k=1}^{\omega_y} v^k {}_kp_y - \sum_{k=1}^{\min(\omega_x,\omega_y)} v^k {}_kp_x {}_kp_y.$$

3 Makeham graduation

In the present paper, we use lifetables based on the Makeham formula and built on the basis of the mortality experienced in Belgium during 1991. Makeham formula gives for $t \in \mathbb{R}^+$,

$${}_tp_x = s_1^t g_1^{c_1^{x+t} - c_1^x}, \quad c_1 > 1, \quad s_1, g_1 \in [0,1], \qquad (1)$$

and

$${}_tp_y = s_2^t g_2^{c_2^{y+t} - c_2^y}, \quad c_2 > 1, \quad s_2, g_2 \in [0,1]. \qquad (2)$$

If we introduce

$$A_i = -\ln(s_i) \text{ and } B_i = -\ln(c_i)\ln(g_i), \quad i = 1,2,$$

the corresponding forces of mortality are given by

$$\mu_{x+t} = A_1 + B_1 c_1^{x+t}, \quad t \in \mathbb{R}^+,$$

for men, and by

$$\mu_{y+t} = A_2 + B_2 c_2^{y+t}, \quad t \in \mathbb{R}^+,$$

for women. The parameters involved in (1)-(2) have been estimated on the basis of data collected by the Belgian NIS during the year 1991. We used the Maximum Likelihood method with initial values provided by the following algorithm, proposed by Frère (1968). Let $\hat{p}_x = 1 - \hat{q}_x$ be the empirical estimator of the one-year survival probability at age x. Let us define

$$\alpha_x = -\ln(\hat{p}_x)$$

and suppose we would like to fit the Makeham model for the ages ν_1 to ν_2. Frère's idea consists in decomposing the age range (ν_1, ν_2) into two parts, namely $(\nu_1, 40)$ et $(41, \nu_2)$. In $(41, \nu_2)$, A_1 may be neglected in

$$-\ln(p_x) = A_1 + \beta_1 c_1^x \text{ where } \beta_1 = -(c_1 - 1)\ln g_1,$$

so that

$$\ln(-\ln(p_x)) \approx \ln \beta_1 + x \ln c_1$$

is approximately linear. On the contrary, in $(\nu_1, 40)$, A_1 may no more be neglected. The method determines the parameters β_1 and c_1 as the solutions of the optimization problem

$$(\beta_1, c_1) = \arg \min_{(\xi_1,\xi_2)} \sum_{x=41}^{\nu_2} \Big\{ \ln(\alpha_x) - \ln(\xi_1) - x\ln(\xi_2) \Big\}^2,$$

whereas A_1 is given by

$$A_1 = \arg\min_{\xi_3} \sum_{x=\nu_1}^{40} \Big\{ \alpha_x - \xi_3 - \beta_1 c_1^x \Big\}^2.$$

The estimations of the parameters A_1, β_1 and c_1 are then given by

$$\ln c_1 = \frac{\sum_{x=41}^{\nu_2} x \sum_{x=41}^{\nu_2} \ln(\alpha_x) - (\nu_2 - 40)\sum_{x=41}^{\nu_2} x\ln(\alpha_x)}{\left(\sum_{x=41}^{\nu_2} x\right)^2 - (\nu_2 - 40)\sum_{x=41}^{\nu_2} x^2},$$

$$\ln \beta_1 = \frac{1}{\nu_2 - 40}\left(\sum_{x=41}^{\nu_2} \ln(\alpha_x) - \ln(c) \sum_{x=41}^{\nu_2} x \right),$$

and

$$A_1 = \frac{1}{41 - \nu_1} \sum_{x=\nu_1}^{40} (\alpha_x - \beta_1 c_1^x).$$

In Table 1, one can find the maximum likelihood estimations of the parameters s, g and c for Belgian men and women.

It should be noted that the data collected by the Belgian NIS relate to the mortality experienced by the Belgian population during 1991. Such data are suitable for the pricing of the widow's pension included in social security systems, but possibly not for contracts issued by private insurance companies. The latter will have to substitute their own observations for those used in the present work.

Parameter	Men ($i = 1$)	Women ($i = 2$)
s_i	0.999 408 439 685	0.999 767 237 352
g_i	0.999 598 683 466	0.999 831 430 984
c_i	1.102 904 035 923	1.106 730 646 873

Table 1. Maximum Likelihood estimators of the Makeham parameters involved in Formulas (1)-(2)

4 Bounds on net single premiums

4.1 Fréchet bounds

Let us first recall the definition of a Fréchet class. The Fréchet class $\mathcal{R}_2(F_1, F_2)$ is the set of all the (bivariate distribution functions $F_{\boldsymbol{X}}$ of) random vectors $\boldsymbol{X} = (X_1, X_2)$ with marginal distribution functions F_1 and F_2, i.e.

$$F_i(x) = \mathbb{P}[X_i \le x], \quad x \in \mathbb{R}, \quad i = 1,2.$$

Henceforth, let $\overline{F}_i \equiv 1 - F_i$ be the decumulative distribution function corresponding to F_i, $i = 1,2$.

In $\mathcal{R}_2(F_1, F_2)$, there are two very particular elements whose definition is recalled next.

Definition 4.1. In $\mathcal{R}_2(F_1,F_2)$, the distribution function W_2 defined by

$$W_2(x_1,x_2)=\min\{F_1(x_1),F_2(x_2)\},\quad (x_1,x_2)\in\mathbb{R}^2,$$

is the Fréchet upper bound; the distribution function M_2 defined by

$$M_2(x_1,x_2)=\max\{F_1(x_1)+F_2(x_2)-1,0\},\quad (x_1,x_2)\in\mathbb{R}^2,$$

is the Fréchet lower bound.

Since Höffding (1940) and Fréchet (1951), it is well-known that the joint distribution function $F_{\boldsymbol{X}}$ of any $\boldsymbol{X}$ in $\mathcal{R}_2(F_1,F_2)$ is constrained from above and below by the Fréchet bounds M_2 and W_2, i.e.

$$M_2(x_1,x_2)\le F_{\boldsymbol{X}}(x_1,x_2)\le W_2(x_1,x_2) \qquad (3)$$

for all $(x_1,x_2)\in\mathbb{R}^2$. Formula (3) directly follows from the following elementary result.

Proposition 4.2. For any events A_1 and A_2, the following inequalities hold:

$$\max\{\mathbb{P}[A_1]+\mathbb{P}[A_2]-1,0\}\le\mathbb{P}[A_1\cap A_2]$$

and

$$\mathbb{P}[A_1\cap A_2]\le\min\{\mathbb{P}[A_1],\mathbb{P}[A_2]\}.$$

Proof. The first inequality follows from

$$1\ge\mathbb{P}[A_1\cup A_2]=\mathbb{P}[A_1]+\mathbb{P}[A_2]-\mathbb{P}[A_1\cap A_2],$$

while the second one is valid since $(A_1\cap A_2)\subseteq A_1$ and $(A_1\cap A_2)\subseteq A_2$. □

In the bivariate case, the Fréchet lower and upper bounds are both reachable within $\mathcal{R}_2(F_1,F_2)$, as it is formally stated in the next result.

Proposition 4.3. In $\mathcal{R}_2(F_1,F_2)$, W_2 is the distribution function of $(F_1^{-1}(U),F_2^{-1}(U))$ and M_2 is the distribution function of $(F_1^{-1}(U),F_2^{-1}(1-U))$, where U is uniformly distributed on $[0,1]$ and the generalized inverses F_1^{-1} and F_2^{-1} are defined as

$$F_i^{-1}(p)=\inf\{x\in\mathbb{R}|F_i(x)\ge p\},\quad 0<p<1,\quad i=1,2.$$

Proof. It is clear that

$$(F_1^{-1}(U),F_2^{-1}(U))\in\mathcal{R}_2(F_1,F_2).$$

For any $(x_1,x_2)\in\mathbb{R}^2$, we have that

$$\begin{aligned}&\mathbb{P}[F_1^{-1}(U)\le x_1,F_2^{-1}(U)\le x_2]\\ =\ &\mathbb{P}[U\le\min\{F_1(x_1),F_2(x_2)\}]\\ =\ &W_2(x_1,x_2).\end{aligned}$$

On the other hand,

$$(F_1^{-1}(U),F_2^{-1}(1-U))=(F_1^{-1}(U),\overline{F}_2^{-1}(U))\in\mathcal{R}_2(F_1,F_2)$$

and for any $(x_1,x_2)\in\mathbb{R}^2$,

$$\begin{aligned}&\mathbb{P}[F_1^{-1}(U)\le x_1,F_2^{-1}(1-U)\le x_2]\\ =\ &\mathbb{P}[U\le F_1(x_1),1-U\le F_2(x_2)]\\ =\ &M_2(x_1,x_2).\end{aligned}$$

This ends the proof □

The distribution of (U,U) has all its mass on the diagonal between (0,0) and (1,1), whereas that of $(U,1-U)$ has all its mass on the diagonal between (0,1) and (1,0). Therefore, it is often said that W_2 and M_2 describe perfect positive and negative dependence, respectively.

4.2 Bounds on multilife premiums obtained from Fréchet bounds

This approach centers on quantifying the maximal impact of a possible dependence on actuarial values by using the bounds for bivariate distribution functions introduced in Definition 4.1. Specifically, applied in the present context, the Fréchet bounds M_2 and W_2 yield

$$\max\{0,{}_tp_x+{}_tp_y-1\}\le{}_tp_{xy}\le\min\{{}_tp_x,{}_tp_y\}, \qquad (4)$$

for all $t\in\mathbb{R}^+$, and that

$$1-\min\{{}_tq_x,{}_tq_y\}\le{}_tp_{\overline{xy}}\le 1-\max\{{}_tq_x+{}_tq_y-1,0\}, \qquad (5)$$

for all $t\in\mathbb{R}^+$. These bounds have been first applied by Carrière and Chan (1986) to different annuities and then placed by Denuit and Lefèvre (1997) and Dhaene, Vanneste and Wolthuis (1997) in the context of bivariate stochastic orderings.

By inserting (4)-(5) in the net single premiums $a_{\overline{xy};\overline{n}|}$ and $a_{xy;\overline{n}|}$, we get

$$a^{\min}_{\overline{xy};\overline{n}|}\le a_{\overline{xy};\overline{n}|}\le a^{\max}_{\overline{xy};\overline{n}|}\text{ and }a^{\min}_{xy;\overline{n}|}\le a_{xy;\overline{n}|}\le a^{\max}_{xy;\overline{n}|} \qquad (6)$$

with

$$\begin{aligned}a^{\min}_{\overline{xy};\overline{n}|}&=\sum_{k=1}^{n}v^k\Big\{1-\min\{{}_tq_x,{}_tq_y\}\Big\}\\ a^{\max}_{\overline{xy};\overline{n}|}&=\sum_{k=1}^{n}v^k\Big\{1-\max\{{}_tq_x+{}_tq_y-1,0\}\Big\}\\ a^{\min}_{xy;\overline{n}|}&=\sum_{k=1}^{n}v^k\max\{0,{}_tp_x+{}_tp_y-1\}\\ a^{\max}_{xy;\overline{n}|}&=\sum_{k=1}^{n}v^k\min\{{}_tp_x,{}_tp_y\}.\end{aligned}$$

To illustrate the accuracy of the bounds (6), we plotted the graphs depicted in Figures 2-3. We considered $x=y=30,40,50$ and 60 years. The continuous line stands for the tariff book premiums (i.e., those computed on the basis of the independence assumption) and the dotted lines represent the lower and upper bounds obtained by (4)-(5). We see that the accuracy of the margins is reasonably good (which means that the impact of a possible dependence on the amounts $a_{\overline{xy};\overline{n}|}$ and $a_{xy;\overline{n}|}$ is moderate) although it decreases with n. For large values of n, $a_{\overline{xy};\overline{n}|} \approx a_{\overline{xy}}$ and $a_{xy;\overline{n}|} \approx a_{xy}$, which explains the horizontal behavior of the rightmost part of the graphs.

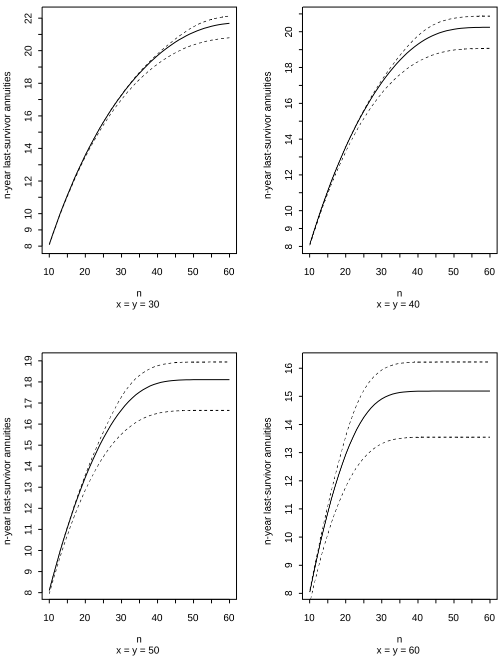

Figure 2. Fréchet bounds $a^{\min}_{xy;\overline{n}|}$ and $a^{\max}_{xy;\overline{n}|}$ (dotted lines) on $a_{xy;\overline{n}|}$ together with $a^{\perp}_{xy;\overline{n}|}$ (solid line) viewed as functions of n, with $\xi = 4\%$, $x = y =$ 30, 40, 50 and 60.

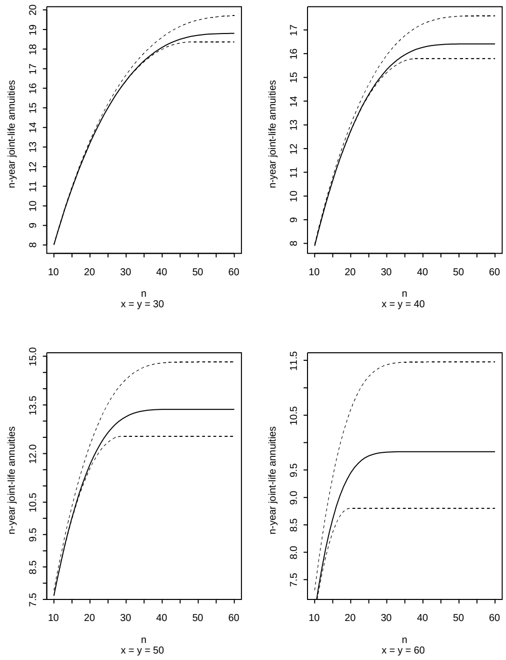

Figure 3. Fréchet bounds $a^{\min}_{\overline{xy};\overline{n}|}$ and $a^{\max}_{\overline{xy};\overline{n}|}$ (dotted lines) on $a_{\overline{xy};\overline{n}|}$ together with $a^{\perp}_{\overline{xy};\overline{n}|}$ (solid line) viewed as functions of n, with $\xi = 4\%$, $x = y =$ 30, 40, 50 and 60.

By inserting (4)-(5) in the net single premium $a_{x|y}$, we get

$$a^{\min}_{x|y} \leq a_{x|y} \leq a^{\max}_{x|y}, \tag{7}$$

with

$$a^{\min}_{x|y} = \sum_{k=1}^{\omega_y} v^k {}_kp_y - \sum_{k=1}^{\min(\omega_x,\omega_y)} v^k \min\{{}_kp_x, {}_kp_y\}$$

and

$$a^{\max}_{x|y} = \sum_{k=1}^{\omega_y} v^k {}_kp_y - \sum_{k=1}^{\min(\omega_x,\omega_y)} v^k \max\{0, {}_kp_x + {}_kp_y - 1\}.$$

In order to figure out the accuracy of the bounds (7), we have drawn the graphs presented in Figure 4. We used three assumptions for the numerical illustrations: firstly, we consider $x = y = 25, 26, \ldots, 90$ (i.e. the husband and his wife both have the same age), then $x = y + 5 = 25, 26, \ldots, 90$ (i.e., the husband is five years older than his wife), and finally $x = y - 5 = 25, 26, \ldots, 90$ (i.e., the husband is five years younger than his wife). The continuous line stands for the tariff book premiums $a^{\perp}_{x|y}$ and the dotted lines represent the lower and upper bounds $a^{\min}_{x|y}$ and $a^{\max}_{x|y}$. The margins for the widow's pension $a_{x|y}$ are of poor quality (especially the lower bounds). For $x = y$, $a^{\min}_{x|y}$ is about 55% to 59% of $a^{\perp}_{x|y}$, while $a^{\max}_{x|y}$ represents 120% to 130% of $a^{\perp}_{x|y}$. The accuracy of the lower bound increases with x, whereas the accuracy of the upper bound decreases with x. When the husband is younger than his wife, the lower bound $a^{\min}_{x|y}$ is about 22% to 1% of $a^{\perp}_{x|y}$ and the upper bound $a^{\max}_{x|y}$ represents 126% to 144% of $a^{\perp}_{x|y}$. Therefore, the lower bound is even worse than in the case $x = y$ and becomes gradually useless (since it decreases to 0). On the contrary, the accuracy tends to increase when the wife is older than her spouse: $a^{\min}_{x|y}$ is about 74% to 82% of $a^{\perp}_{x|y}$ and $a^{\max}_{x|y}$ represents 114% to 118% of $a^{\perp}_{x|y}$. In conclusion, we could say that, for the widow's pension, the independence assumption may lead to a significantly erroneous amount of premium.

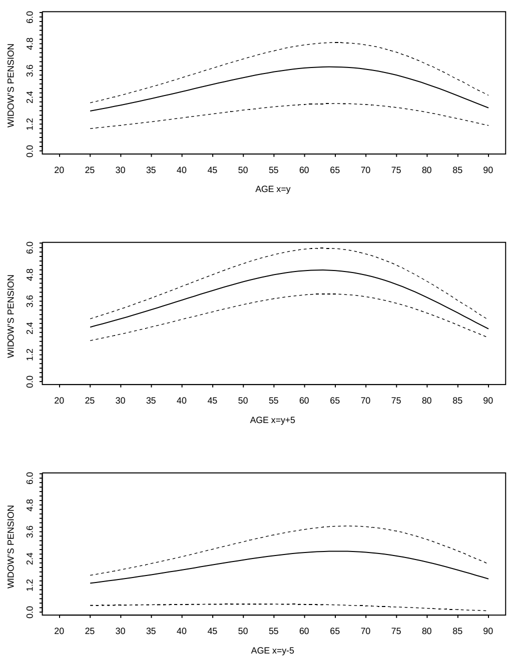

Figure 4. Fréchet bounds $a^{\min}_{x|y}$ and $a^{\max}_{x|y}$ (dotted lines) on $a_{x|y}$ together with $a^{\perp}_{x|y}$ (continuous line) viewed as functions of x and y, with $\xi = 4\%$.

4.3 Positive quadrant dependence

A number of notions of positive dependence among two random lifelengths T_x and T_y have been introduced in the literature in an effort to mathematically describe the property that "large (resp. small) values of T_x go together with large (resp. small) values of T_y". In this Section, we will assume that T_x and T_y are Positive Quadrant Dependent (PQD, in short).

Let $\boldsymbol{X} = (X_1, X_2)$ be a random couple with distribution function $F_{\boldsymbol{X}}$ in $\mathcal{R}_2(F_1, F_2)$. According to Lehmann (1966), $\boldsymbol{X}$ is said to be PQD if

$$\mathbb{P}[X_1 > a_1, X_2 > a_2] \geq \mathbb{P}[X_1 > a_1]\mathbb{P}[X_2 > a_2] \quad (8)$$

for all $a_1, a_2 \in \mathbb{R}$. Since

$$\begin{aligned}\mathbb{P}[X_1 > a_1, X_2 > a_2] &= 1 - \mathbb{P}[X_1 \leq a_1] - \mathbb{P}[X_2 \leq a_2] \\ &\quad + \mathbb{P}[X_1 \leq a_1, X_2 \leq a_2]\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[X_1 > a_1]\mathbb{P}[X_2 > a_2] &= 1 - \mathbb{P}[X_1 \leq a_1] - \mathbb{P}[X_2 \leq a_2] \\ &\quad + \mathbb{P}[X_1 \leq a_1]\mathbb{P}[X_2 \leq a_2]\end{aligned}$$

both hold true, condition (8) is equivalent to

$$\mathbb{P}[X_1 \leq a_1, X_2 \leq a_2] \geq \mathbb{P}[X_1 \leq a_1]\mathbb{P}[X_2 \leq a_2] \quad (9)$$

for all $a_1, a_2 \in \mathbb{R}$. The reason why (8) or (9) defines a positive dependence concept is that X_1 and X_2 are more likely to be large together or to be small together compared with the theoretical situation where X_1 and X_2 are independent of each other.

Remark 4.4. Note that the population version of Spearman's ρ is given by

$$\begin{aligned}\rho &= 12 \int_{x_1=-\infty}^{+\infty} \int_{x_2=-\infty}^{+\infty} \Big\{ F_{\boldsymbol{X}}(x_1, x_2) \\ &\qquad - F_1(x_1)F_2(x_2) \Big\} dF_1(x_1) dF_2(x_2)\end{aligned}$$

and hence $\rho/12$ represents a measure of average quadrant dependence, where the average is with respect to the marginal distribution of X_1 and X_2.

Proposition 4.5. (8)-(9) are satisfied if, and only if,

$$\text{Cov}[\phi_1(X_1), \phi_2(X_2)] \geq 0 \quad (10)$$

holds for any non-decreasing functions ϕ_1 and $\phi_2 : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist.

Proof. For a proof of the equivalence of (8) and (10), proceed as for Theorem 1 in Dhaene and Goovaerts (1996). □

Example 4.6. Let us consider the special case that F_i is a two-point distribution in 0 and $\alpha_i > 0$, $i = 1, 2$. For any $(X_1, X_2) \in \mathcal{R}_2(F_1, F_2)$ with Cov$[X_1, X_2] \geq 0$, we have that

$$\mathbb{P}[X_1 = \alpha_1, X_2 = \alpha_2] \geq \mathbb{P}[X_1 = \alpha_1]\,\mathbb{P}[X_2 = \alpha_2].$$

The latter inequality can be transformed into

$$\mathbb{P}[X_1 = 0, X_2 = 0] \geq \mathbb{P}[X_1 = 0]\,\mathbb{P}[X_2 = 0]$$

from which we find

$$\mathbb{P}[X_1 \leq x_1, X_2 \leq x_2] \geq \mathbb{P}[X_1 \leq x_1]\,\mathbb{P}[X_2 \leq x_2]$$

for any $x_1 \geq 0$, $x_2 \geq 0$. We can conclude that in this special case

$$X_1 \text{ and } X_2 \text{ are PQD } \Leftrightarrow \mathbb{C}\text{ov}[X_1, X_2] \geq 0.$$

4.4 The PQD assumption for remaining lifetimes

Let us prove the following result, which gives an intuitive meaning of PQD for remaining lifetimes T_x and T_y.

Property 4.7. If T_x and T_y are PQD then the inequalities

$$\mathbb{E}[T_y|T_x > t] \geq \mathbb{E}[T_y] \text{ for all } t \in \mathbb{R}^+$$

and

$$\mathbb{E}[T_x|T_y > s] \geq \mathbb{E}[T_x] \text{ for all } s \in \mathbb{R}^+$$

both hold true.

Proof. Let us prove the first inequality; the reasoning for the second one is similar. It comes from

$$\begin{aligned}\mathbb{E}[T_y|T_x > t] &= \int_{s=0}^{+\infty} \mathbb{P}[T_y > s|T_x > t]ds \\ &= \frac{1}{\mathbb{P}[T_x > t]} \int_{s=0}^{+\infty} \mathbb{P}[T_y > s, T_x > t]ds \\ &\geq \frac{1}{\mathbb{P}[T_x > t]} \int_{s=0}^{+\infty} \mathbb{P}[T_y > s]\mathbb{P}[T_x > t]ds \\ &= \int_{s=0}^{+\infty} \mathbb{P}[T_y > s]ds = \mathbb{E}[T_y],\end{aligned}$$

where the inequality is a consequence of the fact that T_x and T_y are PQD. □

Property 4.7 indicates that when PQD remaining lifetimes are involved, the knowledge that one of the two spouses is still alive at some time increases the expected remaining lifetime of the other one. From the introduction, the PQD assumption for the remaining lifetimes of married couples appears as rather natural.

When T_x and T_y are PQD, we have for the joint-life status that

$$a_{xy,\overline{n}|} \geq a^{\perp}_{xy,\overline{n}|}, \tag{11}$$

while for the last-survivor status, the reverse inequality holds, i.e.

$$a_{\overline{xy},\overline{n}|} \leq a^{\perp}_{\overline{xy},\overline{n}|}. \tag{12}$$

Moreover, it is easily seen that

$$a_{x|y} \leq a^{\perp}_{x|y}. \tag{13}$$

For the contracts (12)-(13), the independence assumption appears therefore as conservative as soon as PQD remaining lifetimes are involved. In other words, the premium in the insurer's price list contains an implicit safety loading in such cases. Moreover, the results (11)-(13) increase the accuracy of the bounds on $a_{\overline{xy},\overline{n}|}$ and $a_{xy,\overline{n}|}$. Indeed, when PQD remaining lifetimes T_x and T_y are involved, the unknown values of $a_{xy,\overline{n}|}$ lie between $a^{\perp}_{xy,\overline{n}|}$ and $a^{\max}_{xy,\overline{n}|}$, respectively, while those of $a_{\overline{xy},\overline{n}|}$ lie between $a^{\min}_{\overline{xy},\overline{n}|}$ and $a^{\perp}_{\overline{xy},\overline{n}|}$, respectively. Finally, $a_{x|y}$ lies between $a^{\min}_{x|y}$ and $a^{\perp}_{x|y}$. However, it is not possible to evaluate the height of the safety loading implicitly contained in the insurer's price list; this will be done in Sections 5 and 6.

5 Markovian model

5.1 Description of the model

Since the seminal lecture given by Amsler (1968) at the 18th International Congress of Actuaries and the paper by Hoem (1969), the Markovian model has become an appreciated tool for the calculation of life contingencies functions.

Given a stochastic process $\mathcal{X} = \{X_t,\ \ t \in \mathbb{R}^+\}$, X_t often represents in actuarial applications the state at time t of an individual, group of individuals, or insurance/annuity contract. In such a case, t usually measures the time since some event such as the birth of an individual or the sale of a policy. Let $\mathcal{F}_t$ be the history of the process $\mathcal{X}$ up to time t. The Markovian model assumes, roughly speaking, that the future of $\mathcal{X}$ is independent of all information contained in $\mathcal{F}_t$, except the state X_t at time t. Markov processes have been extensively discussed in the actuarial literature; see, for example, the papers by Amsler (1988), Davis and Vellekoop (1995), Haberman (1983,1984,1988,1995), Hoem (1972,1977,1988), Hoem and Aalen (1978), Jones (1994,1995,1996,1997a,1997b), Moller (1990,1992), Norberg (1988,1989), Panjer (1988), Pitacco (1995), Ramlau-Hansen (1988a,1988b,1991), Ramsay (1989), Tolley and Manton (1991), Waters (1984), Wilkie (1988) and Wolthuis and Van Hoeck (1986), as well as the references therein. An excellent overview is provided by the book of Haberman and Pitacco (1998). The present Section is based for the most part on Denuit and Cornet (1999a,b).

Norberg (1989) and Wolthuis (1994), in a first attempt to take into account a possible dependence among insured lifelengths, proposed a Markovian model with forces of mortality depending on marital status in order to evaluate the amount of premium relating to an insurance contract issued to a married couple. More precisely, assume that the husband's force of mortality at age $x + t$ is $\mu_{01}(t)$ if he is then still married and $\mu_{23}(t)$ if he is a widower. Likewise, the wife's force of mortality at age $y + t$ is $\mu_{02}(t)$ if she is then still married and $\mu_{13}(t)$ if she is a widow. The future development of the marital status for a x-year-old husband and a y-year-old wife may be regarded as a Markov process with state space and forces of transitions as represented in Figure 5.

To be specific, let us denote by $p_{ij}(t, t + \Delta t)$, t, $\Delta t \geq 0$ the transition probabilities of the Markov process described in Figure 5, i.e. $p_{ij}(t, t+\Delta t)$ is the conditional probability that the married couple under interest is in state j at time $t+\Delta t$, given that it was in state i at time t. Obviously, for any $0 \leq t_1 \leq t_2$, $0 \leq p_{ij}(t_1, t_2) \leq 1$ for all i and j, $p_{ij}(t_1, t_1) = 1$ if $i = j$ and 0 otherwise, and $\sum_j p_{ij}(t_1, t_2) = 1$ for all i. Moreover, the functions $t_1 \mapsto p_{ij}(t_1, t_2)$ for fixed t_2 and $t_2 \mapsto p_{ij}(t_1, t_2)$ for fixed t_1 are assumed to be continuously differentiable on $[0, t_2]$ and on $[t_1, +\infty[$, respectively. For $i \neq j$, forces of transition are related to transition probabilities through

$$\lim_{\Delta t \to 0} \frac{p_{ij}(t, t + \Delta t)}{\Delta t} = \frac{\partial}{\partial h} p_{ij}(t, t + h)\Big|_{h=0} = \mu_{ij}(t),$$

so that

$$p_{ij}(t, t + \Delta t) = \mu_{ij}(t)\Delta t + o(\Delta t),$$

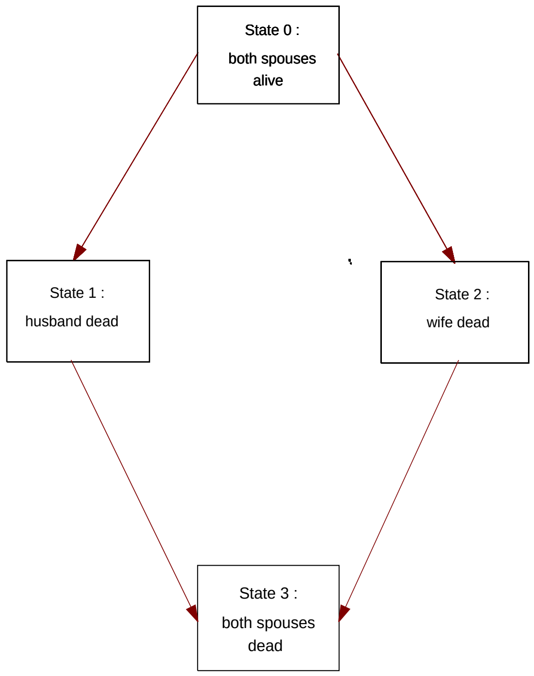

Figure 5. Markovian model with forces of mortality depending on the marital status.

where $o(.)$ is a function such that $\lim_{h\to 0} o(h)/h = 0$. In the model described in Figure 5, using the Markov property, we can write

$$p_{00}(t_1, t_2 + \Delta t_2) = p_{00}(t_1, t_2)p_{00}(t_2, t_2 + \Delta t_2),$$

$0 \le t_1 \le t_2 \le t_2 + \Delta t_2$, whence

$$p_{00}(t_1, t_2 + \Delta t_2) = p_{00}(t_1, t_2)\{1 - (\mu_{01}(t_2) + \mu_{02}(t_2))\Delta t_2\} + o(\Delta t_2).$$

Letting $\Delta t_2 \to 0$, we obtain the differential equation

$$\frac{\partial}{\partial t_2} p_{00}(t_1, t_2) = -\Big\{\mu_{01}(t_2) + \mu_{02}(t_2)\Big\} p_{00}(t_1, t_2).$$

The solution has to satisfy the boundary condition $p_{00}(t_1, t_1) = 1$ and is therefore given by

$$p_{00}(t_1, t_2) = \exp\left\{ - \int_{t_1}^{t_2} \{\mu_{01}(\tau) + \mu_{02}(\tau)\} d\tau \right\},$$

$0 \le t_1 \le t_2$. Similarly,

$$p_{11}(t_1, t_2) = \exp\left\{ - \int_{t_1}^{t_2} \mu_{13}(\tau) d\tau \right\}$$

and

$$p_{22}(t_1, t_2) = \exp\left\{ - \int_{t_1}^{t_2} \mu_{23}(\tau) d\tau \right\}, \quad 0 \le t_1 \le t_2.$$

Obviously, $p_{33}(t_1, t_2) = 1$ for any $t_1 \le t_2$. Now, from the Kolmogorov differential equations, it can be shown that for $j = 1, 2$,

$$p_{0j}(t_1, t_2) = \int_{t_1}^{t_2} p_{00}(t_1, \tau)\mu_{0j}(\tau)p_{jj}(\tau, t_2) d\tau, \quad 0 \le t_1 \le t_2.$$

The latter formula has an intuitive derivation: it is obtained by conditioning on the instant τ when the transition from state 0 to state j occurs.

Now, the joint survival function of (T_x, T_y) is given by

$$\mathbb{P}[T_x > t_1, T_y > t_2] = \begin{cases} p_{00}(0, t_2) + p_{00}(0, t_1)p_{01}(t_1, t_2) & \text{if } 0 \le t_1 \le t_2, \\ p_{00}(0, t_1) + p_{00}(0, t_2)p_{02}(t_2, t_1) & \text{if } 0 \le t_2 < t_1. \end{cases}$$

The marginal survival functions of T_x and T_y are respectively given by

$$\mathbb{P}[T_x > t_1] = p_{00}(0, t_1) + p_{02}(0, t_1)$$

and

$$\mathbb{P}[T_y > t_2] = p_{00}(0, t_2) + p_{01}(0, t_2),$$

for t_1, $t_2 \ge 0$.

Norberg (1989) proved the following result in this model.

Proposition 5.1. In the model described above,

$$\mu_{01} \equiv \mu_{23} \text{ and } \mu_{02} \equiv \mu_{13} \Leftrightarrow T_x \text{ and } T_y \text{ are independent,} \qquad (14)$$

while

$$\mu_{01} \le \mu_{23} \text{ and } \mu_{02} \le \mu_{13} \Rightarrow T_x \text{ and } T_y \text{ are } PQD. \qquad (15)$$

Proof. Norberg's proof is based on the following argument: the idea is to compute

$$\frac{\partial}{\partial t_2}\mathbb{P}[T_x > t_1 | T_y > t_2] = \begin{cases} \frac{\partial}{\partial t_2}\left(\frac{p_{00}(0,t_2)+p_{00}(0,t_1)p_{01}(t_1,t_2)}{p_{00}(0,t_2)+p_{01}(0,t_2)}\right) \\ \text{if } 0 \le t_1 \le t_2, \\ \frac{\partial}{\partial t_2}\left(\frac{p_{00}(0,t_1)+p_{00}(0,t_2)p_{02}(t_2,t_1)}{p_{00}(0,t_2)+p_{01}(0,t_2)}\right) \\ \text{if } 0 \le t_2 < t_1, \end{cases}$$

and to show by a straightforward but tedious calculation that it is always non-negative if $\mu_{01} \le \mu_{23}$ and $\mu_{02} \le \mu_{13}$, while it vanishes when $\mu_{01} \equiv \mu_{23}$ and $\mu_{02} \equiv \mu_{13}$. Therefore, since $\mathbb{P}[T_y = 0] = 0$, we get

(i) in the first case that the inequality

$$\mathbb{P}[T_x > t_1 | T_y > t_2] \ge \mathbb{P}[T_x > t_1 | T_y > 0] = \mathbb{P}[T_x > t_1],$$

holds for any $t_1, t_2 \in \mathbb{R}^+$, which reduces to (15), so that (15) is valid,

(ii) while in the second case, the equality

$$\mathbb{P}[T_x > t_1 | T_y > t_2] = \mathbb{P}[T_x > t_1],$$

holds for any $t_1, t_2 \in \mathbb{R}^+$, whence (14) follows.

This ends the proof. □

From the introduction and in view of equations (14)-(15), it seems natural to propose that, for $t \in \mathbb{R}^+$,

$$\mu_{01}(t) = (1-\alpha_{01})\mu_{x+t}, \quad \mu_{23}(t) = (1+\alpha_{23})\mu_{x+t}, \quad (16)$$

and

$$\mu_{02}(t) = (1-\alpha_{02})\mu_{y+t}, \quad \mu_{13}(t) = (1+\alpha_{13})\mu_{y+t}, \quad (17)$$

where the α_{ij}'s are non-negative and the α_{0j}'s are less than 1. Therefore, the model (16)-(17) comes down to assume that the mortality intensities are lower than the forces of mortality in the entire Belgian population as long as both spouses are alive and are higher when one of the couple is deceased. Setting $\alpha_{ij} \equiv \alpha$, we find the model proposed by Wolthuis (1994, page 62).

5.2 Estimation of the parameters

We are now going to estimate the four parameters α_{01}, α_{02}, α_{13} and α_{23} involved in (16)-(17). To this end, we use data collected by the Belgian NIS and we follow the method of least squares proposed in Wolthuis (1994, chapter VI). The estimators $\hat{\alpha}_{ij}$ of the parameters α_{ij} minimize the sum of the squared differences between the increments $\Delta\Omega_{ij}$ of the transition functions

$$\Omega_{ij}(t) = \int_{\tau=0}^{t} \mu_{ij}(\tau)d\tau, \quad t \geq 0,$$

and their estimations $\Delta\hat{\Omega}_{ij}$, i.e.

$$\hat{\alpha}_{ij} = \arg\min \sum_k \left(\Delta\hat{\Omega}_{ij}(k) - \int_{t=0}^{1} \mu_{ij}(k+t)dt \right)^2 . \quad (18)$$

Let us now briefly expand on the estimation of $\Delta\Omega_{ij}$. Let $L_i(t)$ be the number of couples in state i at age $t-$ (i.e. $L_i(t)$ can be thought of as the number of couples "at risk" just prior to age t of a transition from state i) and let $L_{ij}(t)$ be the number of transitions from state i to state j over $[0,t]$. The Nelson-Aalen non-parametric estimator of $\Omega_{ij}(t)$ is

$$\hat{\Omega}_{ij}(t) = \int_{\tau=0}^{t} \frac{\mathbb{I}[L_i(\tau)>0]}{L_i(\tau)} dL_{ij}(\tau),$$

where $\mathbb{I}[A]$ is the indicator function of the event A, and with the convention that the integrand is defined to be zero when $L_i(\tau) = 0$; for more details, see e.g. Jones (1997b) and the references therein.

For estimation purposes, we had at our disposal data about the population living in Belgium during 1991 (total population of the kingdom splitted by age, sex and marital status at January 1, 1991 and January 1, 1992, as well as the number of the deceases, the number of weddings and divorces in 1991 by age, sex, year of birth and marital status). Since the number of transitions is only available for a year, we use the linearity assumption, i.e. we assume that for any integer k and $0 \leq t < 1$,

$$L_{ij}(k+t) = L_{ij}(k) + t\left\{L_{ij}(k+1) - L_{ij}(k)\right\}$$

and

$$L_i(k+t) = L_i(k) + t\left\{L_i(k+1) - L_i(k)\right\}.$$

These approximations yield

$$\begin{aligned} &\Delta\hat{\Omega}_{ij}(k) \\ = &\int_{\tau=0}^{1} \frac{\mathbb{I}[L_i(k+\tau)>0]}{L_i(k)+\tau\left\{L_i(k+1)-L_i(k)\right\}} \\ &\qquad \Big\{L_{ij}(k+1)-L_{ij}(k)\Big\}d\tau \\ = &\frac{L_{ij}(k+1)-L_{ij}(k)}{L_i(k+1)-L_i(k)}\Big\{\ln L_i(k+1) - \ln L_i(k)\Big\} \\ \equiv &\frac{L_{i:j}(k)}{L_i(k+1)-L_i(k)}\Big\{\ln L_i(k+1) - \ln L_i(k)\Big\}, \quad (19) \end{aligned}$$

where $L_i(k)$ represents the number of couples in state i at age k and

$$L_{i:j}(k) = L_{ij}(k+1) - L_{ij}(k)$$

is the number of transitions from state i to state j observed for k-year-old individuals. Relation (19) is in accordance with formula (33) in Wolthuis (1994, page 108).

Let us explain precisely how we estimate $\Delta\hat{\Omega}_{01}(k)$ and $\Delta\hat{\Omega}_{13}(k)$ (the interested reader will easily deduce the estimations of $\Delta\hat{\Omega}_{02}(k)$ and $\Delta\hat{\Omega}_{23}(k)$ by switching the roles of the two spouses). Let us start with $\Delta\hat{\Omega}_{01}(k)$ and examine the different elements constituting (19):

1. the numerator $L_{0:1}(k)$ is the number (#, in short) of k-year-old married men dying during 1991 (this number is directly available from the NIS);
2. the denominator $L_0(k+1) - L_0(k)$ is equal to

   – # of k-year-old married men dying during 1991
   – # of k-year-old married men
   whose wife died during 1991
   + # k-year-old men getting married during 1991
   – # k-year-old married men
   getting divorced during 1991.

   The number of couples with a k-year-old man whose wife died during 1991 cannot be obtained from the NIS. Therefore, we estimate it as follows :

   # of $(k+1)$-year-old widowers at January 1, 1992
   –(# of k-year-old widowers at January 1, 1991
   – # of k-year-old widowers dying during 1991
   – # of k-year-old widowers
   getting married during 1991);

3. finally, concerning the difference of the logarithms in (19), $L_0(k)$ is the number of k-year-old married men at January 1, 1991, and $L_0(k+1)$ is easily deduced from above.

Let us now examine $\Delta\hat{\Omega}_{13}(k)$:

1. the numerator $L_{1:3}(k)$ is the number of k-year-old widows dying during 1991;
2. the denominator $L_1(k+1) - L_1(k)$ is equal to

   – # of k-year-old widows dying during 1991
   + # of k-year-old women whose husband died during 1991
   – # of k-year-old widows getting married during 1991.

   The number of couples with a k-year-old woman whose husband died during 1991 is not available from the NIS. Therefore, we estimate it as follows :

   # of $(k+1)$-aged widows at January 1, 1992
   –(# of k-year-old widows at January 1, 1991
   – # number of k-year-old widows dying during 1991
   – # of k-year-old widows getting married during 1991);

3. finally, concerning the difference between the logarithms in (19), $L_1(k)$ is the number of k-year-old widows at January 1, 1991, and $L_1(k+1)$ is easily obtained from above.

In the model (16)-(17), the estimations $\hat{\alpha}_{ij}$ of the parameters α_{ij} are explicitly given by

$$\hat{\alpha}_{01} = 1 - \frac{\sum_k \left(A_1 + B_1 c_1^k \frac{c_1-1}{\ln c_1}\right)\Delta\hat{\Omega}_{01}(k)}{\sum_k \left(A_1 + B_1 c_1^k \frac{c_1-1}{\ln c_1}\right)^2}, \tag{20}$$

$$\hat{\alpha}_{02} = 1 - \frac{\sum_k \left(A_2 + B_2 c_2^k \frac{c_2-1}{\ln c_2}\right)\Delta\hat{\Omega}_{02}(k)}{\sum_k \left(A_2 + B_2 c_2^k \frac{c_2-1}{\ln c_2}\right)^2}, \tag{21}$$

$$\hat{\alpha}_{13} = \frac{\sum_k \left(A_2 + B_2 c_2^k \frac{c_2-1}{\ln c_2}\right)\Delta\hat{\Omega}_{13}(k)}{\sum_k \left(A_2 + B_2 c_2^k \frac{c_2-1}{\ln c_2}\right)^2} - 1, \tag{22}$$

and

$$\hat{\alpha}_{23} = \frac{\sum_k \left(A_1 + B_1 c_1^k \frac{c_1-1}{\ln c_1}\right)\Delta\hat{\Omega}_{23}(k)}{\sum_k \left(A_1 + B_1 c_1^k \frac{c_1-1}{\ln c_1}\right)^2} - 1. \tag{23}$$

On the basis of (20)-(23) with the aid of the NIS data concerning individuals aged from 30 to 80 years, we get

$$\hat{\alpha}_{01} = 0.092\ 945\ 871, \quad \hat{\alpha}_{02} = 0.121\ 655\ 037,$$

$$\hat{\alpha}_{13} = 0.041\ 349\ 449 \text{ and } \hat{\alpha}_{23} = 0.241\ 032\ 536.$$

In other words, there is (on the basis of the NIS data collected during 1991) an under-mortality of about 9% for married men, of 12% for married women, and an over-mortality of about 4% for the widows and of 24% for the widowers, compared to the mortality experienced by the entire Belgian population.

5.3 Premium calculation in the markovian model

In order to price the widow's pension, we only need the probabilities $p_{00}(t, t+\Delta t)$, $p_{01}(t, t+\Delta t)$ and $p_{11}(t, t+\Delta t)$ for integers t and Δt. The p_{00}'s and p_{11}'s can be calculated recursively since they satisfy the recurrence scheme

$$\begin{cases} p_{00}(0, k+1) = p_{00}(0, k) p_{00}(k, k+1), \\ p_{11}(0, k+1) = p_{11}(0, k) p_{11}(k, k+1), \end{cases}$$

starting with $p_{00}(0,0) = p_{11}(0,0) = 1$. We further assume that the transition intensities $\mu_{ij}(.)$ are constant for each year of age, i.e.

$$\mu_{ij}(k+\tau) = \mu_{ij}(k) \text{ for } 0 \le \tau < 1. \tag{24}$$

This reduces to consider that for each integer age $x+k$ and $y+k$, we have

$$\mu_{x+k+\tau} = \mu_{x+k} \text{ and } \mu_{y+k+\tau} = \mu_{y+k} \text{ for } 0 \le \tau < 1.$$

The one-year probabilities $p_{00}(k, k+1)$ and $p_{11}(k, k+1)$ are then respectively given by

$$p_{00}(k, k+1) = \exp\left\{-\mu_{01}(k) - \mu_{02}(k)\right\},$$

and

$$p_{11}(k, k+1) = \exp\left\{-\mu_{13}(k)\right\},$$

while the one-year transition probabilities $p_{01}(k, k+1)$ can be expressed as

$$p_{01}(k, k+1) = \frac{\mu_{01}(k)\Big(\exp\left\{-\mu_{01}(k) - \mu_{02}(k)\right\} - \exp\left\{-\mu_{13}(k)\right\}\Big)}{\mu_{13}(k) - \mu_{01}(k) - \mu_{02}(k)}.$$

Reformulated in the Markov model, the net single premium $a_{x|y}^{\text{mark}}$ relating to the widow's pension is given

$$a_{x|y}^{\text{mark}} = \sum_{k=0}^{\min(\omega_x, \omega_y)} p_{00}(0, k) p_{01}(k, k+1) \sum_{j=0}^{\omega_y - k} p_{11}(k+1, k+1+j) v^{k+1+j}.$$

Remark 5.2. The assumption (24) has been done to facilitate computations. More rigorously, from specifications (16)-(17), it is easily seen that for $s < t$

$$\begin{aligned} p_{00}(s,t) &= \exp\left\{-\int_{\tau=s}^{t} \big(\mu_{01}(\tau) + \mu_{02}(\tau)\big) d\tau\right\} \\ &= \exp\left\{-(1-\alpha_{01}) \int_{\tau=s}^{t} \big(A_1 + B_1 c_1^{x+\tau}\big) d\tau\right\} \\ &\quad \exp\left\{-(1-\alpha_{02}) \int_{\tau=s}^{t} \big(A_2 + B_2 c_2^{y+\tau}\big) d\tau\right\} \\ &= \exp\left\{-(1-\alpha_{01}) \left(A_1 + \frac{B_1}{\ln c_1}\big(c_1^{x+t} - c_1^{x+s}\big)\right) \right. \\ &\quad \left. -(1-\alpha_{02}) \left(A_2 + \frac{B_2}{\ln c_2}\big(c_2^{y+t} - c_2^{y+s}\big)\right)\right\} \end{aligned}$$

with similar expressions for $p_{11}(s,t)$, $p_{22}(s,t)$. Tedious calculations then yield expressions for $p_{01}(s,t)$ and $p_{02}(s,t)$.

5.4 Numerical illustrations

To illustrate the possible use of the above Markovian model, we plotted the graphs of Figure 6 on the basis of the set of the three assumptions $x = y$, $x = y - 5$ and $x = y + 5$. The fact that the net single premiums $a_{x|y}^{\text{mark}}$ are indeed lower for the calculation based on the assumption of dependent remaining lifetimes can be explained as follows. Since the $\hat{\alpha}_{ij}$'s are all non-negative, we deduce from (15) that the time-until-death random variables T_x and T_y are PQD. With PQD remaining lifetimes, the policy stays longer in state 0 (thus there is a longer time until possible annuity payements) and shorter in state 1 (less annuity payements). *Grosso modo*, the Markovian model provides net single premiums $a_{x|y}^{\text{mark}}$ of about 90% of those computed on the independence assumption (i.e. $a_{x|y}^{\perp}$).

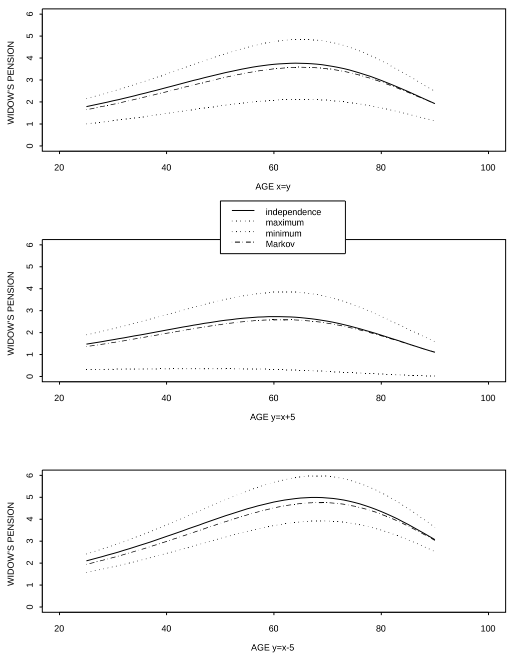

Figure 6. $a_{x|y}^{\perp}$ and $a_{x|y}^{\text{mark}}$.

In conclusion, the Markovian model allows the actuary to determine an "exact" value of the premium $a_{x|y}$ under interest. This "exact" value offers the actuary a yardstick in order to decide whether or not to grant a discount to the assured persons, as well as to select the amount of this discount, or to evaluate the level of the mortality benefits in profit testing. Finally, the value $a_{x|y}^{\text{mark}}$ is also of primordial importance when the level of the safety loading is to be selected. Indeed, the manual premium $a_{x|y}^{\perp}$ itself contains an implicit safety loading of about 10%. This has to be taken into account in order to avoid excessive safety margins.

6 Copula models of ratemaking

6.1 Notion

One of the most useful tools for handling multivariate distributions with given univariate marginals is the copula function (also named uniform representation or dependence functions). The concept of "Copulas" or "copula functions" originates in the late 50's in the context of probabilistic metric spaces. The idea behind this concept is the following: for multivariate distributions, the univariate marginals and the dependence structure can be separated and the latter may be represented by a copula. The word copula is a latin noun which means "a link", and is used in grammar and logic to describe that part of a proposition which connects the subject and predicate. In statistics, it now describes the function that "joins together" one-dimensional ditribution functions to form multivariate distribution functions.

Let us define the notion of copula in the bidimensional case.

Definition 6.1. A bivariate copula C is the joint distribution function for a bivariate distribution with unit uniform marginals. More precisely, C is a function mapping the unit square $[0,1] \times [0,1]$ to the unit interval $[0,1]$ which is non-decreasing and continuous, and satisfies

(i) $\lim_{u_i \to 0} C(u_1,u_2) = 0$ for $i = 1, 2$;
(ii) $\lim_{u_1 \to 1} C(u_1,u_2) = u_2$ and $\lim_{u_2 \to 1} C(u_1,u_2) = u_1$;
(iii) $C(v_1,v_2) - C(u_1,v_2) - C(v_1,u_2) + C(u_1,u_2) \geq 0$ for any $u_1 \leq v_1$, $u_2 \leq v_2$.

Then, it can be proved that every bivariate distribution function $F_{\boldsymbol{X}}$ in $\mathcal{R}_2(F_1,F_2)$ can be represented in terms of a copula C through

$$F_{\boldsymbol{X}}(x_1,x_2) = C\left(F_1(x_1),F_2(x_2)\right), \quad (x_1,x_2) \in \mathbb{R}_2. \quad (25)$$

When the marginals F_1 and F_2 are continuous, the copula C in (25) is unique and coincides with the distribution function of the pair $(F_1(X_1),F_2(X_2))$; this is nevertheless no more true when at least one of the F_i's possesses some discontinuity points. Henceforth, we restrict ourselves to continuous marginals. Considering (25), C "couples" the marginal distributions F_1 and F_2 to get the joint-distribution $F_{\boldsymbol{X}}$ of the pair $\boldsymbol{X}$. The dependence structure is entirely described by C and dissociated of the marginal distributions F_1 and F_2. Thus, the manner in which X_1 and X_2 "move together" is captured by the copula, regardless of the scale in which the variable is measured.

The marginals F_1 and F_2 can be inserted in any copula, so they carry no direct information about the coupling. At the same time any pair of marginals can be inserted into C so C carries no direct information about the marginals. This being the case, it may seem reasonable to expect that the connections between the marginals of $F_{\boldsymbol{X}}$ are determined by C alone, and any question about the dependence structure can be answered with the knowledge of C alone.

Example 6.2. The population version of Kendall's τ for a random couple (X_1, X_2) of continuous rv's with copula C is expressible as

$$\begin{aligned} &\tau(X_1, X_2) \\ =\ & 4\int_{u_1=0}^{1}\int_{u_2=0}^{1} C(u_1,u_2)dC(u_1,u_2) - 1 \\ =\ & 1 - 4\int_{u_1=0}^{1}\int_{u_2=0}^{1} \frac{\partial}{\partial u_1}C(u_1,u_2)\frac{\partial}{\partial u_2}C(u_1,u_2)du_1du_2 \\ =\ & 4\mathbb{E}[C(U_1,U_2)] - 1, \end{aligned} \quad (26)$$

where (U_1, U_2) stands for a couple of random variables uniformly distributed over $[0, 1]$ with joint distribution function C. Similarly, Spearman's ρ can be expressed as

$$\begin{aligned} &\rho(X_1, X_2) \\ =\ & 12\int_{u_1=0}^{1}\int_{u_2=0}^{1} u_1u_2dC(u_1,u_2) - 3 \\ =\ & 12\int_{u_1=0}^{1}\int_{u_2=0}^{1} C(u_1,u_2)du_1du_2 - 3. \end{aligned} \quad (27)$$

6.2 Some classical copula families

We give below the most common copulas, together with their Kendall's τ. Let us now point out some remarkable copulas related to independence and Fréchet bounds:

- the Fréchet upper bound copula, denoted by C_U, is

  $$C_U(u_1,u_2) = \min(u_1,u_2), \quad (u_1,u_2) \in [0,1]\times[0,1],$$

  it corresponds to a unit mass spread over the main diagonal $u_1 = u_2$ of the unit square and has $\tau = 1$;
- the Fréchet lower bound copula, denoted by C_L, is

  $$C_L(u_1,u_2) = \max(0, u_1+u_2-1), \quad (u_1,u_2) \in [0,1]\times[0,1],$$

  it corresponds to a unit mass spread over the secondary diagonal $u_1 = 1 - u_2$ of the unit square and has $\tau = -1$;
- finally, the independence copula, denoted by C_I, is

  $$C_I(u_1,u_2) = u_1u_2, \quad (u_1,u_2) \in [0,1]\times[0,1],$$

  $\tau = 0$.

Let us mention that if X_1 and X_2 possess the distribution function (25) then they are independent if, and only if, $C \equiv C_I$, X_2 is almost surely a non-decreasing function of X_1 if, and only if, $C \equiv C_U$, and X_2 is almost surely a non-increasing function of X_1 if, and only if, $C \equiv C_L$.

Fréchet (1958) proposed a family of copulas consisting in two-parameters convex linear combinations of C_L, C_I and C_U, known in the literature as the Fréchet family. More precisely, the copulas $C_{\alpha,\beta}$ of the Fréchet family are of the form

$$C_{\alpha,\beta} = \alpha C_U + (1-\alpha-\beta)C_I + \beta C_L,$$

with $\alpha, \beta \geq 0$ such that $\alpha + \beta \leq 1$. The corresponding Kendall's τ is given by

$$\tau_{\alpha,\beta} = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3}$$

and the Spearman's ρ equates $\rho_{\alpha,\beta} = \alpha - \beta$. Then, Mardia (1970) introduced a subfamily of the Fréchet family that can be seen as one-parameter mixtures of C_L, C_I and C_U (see also Carrière and Chan (1986) for a generalization of this model).

There are also many parametric families of bivariate copulas, as those listed hereafter (in each case, the parameter α measures the degree of association):

- Cook-Johnson

  $$C(u_1,u_2) = \left(u_1^{-\alpha} + u_2^{-\alpha} - 1\right)^{-1/\alpha}, \quad \alpha \geq 0;$$

  $\tau = \alpha/(\alpha+2)$;
- Farlie-Gumbel-Morgenstern

  $$C(u_1,u_2) = u_1u_2\Big(1 + \alpha(1-u_1)(1-u_2)\Big), \quad \alpha \in [-1,1].$$

  This family has $\tau = \frac{2}{9}\alpha \in [-\frac{2}{9}, \frac{2}{9}]$ and can thus be used only in situations with weak dependence;
- Frank

  $$C(u_1,u_2) = \frac{1}{\alpha}\ln\left(1 + \frac{(\exp(\alpha u_1)-1)(\exp(\alpha u_2-1)}{\exp(\alpha)-1}\right),$$

  $\alpha \neq 0, \tau_\alpha = 1+4(D_1(\alpha)-1)/\alpha$, where $D_1(.)$ is the Debye function of order 1, i.e.

  $$D_1(\alpha) = \int_{\xi=0}^{\alpha} \frac{\xi}{\alpha(\exp\xi - 1)}d\xi, \quad \alpha \in \mathbb{R};$$
- Gumbel-Hougaard or logistic copula

  $$C(u_1,u_2) = \exp\left(-\{(-\ln u_1)^\alpha + (-\ln u_2)^\alpha\}^{1/\alpha}\right),$$

  $\alpha \geq 1, \tau = 1 - \alpha^{-1}$. The parameter α controls the amount of dependence between the two components: $\alpha = 1$ gives independence and the limit for $\alpha \to +\infty$ leads to perfect dependence. This copula is consistent with bivariate extreme value theory and can be used to model the limiting dependence structure of componentwise maxima of bivariate random couples.

Note that all the bivariate copulas examined above satisfy the exchangeability condition $C(u_1,u_2) = C(u_2,u_1)$. In a situation where the appropriateness of this symmetry condition is doubtful, one may wish to have non-exchangeable models. Asymmetric copulas have been proposed in Genest, Ghoudi and Rivest (1998).

6.3 Archimedean family

Let us now present the family of the Archimedean copulas introduced by Genest and MacKay (1986a,b).

Definition 6.3. Consider a function $\phi : [0,1] \to \mathbb{R}^+$ having two continuous derivatives $\phi^{(1)}$ and $\phi^{(2)}$ on $]0,1[$ and satisfying

$$\phi(1) = 0, \quad \phi^{(1)}(\tau) < 0 \text{ and } \phi^{(2)}(\tau) > 0 \qquad (28)$$

for all $\tau \in]0,1[$. Conditions (28) are enough to guarantee that ϕ has an inverse ϕ^{-1} having also two derivatives. Every function ϕ satisfying (28) generates a bivariate distribution function C_ϕ whose marginals are uniform on the unit interval (i.e. a copula) given by

$$C_\phi(u_1, u_2) = \begin{cases} \phi^{-1}\{\phi(u_1) + \phi(u_2)\}, \\ \text{ if } \phi(u_1) + \phi(u_2) \leq \phi(0), \\ 0, \text{ otherwise,} \end{cases} \qquad (29)$$

for $0 \leq u_1, u_2 \leq 1$. Copulae C_ϕ of the form (29) with ϕ satisfying (28) are referred to as Archimedean copulae.

The Archimedean representation (29) allows us to reduce the study of a multivariate copula to a single univariate function. As an example, C_I is the Archimedean copula associated to $\phi(t) = -\ln(t)$.

In general evaluating the population version of Kendall's τ requires the evaluation of a double integral. For an Archimedean copula, the situation is simpler in that τ can be evaluated directly from the generator ϕ as follows: Kendall's τ associated to C_ϕ is given by

$$\tau_\phi = 4 \int_{t=0}^{1} \frac{\phi(t)}{\phi^{(1)}(t)} dt + 1.$$

One of the reasons Archimedean copulas are easy to work with is that often expressions with a one-dimensional function (the generator) can be employed rather than expressions with a two-dimensional function (the copula).

Now, a bivariate distribution function $F_{\boldsymbol{X}}$ in $\mathcal{R}_2(F_1, F_2)$ is said to be generated by an Archimedean copula if, and only if, it can be expressed in the form

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C_\phi(F_1(x_1), F_2(x_2)), \quad \boldsymbol{x} \in \mathbb{R}^2,$$

for some C_ϕ satisfying (28)-(29). Several one-parameter systems of bivariate distributions with fixed marginals can be seen to have Archimedean copulas as their dependence functions. Three key examples of generators are given next (these can be found in Genest and Rivest (1993)):

1. Cook-Johnson:
$$\phi_\alpha = t^{-\alpha} - 1, \ \ \alpha > 1;$$
2. Frank:
$$\phi_\alpha = \ln\left\{\frac{\exp(\alpha t) - 1}{\exp(\alpha) - 1}\right\}, \ \ \alpha \in \mathbb{R};$$
3. Gumbel-Hougaard:
$$\phi_\alpha = \Big(-\ln(t)\Big)^\alpha, \ \ \alpha \geq 1.$$

The Archimedean family provides a host of models that are extremely versatile and that have been successfully used in a number of data modeling contexts (see the references provided in Section 2 of Genest, Ghoudi and Rivest (1998)). Moreover, this class of dependence functions is mathematically tractable and its elements have stochastic properties that make these functions attractive for the statistical treatment of data.

6.4 Data for numerical illustrations

In order to fit these models, we need a bivariate data set. The required data were not available at the NIS (National Institute of Statistics) since the Belgian official statistics are based on the death certificates fulfilled by the family practitioner or a physician with a hospital appointment and the latter do not mention the age of death (if applicable) of the spouse of the deceased. In other words, NIS can provide detailed statistics about the mortality of Belgian males and females split by age and marital status, but not bivariate data sets. Therefore, we selected at random two cemeteries in Brussels (namely Koekelberg and Ixelles-Elsene) and we collected the ages at death of 533 couples buried there. Of course, this methodology is open to criticism and we do not claim at all that the present data set is the best sample a statistician could find. These data are only used in order to illustrate the techniques examined in the paper. The interested actuary will substitute his own data set for the present one.

Let us now provide some data characteristics. The two variables of interest are called “Age at death/Man" and “Age at death/Woman". First, Table 2 provides some descriptive statistics about these two variables, namely, the mean, the mode, the standard deviation, the standard error (calculated by dividing the standard deviation of the observations by the square root of the number of observations; it estimates the variability one expects if repeated samples of the same size are taken from the population), the smallest and largest values in the set of observations, the range, the 10th, 25th, 50th, 75th and 90th percentiles, the skewness (the negative value of the skewness indicates that the left tail spreads out further than the right), the kurtosis (the positive kurtosis value indicates that the data is squeezed into the middle of the distribution). The histograms of “Age at death/Man" and “Age at death/Woman" are displayed in Figures 7 and 8, while the scattergram of the pair “Age at death/Man" – “Age at death/Woman" is depicted at Figure 9. The results clearly suggest higher mortality rates for males than for females and moderate association.

The box plots for “Age at death/Man" and “Age at death/Woman" are depicted in Figure 10. The central box is composed of three horizontal lines that display the 25th, 50th and 75th percentiles of the variable. In addition, two segments of 1.5 times the inter-quartile interval are plotted above the 75th percentile and below the 25th one.

Considering Table 3, the positive value of Spearman's ρ indicates that high ranks of one variable occur with high ranks of the other variable. The null hypothesis tested in Table 3 (first column) is that the two variables “Age at death/Man" and

	“Age at death/Man"	“Age at death/Woman"
Mean	73,083	78,340
Mode	81,000	81,000
Std. deviation	12,268	11,074
Std. error	0,531	0,480
Minimum	24,000	22,000
Maximum	98,000	103,000
Range	74,000	81,000
10th percentile	56,000	64,000
25th percentile	67,000	72,000
50th percentile	74,000	80,000
75th percentile	82,000	86,000
90th percentile	88,000	91,000
Skewness	-0,752	-0,949
Kurtosis	0,711	1,824

Table 2. Descriptive statistics for “Age at death/Man" and “Age at death/Woman".

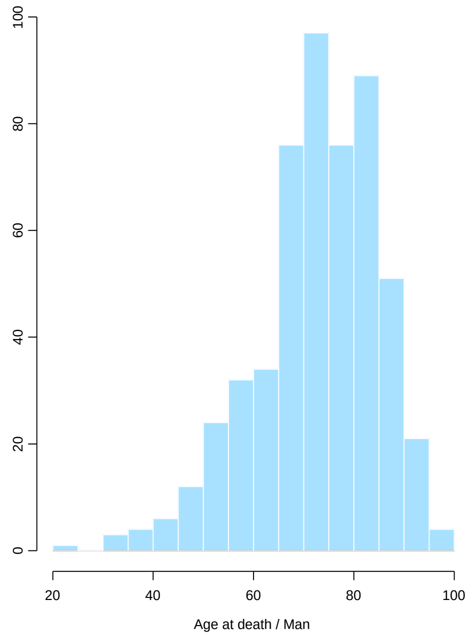

Figure 8. Histogram of “Age at death/Man"

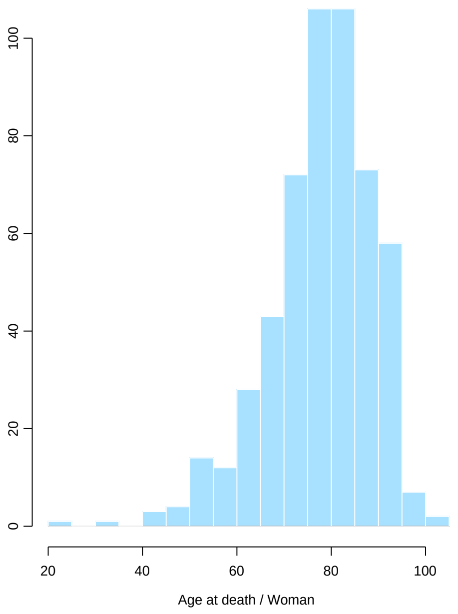

Figure 7. Histogram of “Age at death/Woman"

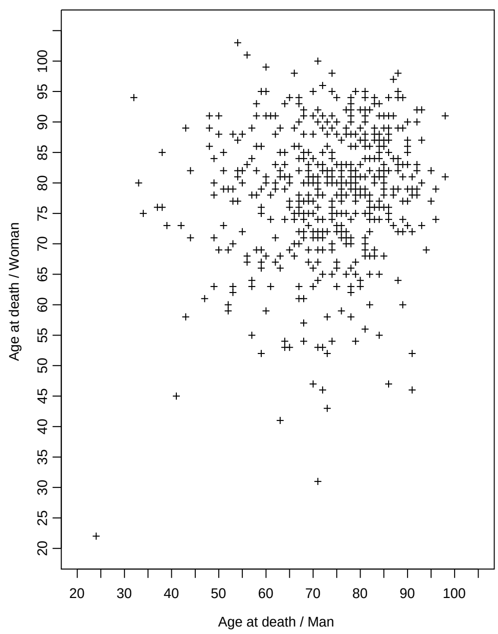

Figure 9. Scattergram of the pair “Age at death/Man" – “Age at death/Woman"

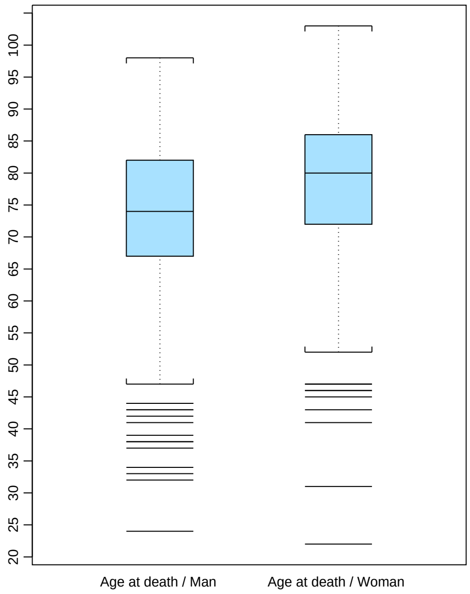

Figure 10. Box plots of “Age at death/Man" and “Age at death/Woman"

“Age at death/Woman" are independent of each other, against an alternative hypothesis that the rank of a variable is correlated with the rank of another variable. The tied p-value is equal to 0,15% so that the independence assumption is clearly rejected at any reasonable confidence level. Similar conclusions are drawn from Kendall’s τ.

	Spearman’s rank correlation	Kendall’s rank correlation
Rank correlation	0,139	0,092
Z-value	3,199	3,169
p-value	0,0014	0,0015
Rank correlation corrected for ties	0,138	0,094
Tied Z-value	3,180	3,254
Tied p-value	0,0015	0,0011

Table 3. Spearman and Kendall rank correlations and independence test.

6.5 Archimedean copula selection

In order to select a copula model, we first restricted ourselves to Archimedean copulas, and we applied the methodology proposed by Genest and Rivest (1993) for identifying a copula form in empirical applications. It assumes that the underlying distribution function $F_{\boldsymbol{X}}$ has an associated copula C_ϕ; the aim is thus to identify ϕ. The idea is to work with an intermediate random variable $Z_i = F_{\boldsymbol{X}}(X_1^{(i)}, X_2^{(i)})$ that has distribution function F_Z. It can be shown that

$$F_Z(z) = z - \frac{\phi(z)}{\phi^{(1)}(z)}.$$

To identify ϕ, the method proceeds in three steps:

1. estimate Kendall’s τ;
2. construct a nonparametric estimate of F_Z as
$$\widehat{F}_n(z) = \frac{1}{n}\#\{i|z_i \leq z\}$$
where
$$z_i = \frac{1}{n-1}\#\left\{(x_1^{(j)}, x_2^{(j)})\Big| x_1^{(j)} < x_1^{(i)}, x_2^{(j)} < x_2^{(i)}\right\};$$
3. construct a parametric estimate of F_Z: for various choices of ϕ use $\widehat{\tau}$ to estimate α and
$$F_{\phi_{\widehat{\alpha}}}(z) = z - \frac{\phi_{\widehat{\alpha}}}{\phi_{\widehat{\alpha}}^{(1)}}.$$

After having repeated Step 3 for several choices of ϕ, it suffices to compare each parametric estimate to the nonparametric estimate constructed in Step 2. The idea is to select ϕ so that the parametric estimate resembles the nonparametric one. Measuring closeness can be done by minimizing a distance such as

$$\int \left\{F_{\phi_{\widehat{\alpha}}}(z) - \widehat{F}_n(z)\right\} d\widehat{F}_n(z),$$

or graphically.

As it can be seen from Figure 11, the empirical distribution $\widehat{F}_n$ is very close to its theoretical couterparts $F_{\phi_{\widehat{\alpha}}}$ so that Cook, Frank and Gumbel models seem appropriate to model our data. Therefore, let us continue with Gumbel copula, for the sake of simplicity.

The parameter α involved in the Gumbel model is estimated via Maximum Likelihood, starting from the following initial value:

$$\frac{\hat{\tau}}{1-\hat{\tau}} = 0,104;$$

we have obtained $\widehat{\alpha} = 0.1015378$. We then get the probability that both T_x and T_y fail before t can be estimated by

$$\begin{aligned} {}_tq_{\overline{xy}} &= C_{\phi_{\hat{\alpha}}}({}_tq_x, {}_tq_y) \\ &= \exp\left(-\left\{(-\ln {}_tq_x)^{\hat{\alpha}+1} + (-\ln {}_tq_y)^{\hat{\alpha}+1}\right\}^{\frac{1}{\hat{\alpha}+1}}\right), \end{aligned}$$

$t \in \mathbb{R}^+$. Probabilities ${}_tp_{xy}$ and ${}_tp_{\overline{xy}}$ can then be estimated by ${}_t\hat{p}_{xy}$ and ${}_t\hat{p}_{\overline{xy}}$ given by

$$ {}_t\hat{p}_{xy} = 1 - {}_tq_x - {}_tq_y + C_{\phi_{\hat{\alpha}}}({}_tq_x, {}_tq_y), \quad t \in \mathbb{R}^+, \qquad (30)$$

and

$$ {}_t\hat{p}_{\overline{xy}} = 1 - C_{\phi_{\hat{\alpha}}}({}_tq_x, {}_tq_y), \quad t \in \mathbb{R}^+. \qquad (31)$$

By inserting the values provided by (30)-(31) in the net single premiums $a_{xy;\overline{n}|}$, $a_{\overline{xy};\overline{n}|}$ and $a_{x|y}$, we get the graphs depicted

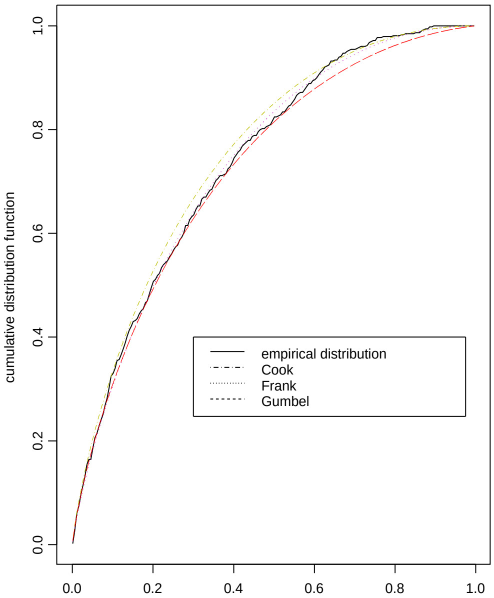

Figure 11. Graphical procedure for selecting the appropriate Archimedean copula.

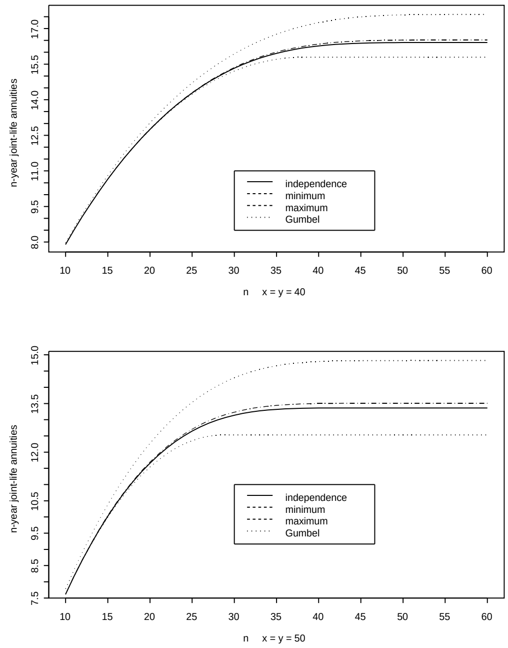

Figure 12. $a_{xy;\overline{n}|}$ as a function of n, with $\xi = 4\%$, $x = y = 40$ and 50, computed on the basis of Gumbel's model.

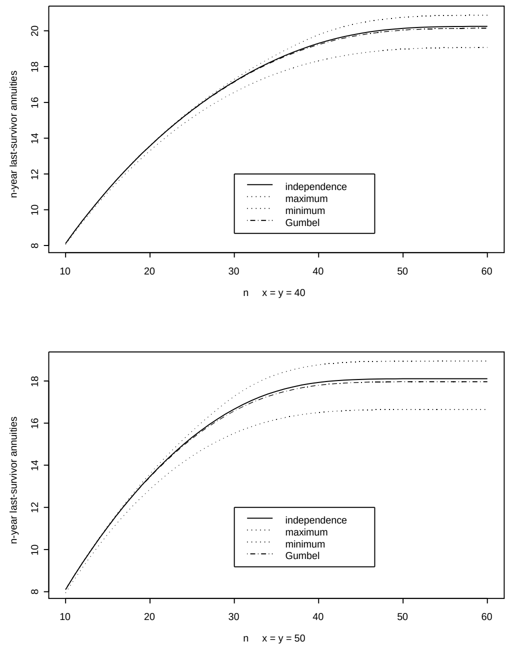

Figure 13. $a_{\overline{xy};\overline{n}|}$ as a function of n, with $\xi = 4\%$, $x = y = 40$ and 50, computed on the basis of Gumbel's model.

in Figures 12-14, where the premiums based on the independence assumption, the Fréchet bounds as well as the premiums based on Gumbel's model are presented. Because of the poor dependence exhibited by the data set, the Gumbel's premiums are very close to the tariff book premiums.

6.6 Mardia's model

In our context, the one-parameter mixture of C_L, C_I and C_U arises from the assumption that the population under interest consists of three groups: the married couples in perfect disagreement (i.e. those with a dependence structure described by C_L) in proportion π_1, those in perfect agreement (i.e. those with a dependence structure described by C_U) in proportion π_3 and those with independent time-until-death random variables in proportion π_2. Regarding these proportions as functions of a unique parameter β, we get the one-parameter copula C_β given by

$$\begin{aligned} C_\beta(u_1,u_2) &= \pi_1(\beta)C_L(u_1,u_2)+\pi_2(\beta)C_I(u_1,u_2) \\ &\quad +\pi_3(\beta)C_U(u_1,u_2), \end{aligned} \tag{32}$$

$0 \le u_1, u_2 \le 1$. The parameter β in (32) is thought of as providing some measure of association between T_x and T_y.

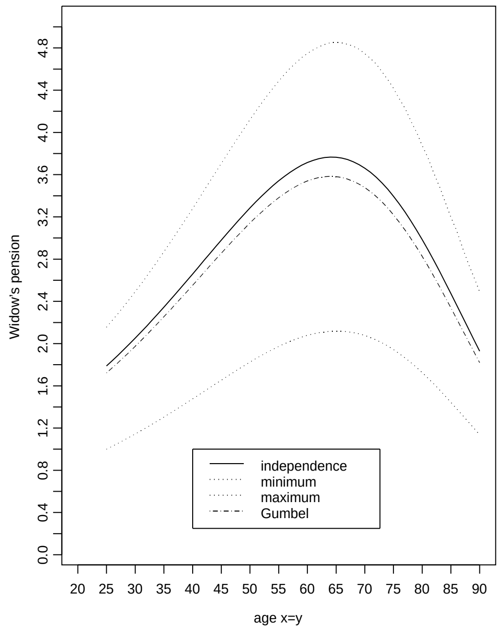

Figure 14. $a_{x|y}$ as a function of $x = y = 25, 26, \ldots, 90$, computed with $\xi = 4\%$ on the basis of Gumbel's model.

Mardia's model is obtained by considering

$$\begin{aligned} \pi_1(\beta) &= \beta^2 \frac{1-\beta}{2} \\ \pi_2(\beta) &= 1-\beta^2 \\ \pi_3(\beta) &= \beta^2 \frac{1+\beta}{2}, \end{aligned} \tag{33}$$

where $\beta \in [-1, 1]$.

We emphasize that the mixture (32) boils down to an insurance premium built as a combination of the tariff book premium (based on the independence assumption, i.e. on C_I) together with the extremal premium amounts (computed with the Fréchet bounds C_L and C_U). This particularly simple model is consequently very interesting, on the one hand because of its intuitive appeal and, on the other hand since it is easily put into insurance practice.

Because of its intuitive meaning, Mardia's model is also an interesting candidate for our data. Let us now work with the model (32)-(33). It can be shown that $\rho = \beta^3$, so that we start the iterative algorithm yielding the maximum likelihood estimation of the parameter β with

$$\hat{\rho}^{1/3} = 0,517;$$

we have obtained $\widehat{\beta} = 0.5170861$. We then have

$$\begin{aligned} {}_t\hat{q}_{\overline{xy}} &= C_{\hat{\beta}}({}_tq_x, {}_tq_y) \\ &= \hat{\pi}_1 \max(0, {}_tq_x + {}_tq_y - 1) + \hat{\pi}_2 \; {}_tq_x \; {}_tq_y \\ &\quad + \hat{\pi}_3 \min({}_tq_x, {}_tq_y) \end{aligned}$$

where $\hat{\pi}_1 = \pi_1(\hat{\beta}) = 0,065$, $\hat{\pi}_2 = \pi_2(\hat{\beta}) = 0,730$ and $\hat{\pi}_3 = \pi_3(\hat{\beta}) = 0,205$. Then, ${}_t\hat{p}_{xy}$ and ${}_t\hat{p}_{\overline{xy}}$ are deduced from (30)-(31). The premiums based on the independence assumption (dotted line), the Fréchet bounds (dotted line) as well as the premiums based on Mardia's model (continuous line) are presented in Figures 15-17. Mardia's premiums therefore consist in approximately 73% of the tariff book premium plus a mixture of the "extreme" Fréchet premiums. Again, Mardia's premiums are close to the tariff book ones because of the weak dependence in the data set.

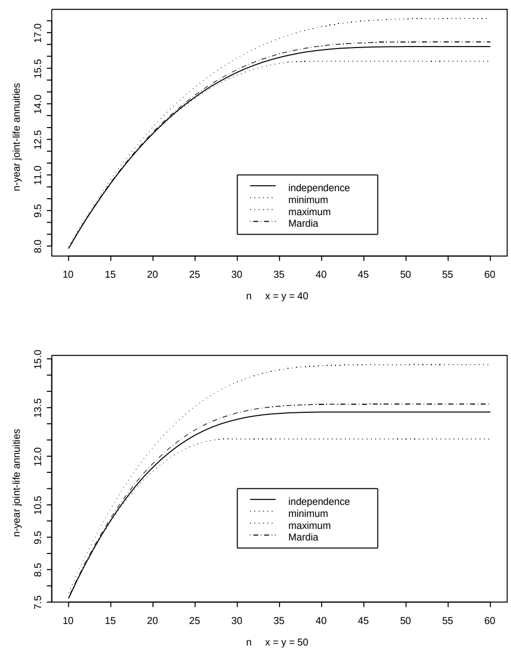

Figure 15. $a_{xy;\overline{n}|}$ as a function of n, with $\xi = 4\%$, $x = y = 40$ and 50, computed on the basis of Mardia's model.

7 Ordering lifetimes with the correlation order

The correlation order is a partial order between the joint distributions of the remaining lifetimes in $\mathcal{R}_2(F_1, F_2)$. It expresses the notion that some elements of $\mathcal{R}_2(F_1, F_2)$ are more positively correlated than others.

Definition 7.1. Let us consider two random couples $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ in $\mathcal{R}_2(F_1, F_2)$. If

$$F_{\boldsymbol{X}}(x_1, x_2) \leq F_{\boldsymbol{Y}}(x_1, x_2), \qquad \text{for all } x_1 \text{ and } x_2, \tag{34}$$

or, equivalently, if

$$\overline{F}_{\boldsymbol{X}}(x_1, x_2) \leq \overline{F}_{\boldsymbol{Y}}(x_1, x_2), \qquad \text{for all } x_1 \text{ and } x_2, \tag{35}$$

then we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the correlation order (denoted by $\boldsymbol{X} \preceq_c \boldsymbol{Y}$).

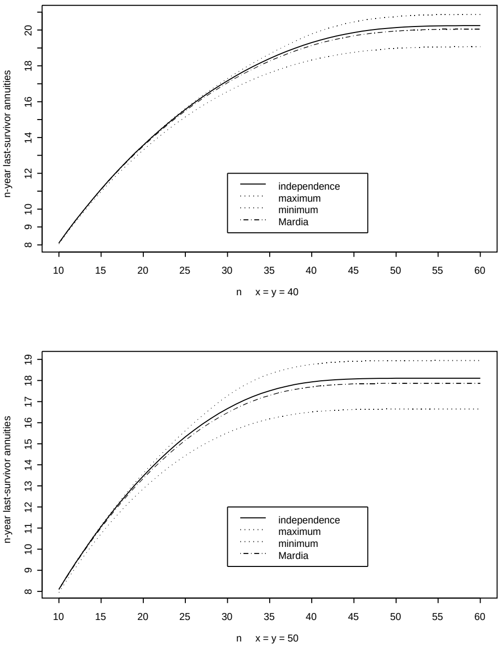

Figure 16. $a_{\overline{xy};\overline{n}|}$ as a function of n, with $\xi = 4\%$, $x = y = 40$ and 50, computed on the basis of Mardia's model.

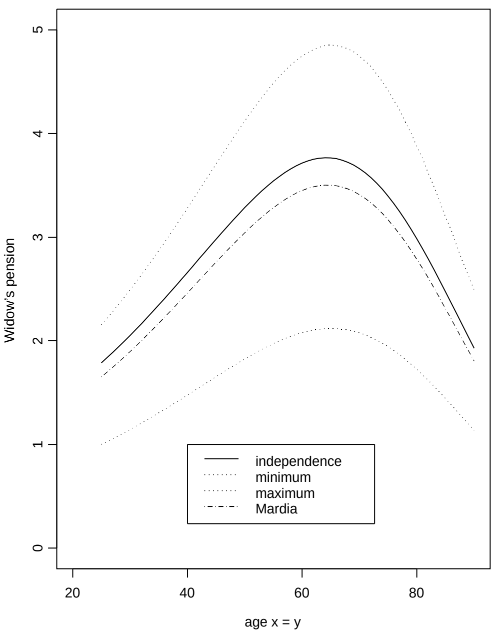

Figure 17. $a_{x|y}$ as a function of $x = y = 25, 26, \ldots, 90$, computed with $\xi = 4\%$ on the basis of Mardia's model.

From (8) it is seen that $\boldsymbol{X}$ is PQD if, and only if, $\boldsymbol{X}^{\perp} \preceq_c \boldsymbol{X}$ where $\boldsymbol{X}^{\perp}$ denotes an independent version of $\boldsymbol{X}$.

By Hoeffding's Lemma (see Lehmann, 1966, page 1139) we see that if (X_1, X_2) has distribution $F_{\boldsymbol{X}}$ in $\mathcal{R}_2(F_1, F_2)$, then

$$\text{Cov}[X_1, X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big\{F_{\boldsymbol{X}}(x_1, x_2) - F_1(x_1)F_2(x_2)\Big\}\,dx_1 dx_2$$

provided the covariance is well-defined. It thus follows from (34) that if (X_1, X_2) and (Y_1, Y_2) both belong to $\mathcal{R}_2(F_1, F_2)$, the stochastic inequality $(X_1, X_2) \preceq_c (Y_1, Y_2)$ implies

$$\text{Cov}[X_1, X_2] \le \text{Cov}[Y_1, Y_2], \tag{36}$$

and therefore, denoting as r the Pearson's correlation coefficient, since $\text{Var}[X_i] = \text{Var}[Y_i]$, $i = 1, 2$, we have that

$$r(X_1, X_2) \le r(Y_1, Y_2),$$

provided the underlying variances are well defined. It is worth mentioning that some other correlation measures, such as Kendall's τ and Spearman's ρ, are preserved under the correlation order.

If $\boldsymbol{X} \preceq_c \boldsymbol{Y}$ then the inequalities

$$\mathbb{P}[X_2 > x_2 \big| X_1 > x_1] \le \mathbb{P}[Y_2 > x_2 \big| Y_1 > x_1] \text{ for all } x_1 \text{ and } x_2,$$

and

$$\mathbb{P}[X_2 \le x_2 \big| X_1 > x_1] \ge \mathbb{P}[Y_2 \le x_2 \big| Y_1 > x_1] \text{ for all } x_1 \text{ and } x_2.$$

Thus, for all x_1 we have

$$\begin{aligned} & \mathbb{E}[X_2 \big| X_1 > x_1] \\ = \; & -\int_{-\infty}^{0} \mathbb{P}[X_2 \le x_2 \big| X_1 > x_1]\, dx_2 \\ & + \int_{0}^{\infty} \mathbb{P}[X_2 > x_2 \big| X_1 > x_1]\, dx_2 \\ \le \; & -\int_{-\infty}^{0} \mathbb{P}[Y_2 \le x_2 \big| Y_1 > x_1]\, dx_2 \\ & + \int_{0}^{\infty} \mathbb{P}[Y_2 > x_2 \big| Y_1 > x_1]\, dx_2 \\ = \; & \mathbb{E}[Y_2 \big| Y_1 > x_1]. \end{aligned}$$

For every $\boldsymbol{X} \in \mathcal{R}_2(F_1, F_2)$, the stochastic inequalities

$$\Big(F_1^{-1}(U), F_2^{-1}(1-U)\Big) \preceq_c \boldsymbol{X} \preceq_c \Big(F_1^{-1}(U), F_2^{-1}(U)\Big) \tag{37}$$

are valid, where U stands for a unit uniform random variable.

The notions of correlation order and positive quadrant dependence can easily be expressed in terms of copulas. Indeed, let (T_x, T_y) and $\Big(\widetilde{T}_x, \widetilde{T}_y\Big)$ be elements of $\mathcal{R}_2(F_1, F_2)$ with respective copula functions C and $\widetilde{C}$. Then

$$(T_x, T_y) \preceq_c \Big(\widetilde{T}_x, \widetilde{T}_y\Big)$$

is equivalent with

$$C(u,v) \le \widetilde{C}(u,v) \text{ for all } u,v \in [0,1].$$

Moreover, saying that T_x and T_y are PQD is equivalent with

$$C(u,v) \ge uv \text{ for all } u,v \in [0,1].$$

Let T be the remaining lifetime of a joint-life or last-survivor status. We will consider life insurances and annuities for which the present value of future benefits (PVFB, in short) is given by $f(T)$ with f a non-decreasing or non-increasing non-negative function. The expectation of $f(T)$ is the (pure) single premium for the insurance or annuity under consideration.

Remark that the PVFB of most of the usual joint-life and last-survivor insurances and annuities can be written as non-decreasing or non-increasing functions of the remaining life time of the joint-life or last-survivor status involved:

(i) the PVFB of pure endowments (${}_nE_{xy}$, ${}_nE_{\overline{xy}}$) and whole life annuities ($\ddot{a}_{xy}$, $\ddot{a}_{\overline{xy}}$, $\overline{a}_{xy}$, $\overline{a}_{\overline{xy}}$, ...) are non-decreasing functions of the multiple life status involved;
(ii) the PVFB of whole life insurances (A_{xy}, $A_{\overline{xy}}$, $\overline{A}_{xy}$, $\overline{A}_{\overline{xy}}$, ...) are non-increasing functions of the remaining life time of the multiple life status involved.

In the following theorem, which states our main result, we will consider two bivariate remaining lifetimes in $\mathcal{R}_2(F_1,F_2)$ which are ordered in the $\preceq_c$-sense. We will show that this implies an ordering of the corresponding multiple life premiums.

Proposition 7.2. Let (T_x,T_y) and $\left(\widetilde{T}_x,\widetilde{T}_y\right)$ be two bivariate remaining life times, both elements of $\mathcal{R}_2(F_1,F_2)$. If

$$(T_x,T_y) \preceq_c \left(\widetilde{T}_x,\widetilde{T}_y\right)$$

then the following inequalities hold for any non-decreasing function ϕ:

$$E\phi\left(\min\{T_x,T_y\}\right) \le E\phi\left(\min\left\{\widetilde{T}_x,\widetilde{T}_y\right\}\right),$$

$$E\phi\left(\max\left\{\widetilde{T}_x,\widetilde{T}_y\right\}\right) \ge E\phi\left(\max\{T_x,T_y\}\right).$$

If ϕ is non-increasing then the opposite inequalities hold.

Proof. Let us show that if

$$(T_x,T_y) \preceq_c \left(\widetilde{T}_x,\widetilde{T}_y\right)$$

then the corresponding joint-life and last-survivor statuses are ordered in the $\preceq_{st}$-sense, specifically the following stochastic order relations hold:

$$\min\{T_x,T_y\} \preceq_{st} \min\left\{\widetilde{T}_x,\widetilde{T}_y\right\},$$

and

$$\max\left\{\widetilde{T}_x,\widetilde{T}_y\right\} \preceq_{st} \max\{T_x,T_y\},$$

where $X \preceq_{st} Y$ means $\mathbb{P}[X>t] \le \mathbb{P}[Y>t]$ for all $t \in \mathbb{R}$. Since

$$\mathbb{P}\left[T_x>t, T_y>s\right] \le \mathbb{P}\left[\widetilde{T}_x>t, \widetilde{T}_y>s\right],$$

we find

$$\begin{aligned}\mathbb{P}\left[\min\{T_x,T_y\}>t\right] &= \mathbb{P}\left[T_x>t,T_y>t\right] \\ &\le \mathbb{P}\left[\widetilde{T}_x>t,\widetilde{T}_y>t\right] \\ &= \mathbb{P}\left[\min\left\{\widetilde{T}_x,\widetilde{T}_y\right\}>t\right],\end{aligned}$$

which proves the first stochastic order relation. The other relation is proven similarly. The first inequality is thus proven. The proof for the other inequality is similar. The inequalities for a non-increasing function ϕ follow immediately by remarking that $-\phi$ is non-decreasing in this case. □

Proposition 7.2 can be interpreted as follows. Assume that the marginal distributions of the remaining lifetimes T_x and T_y are given. If the bivariate remaining life time of the couple increases in the sense of the correlation order, then the single premiums of endowment insurances and annuities on the joint-life status increase, while the single premiums of endowment insurances and annuities on the last-survivor status decrease. For whole life insurances, the opposite conclusions hold.

Remark that Proposition 7.2 can also be used for ordering single premiums of more complex multiple life functions. Consider e.g. an annuity which pays one per year while both T_x and T_y are alive, and α per year while T_y is alive and T_x has died. The discounted value of the benefits involved is given by

$$\begin{aligned}&\int_0^{\min\{T_x,T_y\}} v^t\,dt + \alpha\int_{\min\{T_x,T_y\}}^{T_y} v^t\,dt \\ =\ &(1-\alpha)\int_0^{\min\{T_x,T_y\}} v^t\,dt + \alpha\int_0^{T_y} v^t\,dt.\end{aligned}$$

Under the conditions of Proposition 7.2, we find from the equality above that

$$(T_x,T_y) \preceq_c \left(\widetilde{T}_x,\widetilde{T}_y\right)$$

implies

$$a_{xy}^{(T_x,T_y)} + \alpha\, a_{x|y}^{(T_x,T_y)} \le a_{xy}^{\left(\widetilde{T}_x,\widetilde{T}_y\right)} + \alpha\, a_{x|y}^{\left(\widetilde{T}_x,\widetilde{T}_y\right)},$$

in obvious notations.

Remark 7.3. A natural measure of dependency between two random variables is the covariance. So, one could wonder whether

$$\text{Cov}\left[T_x,T_y\right] \le \text{Cov}\left[\widetilde{T}_x,\widetilde{T}_y\right]$$

is a sufficient condition for the ordering relations in Proposition 7.2 to hold. In the following example, we will show that the ordering of the covariances is not a sufficient condition.

Example 7.4. Let F be the cumulative distribution function of a remaining life time that can be equal to 1/2, 3/2 or 5/2, each with probability 1/3. Now, we consider the couples (T_x, T_y) and $\left(\widetilde{T}_x, \widetilde{T}_y\right)$, both elements of $\mathcal{R}(F, F)$. Further, we assume that T_x and T_y are mutually independent, while the dependency structure of $\left(\widetilde{T}_x, \widetilde{T}_y\right)$ is described by the following relations:

$$\begin{aligned} P\left[\widetilde{T}_y = 1/2 \mid \widetilde{T}_x = 1/2\right] &= 1, \\ P\left[\widetilde{T}_y = 3/2 \mid \widetilde{T}_x = 5/2\right] &= 1, \\ P\left[\widetilde{T}_y = 5/2 \mid \widetilde{T}_x = 3/2\right] &= 1. \end{aligned}$$

We have that $\text{Cov}\,[T_x, T_y] = 0$ and $\text{Cov}\left[\widetilde{T}_x, \widetilde{T}_y\right] = 1/3$. On the other hand, we find

$$P\left[\max\{T_x, T_y\} \le t\right] = \begin{cases} 1/9 & \text{for } t < 3/2, \\ 4/9 & \text{for } 3/2 \le t < 5/2, \\ 1 & \text{for } t \ge 5/2. \end{cases}$$

and

$$P\left[\max\left\{\widetilde{T}_x, \widetilde{T}_y\right\} \le t\right] = \begin{cases} 1/3 & \text{for } t < 5/2, \\ 1 & \text{for } t \ge 5/2. \end{cases}$$

From the distribution functions of $\max\{T_x, T_y\}$ and $\max\left\{\widetilde{T}_x, \widetilde{T}_y\right\}$ we find that

$${}_1E_{\overline{xy}}^{(T_x,T_y)} > {}_1E_{\overline{xy}}^{(\widetilde{T}_x,\widetilde{T}_y)},$$

but

$${}_2E_{\overline{xy}}^{(T_x,T_y)} < {}_2E_{\overline{xy}}^{(\widetilde{T}_x,\widetilde{T}_y)}.$$

Although it is customary to compute covariances in relation with dependency considerations, one number alone cannot reveal the nature of dependency adequately. From the example above, we see that the order induced by comparing only the covariances of (T_x, T_y) and $\left(\widetilde{T}_x, \widetilde{T}_y\right)$ will not imply a consistent ordering between the single premiums of endowment insurances on the last-survivor status. Hence, the results of Proposition 7.2 cannot be generalized in this way.

Instead of comparing

$$\text{Cov}\,[T_x, T_y] \text{ and } \text{Cov}\left[\widetilde{T}_x, \widetilde{T}_y\right]$$

one could compare

$$\text{Cov}\,[\phi_1(T_x), \phi_2(T_y)] \text{ with } \text{Cov}\left[\phi_1(\widetilde{T}_x), \phi_2(\widetilde{T}_y)\right]$$

for all non-decreasing functions ϕ_1 and ϕ_2. The order induced in this way is $\preceq_c$. As we see from Proposition 7.2, this generalization of an order based on comparing covariances implies a consistent ordering between single premiums of joint life and last-survivor annuities and insurances.

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