DOES POSITIVE DEPENDENCE BETWEEN INDIVIDUAL RISKS INCREASE STOP-LOSS PREMIUMS?

MICHEL DENUIT
Institut de Statistique
Université Catholique de Louvain
Voie du Roman Pays, 20
B-1348 Louvain-la-Neuve, Belgium
denuit@stat.ucl.ac.be

JAN DHAENE
Departement Toegepaste Economische Wetenschappen
Faculteit Economische and Toegepaste Economische Wetenschappen
Katholieke Universiteit Leuven
Minderbroedersstraat, 5
B-3000 Leuven, Belgium
Jan.Dhaene@econ.kuleuven.ac.be

CARMEN RIBAS
Dpt. de Matemàtica Econòmica, Financiera I Actuarial
Facultat de Ciènces Econòmiques I Empresarials
Universitat De Barcelona
Avda Diagonal, 690
08034 Barcelona, Spain
cribas@eco.ub.es

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Abstract

Actuaries intuitively feel that positive correlations between individual risks reveal a more dangerous situation compared to independence. The purpose of this short note is to formalize this natural idea. Specifically, it is shown that the sum of risks exhibiting a weak form of dependence known as positive cumulative dependence is larger in convex order than the corresponding sum under the theoretical independence assumption.

Key words and phrases: dependence, risk theory, convex order

1 Introduction

The study of the impact of dependence among risks has become a major topic in actuarial science nowadays; see e.g. Dhaene, Wang, Young and Goovaerts (2000), Goovaerts, Dhaene and De Schepper (2000), Kaas, Dhaene and Goovaerts (2000), Simon, Goovaerts and Dhaene (2000), Vyncke, Goovaerts and Dhaene (2000) as well as the references therein. It has been recognized that the assumption of mutual independence of risks is often violated in insurance practice. In many lines of business, the introduction of common shocks at the portfolio level is needed to represent the effects of catastrophes hitting several (or a large number of) policies simultaneously, like earthquakes, tornados, epidemics and so on. Consequently, the risks in the individual model are certainly not independent but merely depend on each other.

Several notions of positive dependence were introduced in the literature to model the fact that large values of one of the components of a multivariate risk $(X_1, X_2, \ldots, X_n)$ tend to be associated with large values of the others. Some of these concepts appear to be relevant in actuarial science. For a review, see e.g. Scarsini and Shaked (1996) or Joe (1997).

In this paper, we consider a weak form of positive dependence, known as positive cumulative dependence. This concept seems to have received little attention until now, compared to the stronger notions of association or conditional increasingness in sequence, for instance. As shown in Section 2, the positive cumulative dependence is particularly appealing for actuaries. In Section 3, we state our main result. Finally, in Section 4 an application is proposed.

Let us briefly specify some notations. Henceforth, a non-negative random variable X with a finite expectation is called a risk. Further, $\mathbb{R}$ denotes the real line $(-\infty, +\infty)$, $\mathbb{R}_+$ the half positive real line $[0, +\infty)$ and $\mathbb{N}$ the set of the non-negative integers $\{0, 1, 2, \ldots\}$. The symbol "$=_d$" means "is equally distributed as". The risks $X_1^\perp, X_2^\perp, \ldots, X_n^\perp$ represent independent versions of $X_1, X_2, \ldots, X_n$, i.e. (i) the random variables $X_1^\perp, X_2^\perp, \ldots, X_n^\perp$ are mutually independent and (ii) for any $i = 1, 2, \ldots, n$, the random variables X_i and $X_i^\perp$ are identically distributed. Furthermore, the risks $X_1^U, X_2^U, \ldots, X_n^U$ represent the comonotonic version of $X_1, X_2, \ldots, X_n$, i.e. $X_1^U = F_1^{-1}(U), X_2^U = F_2^{-1}(U), \ldots, X_n^U = F_n^{-1}(U)$ where U denotes a random variable uniformly distributed on the unit interval $[0, 1]$ and F_i^{-1} is the quantile function associated to the distribution function F_i of X_i, i.e.

$$F_i^{-1}(p) = \inf\{x \in \mathbb{R} | F_i(x) \geq p\}, \ \ 0 < p < 1.$$

Given two risks X and Y, X is said to precede Y in the stop-loss order, written as $X \preceq_{s\ell} Y$, if $E\phi(X) \leq E\phi(Y)$ holds for all the non-decreasing and convex functions ϕ for which the expectations exist. It is worth mentioning that $X \preceq_{s\ell} Y$ and $EX = EY$ if, and only if, $E\phi(X) \leq E\phi(Y)$ holds for all the convex functions ϕ for which the expectations exist.

2 Positive cumulative dependence

As far as random couples are concerned ($n = 2$), positive quadrant dependence (PQD, in short) has been extensively used in actuarial sciences, e.g. by Dhaene and Goovaerts (1996) and Denuit, Lefèvre and Mesfioui (1999). Let us recall that two risks X_1 and X_2 are said to be PQD if the inequality

$$P[X_1 > x_1, X_2 > x_2] \geq P[X_1 > x_1]P[X_2 > x_2] \tag{2.1}$$

holds for any reals $x_1, x_2 \in I\!\!R_+$. Considering (2.1), the intuitive meaning of PQD is clear: if X_1 and X_2 are PQD then the probability that they both assume "large" values is greater than if they were independent. Note that (2.1) can be cast into $P[X_1 > x_1 | X_2 > x_2] \geq P[X_1 > x_1]$ for any x_2 such that $P[X_2 > x_2] > 0$, which is also very intuitive: the knowledge that X_2 is "large" (i.e. exceeds some treshold x_2) increases the probability for X_1 to be "large". It is known from Dhaene and Goovaerts (1996, Theorem 2) that

$$X_1, X_2 \text{ are PQD } \Rightarrow X_1^{\perp} + X_2^{\perp} \preceq_{s\ell} X_1 + X_2. \tag{2.2}$$

Our aim is to extend the stochastic inequality (2.2) to the case of n risks $X_1, X_2, \ldots, X_n$. For this purpose, we need to introduce a positive dependence notion involving more than two risks.

For $\mathcal{I} \subset \{1, 2, \ldots, n\}$, let us define $S_{\mathcal{I}}$ as the sum of the X_i's whose index is in $\mathcal{I}$, i.e. $S_{\mathcal{I}} = \sum_{i \in \mathcal{I}} X_i$. The positive cumulative dependence (PCD, in short) is defined as follows: the risks $X_1, X_2, \ldots, X_n$ are PCD if for any $\mathcal{I}$ and $j \notin \mathcal{I}$, $S_{\mathcal{I}}$ and X_j are PQD. This weak form of dependence extends the bivariate PQD to arbitrary dimension and keeps the intuitive meaning of PQD: if the X_i's are PCD, the probability that $S_{\mathcal{I}}$ and X_j both assume "large" values is greater than if the X_i's were independent. In particular, the inequality

$$P\left[\sum_{i \neq j} X_i > t_1 \Big| X_j > t_2\right] \geq P\left[\sum_{i \neq j} X_i > t_1\right]$$

holds true for any $j = 1, 2, \ldots, n$ with $P[X_j > t_2] > 0$ provided all the risks $X_1, X_2, \ldots, X_n$ are PCD, whence it follows that

$$E\left[\left(\sum_{i \neq j} X_i - t_1\right)_+ \Big| X_j > t_2\right] \geq E\left(\sum_{i \neq j} X_i - t_1\right)_+ .$$

This means that the knowledge that one of the individual risks, X_j say, is large (i.e. $X_j > t_2$ for some $t_2 \in I\!\!R_+$) increases the probability that the aggregate claim produced by the $n-1$ remaining risks of the portfolio is also large, as well as the stop-loss premiums relating to them.

3 Main result

Let us now prove the following result which enhances the interest of PCD in the study of dependent risks. More precisely, we provide hereafter a multivariate generalization of (2.2).

Theorem 3.1. *Let us consider PCD risks $X_1, X_2, \ldots, X_n$ with marginal distribution functions $F_1, F_2, \ldots, F_n$. Then, we have*

$$X_1^{\perp} + X_2^{\perp} + \ldots + X_n^{\perp} \preceq_{s\ell} X_1 + X_2 + \ldots + X_n \preceq_{s\ell} X_1^U + X_2^U + \ldots + X_n^U .$$

Proof. The second stop-loss inequality is true in general, for risks $X_1, X_2, \ldots, X_n$ with distribution function $F_1, F_2, \ldots, F_n$; see, e.g., Dhaene, Wang, Young and Goovaerts (2000). Let us prove the first stop-loss inequality. Without loss of generality, the random vectors $(X_1^\perp, X_2^\perp, \ldots, X_n^\perp)$ and $(X_1, X_2, \ldots, X_n)$ may be considered independent. Now, proceed by induction. First, $X_1^\perp \preceq_{s\ell} X_1$ trivially holds. Now, assume that

$$X_1^\perp + X_2^\perp + \cdots + X_k^\perp \preceq_{s\ell} X_1 + X_2 + \cdots + X_k$$

holds true for $k = 1, 2, \cdots, n-1$. Then, by the closure of $\preceq_{s\ell}$ under convolution, the latter stochastic inequality yields

$$X_1^\perp + X_2^\perp + \cdots + X_{n-1}^\perp + X_n^\perp \preceq_{s\ell} X_1 + X_2 + \cdots + X_{n-1} + X_n^\perp. \tag{3.1}$$

Now, since the X_i's are PCD, X_n and $X_1 + X_2 + \cdots + X_{n-1}$ are PQD, and we get

$$X_1 + X_2 + \cdots + X_{n-1} + X_n^\perp \preceq_{s\ell} X_1 + X_2 + \cdots + X_{n-1} + X_n. \tag{3.2}$$

Combining (3.1) and (3.2) yields the announced result by the transitivity property of $\preceq_{s\ell}$. □

Note that the conclusions of Theorem 3.1 *a fortiori* hold when the X_i's are associated, linear positive quadrant dependent or conditionally increasing in sequence as these positive dependence notions imply PCD (see Joe (1997) for further details about these concepts). Therefore, Theorem 3.1 can be applied in many situations. As a few examples, let us mention the class of counting distributions introduced by Ambagaspityia (1998) or the models recently defined by Cossette and Marceau (2000); for more details, see Denuit, Dhaene and Ribas (1999).

From the above result, once the marginal distributions of the X_i's are fixed, the best lower and upper bounds in the $\preceq_{s\ell}$-sense on the aggregate claims $X_1 + X_2 + \ldots + X_n$ of PCD risks are provided by $X_1^\perp + X_2^\perp + \ldots + X_n^\perp$ and $X_1^U + X_2^U + \ldots + X_n^U$, respectively. Therefore, any risk-averse decision-maker will prefer $X_1^\perp + X_2^\perp + \ldots + X_n^\perp$ over $X_1 + X_2 + \ldots + X_n$ when the risks $X_1, X_2, \cdots, X_n$ are PCD. This conclusion holds both in Von Neumann and Morgenstern expected utility theory, as well as in Yaari's dual theory of choice under risk. It also follows from Theorem 3.1 that making the assumption of mutual independence for PCD risks $X_1, X_2, \ldots, X_n$ leads to an underestimation of the stop-loss premiums.

For PCD risks, the safest dependence structure is provided by mutual independence, for fixed marginals. When the risks are not known to be PCD, the safest dependence structure does not always exist; see Dhaene and Denuit (1999) for more details.

4 An application to premium calculation principles

Let us consider a premium calculation principle $H[.]$, that assigns a premium amount $H[X]$ to any risk X. We assume that the distribution function of X completely determines the premium for X. Assume further that $H[.]$ preserves the stop-loss order, i.e. given two risks X and Y,

$$X \preceq_{s\ell} Y \Rightarrow H[X] \leq H[Y].$$

Consider PCD risks $X_1, X_2, \cdots, X_n$. The stop-loss preserving property together with Theorem 3.1 yields

$$H\left[\sum_{i=1}^{n} X_i^{\perp}\right] \leq H\left[\sum_{i=1}^{n} X_i\right] \leq H\left[\sum_{i=1}^{n} X_i^{U}\right]. \tag{4.1}$$

The inequality above states that for a stop-loss preserving premium principle, the premium of a sum of PCD risks is maximal if the risks are comonotonic and minimal if the risks are mutually independent. We remark that the second inequality holds in general for all risks $X_1, X_2, \ldots, X_n$ (not necessarily PCD); see e.g. Wang and Dhaene (1998). From (4.1), we find that if a premium principle preserves stop-loss order and is additive for independent risks, then it is super-additive for PCD risks. This result is a generalization of the bivariate case considered in Wang and Dhaene (1998).

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