The valuation of cash-flows in the presence of dividend barriers

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Abstract

A subject often recurring in financial and actuarial papers is the pricing of stocks and securities when the rate of return is stochastic. In most cases, the stocks considered are assumed not to pay out any dividend. In the present contribution we show how it is possible to obtain upper and lower bounds for the (distribution of the) accumulated value of a cash-flow in the presence of dividend barriers at a future time t, when the logarithm of the stock price is modelled by means of a Wiener process.

1 Description of the problem

For $t \geq 0$, let $S(t)$ denote the price of a non-dividend paying stock or security at time t. We then have

$$S(t) = S_0 e^{X(t)}, \tag{1}$$

assuming that there exists a stochastic process $X(t)$ with stationary and independent increments, representing the stochastic continuous compounded rate of return over the period $[0, t]$. In the classical model, stock prices are assumed to be log-normally distributed, and the process $X(t)$ is a Wiener process.

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If we look at the total period $[0,t]$ as a number of subperiods, say years, months, weeks etc., it is useful to write the value of $X(t)$ at time t by means of the increments per period until time t. In that case the price of the security can be rewritten as

$$S(t) = S_0 \exp\left[\sum_{i=1}^{n} [X(t_i) - X(t_{i-1})]\right] , \tag{2}$$

with $0 = t_0 < t_1 < \ldots < t_n = t$ and $X(0) = 0$.

In the present contribution, we will assume that a dividend is paid out whenever the increment of the rate of return in one of the periods exceeds a certain value β. We will generalize the basic variable $S(t)$ of (2) to the accumulated value of a stochastic cash-flow

$$V = \sum_{j=1}^{n} \alpha_j e^{Y_j} , \tag{3}$$

with

$$Y_j = \sum_{i=j+1}^{n} \min\left[X(t_i) - X(t_{i-1}), \beta\right] , \tag{4}$$

and we will look for as much information as possible about the distribution. The positive value α_j $(j = 1, \ldots , n)$ in (3) represents the deterministic cash-flow at time t_j, and e^{Y_j} $(j = 1, \ldots , n)$ is the stochastic accumulation factor for a payment made at time t_j.

In order to solve this problem, we will use some rather new results concerning the distribution of sums of variables. Looking at the variable V above, we see that this variable can be written as

$$V = \sum_{j=1}^{n} \phi_j(Y_j). \tag{5}$$

The variable Y_j is used to denote the real (compounded) rate of return over the period $[t_j, t]$, and the real functions ϕ_j are convex increasing functions, for the present problem mainly exponential.

The main body of this contribution is divided into two parts. In section 2, we will explain the methodology that is used in getting the desired answers. In order to make this paper self-contained, we repeat all the main results. Afterwards, we will apply these techniques to the problem at hand in section 3.

2 Methodology

In case the distributions of the random variables Y_j in (5) are known, the problem of finding a distribution function for random variables of the form of (5) looks rather trivial. This, however, is not true.

The most important difficulty arises from the fact that the random variables Y_j are not mutually independent. So, a "simple" convolution of the different individual distribution functions is not correct, since also the dependency structure of the random vector $(Y_1, \ldots, Y_n)$ has to be taken into account. And this, unfortunately, is almost impossible to obtain in most cases.

Therefore, instead of calculating the exact distribution of the variable V, we will look for bounds, in the sense of "more favourable/less dangerous" and "less favourable/more dangerous", with a simpler structure. This technique is rather common in the actuarial literature. When lower and upper bounds are close to each other, together they can provide reliable information about the original and more complex variable V.

We will briefly repeat the meaning and most important results of this technique, presenting it in a form that is useful when handling the problem of the distribution of the variable V of (3). For proofs and more details, we refer to the recent literature.

2.1 Convex ordering

The notion "less favourable" or "more dangerous" variable will be defined by means of the convex ordering, see e.g. [4]:

Definition 2.1 *If two variables V and W are such that for each real convex function $u(\cdot)$ the expected values (provided they exist) are ordered as*

$$E\left[u(V)\right] \leq E\left[u(W)\right], \tag{6}$$

the variable V is said to be smaller in convex ordering than the variable W, which is denoted as

$$V \leq_{cx} W. \tag{7}$$

Since convex functions are functions that take on their largest values in the tail(s), this means that the variable W is more likely to take on extreme values than the variable V, and thus it is more dangerous.

Condition (6) on the expectations can be rewritten as

$$E\left[u(-V)\right] \geq E\left[u(-W)\right] \tag{8}$$

for arbitrary concave utility functions $u(\cdot)$. Thus, for any risk averse decision maker, the expected utility of the loss W is smaller than the expected utility of the loss V. This means that replacing the unknown distribution function of the variable V by the distribution function of the variable W is a prudent stategy.

Since the functions $u(x) = x$, $u(x) = -x$ and $u(x) = x^2$ are all convex functions, it follows immediately that $V \leq_{cx} W$ implies $E[V] = E[W]$ and $Var[V] \leq Var[W]$.

The following lemma provides an interesting and useful characterization of convex order, a proof of which can be found in [4]:

Lemma 2.2 *If two variables V and W are such that $E[V] = E[W]$, then*

$$V \leq_{cx} W \Leftrightarrow E[(V-k)_+] \leq E[(W-k)_+] \text{ for all } k, \tag{9}$$

with $(x)_+ = max(0, x)$.

The expectation $E[(V-k)_+]$ is called the stop-loss premium for the variable V. Since more dangerous risks will correspond to higher stop-loss premiums, again it can be seen that the notion of convex order is very adequate to describe an ordering in dangerousness. If all stop-loss premiums of a variable V are smaller than (or equal to) those of W, then V is said to be smaller in stop-loss ordering, denoted by $V \leq_{s\ell} W$.

2.2 Application to sums of variables

In some former contributions, see e.g. [1, 2, 3], the notion of convex ordering of two single variables was expanded to two sums of variables. We will summarize the main results in the following propositions, a proof of which can be found in [1, 2, 3].

We will make use of the notation

$$F_X(x) = Prob(X \leq x) \tag{10}$$

for the distribution of a random variable X, where $x \in \mathbb{R}$, and of

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\} \tag{11}$$

for the inverse distribution of X, where $p \in [0,1]$.

Proposition 2.3 *Consider an arbitrary sum of random variables*

$$V = X_1 + X_2 + \ldots + X_n, \tag{12}$$

and define the related stochastic quantities

$$V_u = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U) \tag{13}$$

$$V_{u*} = F_{X_1|Z}^{-1}(U) + F_{X_2|Z}^{-1}(U) + \ldots + F_{X_n|Z}^{-1}(U) \tag{14}$$

$$V_l = E[X_1|Z] + E[X_2|Z] + \ldots + E[X_n|Z] \ , \tag{15}$$

with U an arbitrary random variable that is uniformly distributed on $[0,1]$, and with Z an arbitrary random variable that is independent of U. The notation $F_{X_i|Z}^{-1}(U)$ is used for the random variable $f_i(U,Z)$, where the function is defined by $f_i(u,z) = F_{X_i|Z=z}^{-1}(u)$.

The following relation then holds:

$$V_l \leq_{cx} V \leq_{cx} V_{u*} \leq_{cx} V_u. \tag{16}$$

For each $j = 1, \ldots, n$, the terms in the original variable V and the corresponding terms in the upper bounds V_u and V_{u*} are all identically distributed, i.e.

$$X_j \overset{d}{=} F_{X_j}^{-1}(U) \overset{d}{=} F_{X_j|Z}^{-1}(U) \ , \tag{17}$$

see [2, 3].

The upper bound V_u in fact is constructed as the most dangerous combination of variables with the same marginal distributions as the original terms X_j. Indeed, the sum now consists of a sum of comonotonous variables all depending on the same random variable U, and thus they are not usable as hedges against each other. The upper bound V_{u*} is an improved bound, which is closer to V due to the extra information through conditioning. The lower bound becomes ‘better’ in the sense of closer to the original variable V when the conditioning variable Z is more related to the sum of the variables X_j.

For more complicated problems, the extension of the previous results for ‘ordinary’ sums of random variables to sums of functions of variables turns out to be very useful. A proof of this second proposition can be found in [2, 3].

Proposition 2.4 *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + \phi_2(X_2) + \ldots + \phi_n(X_n), \tag{18}$$

where each function $\phi_j(\cdot)$ is increasing. Define the related stochastic quantities

$$\begin{aligned} V_u &= \phi_1(F^{-1}_{X_1}(U)) + \phi_2(F^{-1}_{X_2}(U)) + \ldots + \phi_n(F^{-1}_{X_n}(U)) && (19) \\ V_{u*} &= \phi_1(F^{-1}_{X_1|Z}(U)) + \phi_2(F^{-1}_{X_2|Z}(U)) + \ldots + \phi_n(F^{-1}_{X_n|Z}(U)) && (20) \\ V_l &= E[\phi_1(X_1)|Z] + E[\phi_2(X_2)|Z] + \ldots + E[\phi_n(X_n)|Z] \ , && (21) \end{aligned}$$

with U an arbitrary random variable that is uniformly distributed on $[0,1]$, and with Z an arbitrary random variable that is independent of U.

The following relation then holds :

$$V_l \leq_{cx} V \leq_{cx} V_{u*} \leq_{cx} V_u. \tag{22}$$

This result is mainly based on the property that for any increasing function ϕ and for any $p \in [0,1]$ we have that

$$F^{-1}_{\phi(X)}(p) = \phi(F^{-1}_X(p)). \tag{23}$$

In the following section, which constitutes the “core” of this contribution, we will rely on this last proposition. The advantage of using this proposition has to be found in the fact that the knowledge of the (inverse of the) distribution functions of the variables X_j and of the conditional distribution functions of the variables X_j given Z provides us with all the necessary ingredients that are needed in order to calculate upper and lower bounds for the original variable V. Furthermore, the presence of a uniform distributed variable U simplifies the computations.

We will use the classical notations

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{24}$$

for the density of the standard normal distribution, and

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y) dy \tag{25}$$

for the probability integral or cumulative values of the standard normal distribution.

3 Calculation of upper and lower bounds

We now return to the main problem of this contribution, the accumulated value of a stochastic cash-flow

$$V = \sum_{j=1}^{n} \alpha_j \exp \left\{ \sum_{i=j+1}^{n} \min \left[X(t_i) - X(t_{i-1}), \beta \right] \right\}, \tag{26}$$

where all cash-flows α_j $(j = 1, \ldots, n)$ are non-negative. For the points in time we assume that $0 = t_0 < t_1 < t_2 < \ldots < t_n = t$ with $t_j - t_{j-1} = \Delta = t/n$ $(j = 1, \ldots, n)$, which corresponds to dividing the interval $[0, t]$ into n years, months, weeks etc.

In order to model the stochastic interest rates, we will use processes with stationary and independent increments – as was mentioned in the introduction – and more specifically a Wiener process.

To start with, we will give expressions for the distributions of the variables 'X_j' (unconditional and conditional given 'Z') of the previous section, followed by results for bounds for V based on Proposition 2.4. Afterwards we will combine both results in order to get explicit formulas for the stop-loss premiums and distributions of these bounds.

3.1 Intermediate distributions

The first lemma recapitulates some well-known results for the Wiener process.

Lemma 3.1 *Consider the process* $\{X(\theta)\}$*, assumed to be a Wiener process with mean* μ *and variance* σ^2 *per unit time. Then the distribution of* $X(\theta)$ *is normal with mean* $\mu\theta$ *and variance* $\sigma^2\theta$*, or*

$$F(x,\theta) = Prob[X(\theta) \leq x] = \Phi\left(\frac{x-\mu\theta}{\sigma\sqrt{\theta}}\right) \tag{27}$$

and

$$f(x,\theta) = \frac{d}{dx}F(x,\theta) = \frac{1}{\sigma\sqrt{\theta}}\varphi\left(\frac{x-\mu\theta}{\sigma\sqrt{\theta}}\right). \tag{28}$$

Since Wiener processes have stationary and independent increments, we can rewrite these results for the distribution of the increments:

Lemma 3.2 *Consider the process* $\{X(\theta)\}$*, assumed to be a Wiener process with mean* μ *and variance* σ^2 *per unit time. With the point of time assumptions as made earlier, the distribution of* $X(t_j)-X(t_{j-1})$ *is normal with mean* $\mu\Delta$ *and variance* $\sigma^2\Delta$*, i.e.*

$$\tilde{F}(x) = Prob[X(t_j) - X(t_{j-1}) \leq x] = F(x,\Delta) = \Phi\left(\frac{x-\mu\Delta}{\sigma\sqrt{\Delta}}\right) \tag{29}$$

and

$$\tilde{f}(x) = \frac{d}{dx}\tilde{F}(x) = f(x,\Delta) = \frac{1}{\sigma\sqrt{\Delta}}\varphi\left(\frac{x-\mu\Delta}{\sigma\sqrt{\Delta}}\right). \tag{30}$$

For the conditional distributions – which is less trivial – we first have to choose the variable Z. Since the results become better as this variable Z is closer to the sum of the original variables (especially for the lower bound), we choose Z to be equal to $X(t)$. This Z indeed seems to be a good choice, because it can be written as

$$Z = X(t) = X(t_n) = \sum_{j=1}^{n}\left[X(t_j) - X(t_{j-1})\right]. \tag{31}$$

Lemma 3.3 *Consider the process* $\{X(\theta)\}$*, assumed to be a Wiener process with mean* μ *and variance* σ^2 *per unit time. We denote the conditional distribution of* $X(t_j) - X(t_{j-1})$ *as*

$$\tilde{F}_{CO}(x|X(t)) = Prob[X(t_j) - X(t_{j-1}) \leq x|X(t)]. \tag{32}$$

For any realization $X(t) = c$*, this conditional distribution is normal with mean* c/n *and variance* $\sigma^2 \frac{n-1}{n}\Delta$*. We have*

$$\begin{aligned}\tilde{F}_c(x) &= Prob[X(t_j) - X(t_{j-1}) \leq x|X(t) = c] \\ &= \Phi\left(\frac{1}{\sigma\sqrt{\frac{n-1}{n}\Delta}}\left(x - \frac{c}{n}\right)\right)\end{aligned} \tag{33}$$

and

$$\begin{aligned}\tilde{f}_c(x) &= \frac{d}{dx}\tilde{F}_c(x) \\ &= \frac{f(x,\Delta).f(c-x,(n-1)\Delta)}{f(c,n\Delta)} \\ &= \frac{1}{\sigma\sqrt{\frac{n-1}{n}\Delta}}\varphi\left(\frac{1}{\sigma\sqrt{\frac{n-1}{n}\Delta}}\left(x - \frac{c}{n}\right)\right).\end{aligned} \tag{34}$$

Proof. By Bayes' rule, we have

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}, \tag{35}$$

which we apply to the variables $X = X(t_j) - X(t_{j-1})$ and $Y = X(t_n)$. Due to the assumptions about the points of time and the stationary and independent increments, this results in

$$\begin{aligned}\tilde{f}_c(x) &= \frac{d}{dx}Prob[X(t_j) - X(t_{j-1}) \leq x|X(t_n) = c] \\ &= \frac{f(x,\Delta) \cdot f(c-x,(n-1)\Delta)}{f(c,n\Delta)},\end{aligned} \tag{36}$$

where we used the notations of Lemma 3.1.

The difficulty with this first and 'intermediate' result of the lemma consists in the fact that the argument x appears twice in the right-hand side. Fortunately, we can work out this right-hand side into a form that explicitly contains the x only once. Indeed, using the results mentioned in Lemma 3.1 and combining the exponential functions in the normal densities, we have

$$\tilde{f}_c(x) = \frac{\sqrt{2\pi\sigma^2 n\Delta}}{\sqrt{2\pi\sigma^2\Delta}\sqrt{2\pi\sigma^2(n-1)\Delta}} \cdot \exp\left\{-\frac{(x-\mu\Delta)^2}{2\sigma^2\Delta} - \frac{(c-x-\mu(n-1)\Delta)^2}{2\sigma^2(n-1)\Delta} + \frac{(c-\mu n\Delta)^2}{2\sigma^2 n\Delta}\right\}; \tag{37}$$

after some rearrangements we get

$$\tilde{f}_c(x) = \frac{1}{\sqrt{2\pi\sigma^2\frac{n-1}{n}\Delta}} \cdot \exp\left\{-\frac{(x-\frac{c}{n})^2}{2\sigma^2\frac{n-1}{n}\Delta}\right\}, \tag{38}$$

which completes the proof.

Q.E.D.

The same techniques lead us to the conditional distribution of $X(t_j)$.

Lemma 3.4 *Consider the process $\{X(\theta)\}$, assumed to be a Wiener process with mean μ and variance σ^2 per unit time. We denote the conditional distribution of $X(t_j)$ as*

$$F_{CO}(x, t_j|X(t)) = Prob[X(t_j) \leq x|X(t)]. \tag{39}$$

For any realization $X(t) = c$, this conditional distribution is normal with mean jc/n and variance $\sigma^2\frac{j(n-j)}{n}\Delta$. We have

$$\begin{aligned} F_c(x, t_j) &= Prob[X(t_j) \leq x|X(t) = c] \\ &= \Phi\left(\frac{1}{\sigma\sqrt{\frac{j(n-j)}{n}\Delta}}\left(x - \frac{jc}{n}\right)\right) \end{aligned} \tag{40}$$

and

$$\begin{aligned} f_c(x, t_j) &= \frac{d}{dx}F_c(x, t_j) \\ &= \frac{1}{\sigma\sqrt{\frac{j(n-j)}{n}\Delta}}\varphi\left(\frac{1}{\sigma\sqrt{\frac{j(n-j)}{n}\Delta}}\left(x - \frac{jc}{n}\right)\right). \end{aligned} \tag{41}$$

Proof. We can repeat the proof of the previous lemma, where in (35) we now choose $X = X(t_j)$ and $Y = X(t_n)$. This results in

$$\begin{aligned} f_c(x, t_j) &= \frac{d}{dx} Prob[X(t_j) \leq x | X(t_n) = c] \\ &= \frac{f(x, j\Delta) \cdot f(c - x, (n - j)\Delta)}{f(c, n\Delta)} . \end{aligned} \tag{42}$$

Combining the exponential functions in the normal densities and rearranging terms, yields the desired result. Q.E.D.

3.2 Bounds

With all the necessary distribution functions at hand, different convex upper and lower bounds for the accumulated value

$$V = \sum_{j=1}^{n} \alpha_j \cdot \exp\left\{ \sum_{i=j+1}^{n} \min\left[X(t_i) - X(t_{i-1}), \beta\right] \right\} \tag{43}$$

can be found in a straightforward way, taking into account

$$Prob\left(\sum_{j=1}^{n} \min\left[X_j, \beta\right] \leq \min\left[\sum_{j=1}^{n} X_j, n\beta \right] \right) = 1. \tag{44}$$

Proposition 3.5 *Define the following four stochastic quantities*

$$V_{u1} = \sum_{j=1}^{n} \alpha_j \cdot e^{\min\left[X(t) - X(t_j), (n - j)\beta\right]} \tag{45}$$

$$V_{u2} = \sum_{j=1}^{n} \alpha_j \cdot e^{\min\left[F^{-1}(U, (n - j)\Delta), (n - j)\beta\right]} \tag{46}$$

$$V_{u3} = \sum_{j=1}^{n} \alpha_j \cdot e^{\min\left[F_{CO}^{-1}(U, (n - j)\Delta | X(t)), (n - j)\beta\right]} \tag{47}$$

$$V_l = \alpha_n + \sum_{j=1}^{n-1} \alpha_j M^{n-j-1}(\beta, \Delta) \cdot e^{\min\left[F^{-1}(U, \Delta), \beta\right]} , \tag{48}$$

with U *an arbitrary random variable that is uniformly distributed on* $[0,1]$*, and with*

$$\begin{aligned} M(\beta,\Delta) &= E\left[e^{\min\,[X(t_1),\,\beta]}\right] \\ &= \int_{-\infty}^{+\infty} dy\; f(y,\Delta)\; e^{\min\,[y,\,\beta]}. \\ &= e^{(\mu+\frac{\sigma^2}{2})\Delta}\;\Phi\left(\frac{\beta-(\mu+\sigma^2)\Delta}{\sigma\sqrt{\Delta}}\right)+e^{\beta}\left[1-\Phi\left(\frac{\beta-\mu\Delta}{\sigma\sqrt{\Delta}}\right)\right] \end{aligned} \tag{49}$$

The following relation then holds :

$$V_l \leq_{cx} V \leq_{s\ell} V_{u1} \leq_{cx} V_{u3} \leq_{cx} V_{u2}. \tag{50}$$

Proof. (a). The first upper bound immediately follows from stochastic dominance criteria, when (44) is applied to (43).

(b). For the upper bounds V_{u2} and V_{u3}, use has been made of Proposition 2.4 with $Z = X(t)$, starting from the result for V_{u1}.

(c). In order to find the lower bound V_l, we start by applying Proposition 2.4 with $Z = Z(t_n)$, where $Z(t_i) = X(t_i) - X(t_{i-1})$. We get

$$V_l = \sum_{j=1}^{n} \alpha_j \cdot E\left[\exp\left\{\sum_{i=j+1}^{n} \min\,[Z(t_i),\beta]\right\}\middle| Z(t_n)\right] \tag{51}$$

$$= \alpha_n + \sum_{j=1}^{n-1} \alpha_j \cdot e^{\min\,[Z(t_n),\,\beta]} \tag{52}$$

Each of the remaining variables is normally distributed (see Lemma 3.2). Therefore, and because of the independency, the expected values in the product all equal $M(\beta,\Delta)$.

Taking $U = \Phi\left(\dfrac{Z(t_n)-\mu\Delta}{\sigma\sqrt{\Delta}}\right) = F(Z(t_n),\Delta)$, so U has a uniform distribution on $[0,1]$, the exponent in the first factor can be written as

$$\min\,[Z(t_n),\beta] = \min\left[F^{-1}(U,\Delta),\beta\right], \tag{53}$$

which gives the desired result.

Q.E.D.

In the next two subsections, explicit calculations for V_{u1} will be omitted, since this variable does not have the required “simple” structure.

3.3 Stop-loss premiums

We now construct formulas for the stop-loss premiums of the different variables of Proposition 3.5. As mentioned earlier, the stop-loss premium for V_a with retention k is defined as the expectation

$$E\left[(V_a - k)_+\right] . \tag{54}$$

The following proposition summarizes the different results for the stop-loss premiums of the boundary variables:

Proposition 3.6 *Consider the stochastic quantities V_{u2}, V_{u3} and V_l as mentioned in Proposition 3.5. The stop-loss premiums for these variables can be calculated as*

$$\begin{aligned} & E\left[(V_{u2} - k)_+\right] && (55) \\ & = \int_{u_k}^{1} du \, \left(\sum_{j=1}^{n} \alpha_j e^{\min\left[(n-j)\mu\Delta + \sigma\sqrt{(n-j)\Delta}\, \Phi^{-1}(u), (n-j)\beta\right]} \; - \; k \right) , && \end{aligned}$$

$$\begin{aligned} & E\left[(V_{u3} - k)_+\right] && (56) \\ & = \int_{-\infty}^{+\infty} dc \; f(c,t) \int_{u_k(c)}^{1} du && \\ & \left(\sum_{j=1}^{n} \alpha_j e^{\min\left[\frac{(n-j)c}{n} + \sigma\sqrt{\frac{j(n-j)}{n}\Delta}\, \Phi^{-1}(u), (n-j)\beta\right]} \; - \; k \right) , && \end{aligned}$$

$$\begin{aligned} & E\left[(V_l - k)_+\right] && (57) \\ & = \int_{v_k}^{1} du \, \left(\alpha_n + \sum_{j=1}^{n-1} \alpha_j M^{\,n-j-1}(\beta, \Delta) \cdot e^{\min\left[\mu\Delta + \sigma\sqrt{\Delta}\, \Phi^{-1}(u), \beta\right]} \; - \; k \right) . && \\ & && (58) \end{aligned}$$

For each value of k, the numbers u_k, v_k and the function $u_k(c)$ in the stop-loss premiums (55)-(57) *are defined implicitly through the equations*

$$\sum_{j=1}^{n} \alpha_j \exp\left\{ \min\left[(n-j)\mu\Delta + \sigma\sqrt{(n-j)\Delta}\, \Phi^{-1}(u_k), (n-j)\beta\right]\right\} = k \; , \tag{59}$$

$$\sum_{j=1}^{n} \alpha_j \exp\left\{\min\left[\frac{(n-j)c}{n} + \sigma\sqrt{\frac{j(n-j)}{n}\Delta}\,\Phi^{-1}(u_k(c)), (n-j)\beta\right]\right\} = k\ , \tag{60}$$

and

$$\alpha_n + \sum_{j=1}^{n-1} \alpha_j M^{\,n-j-1}(\beta,\Delta) \exp\left\{\min\left[\mu\Delta + \sigma\sqrt{\Delta}\,\Phi^{-1}(v_k), \beta\right]\right\} = k\ . \tag{61}$$

Proof.
(a). Making use of the second upper bound in Proposition 3.5, we can write

$$\begin{aligned} &E\left[(V_{u2}-k)_+\right] \\ &= E_U\left[\left(\sum_{j=1}^{n} \alpha_j \exp\left\{\min\left[F^{-1}(U,(n-j)\Delta),(n-j)\beta\right]\right\}\ -\ k\right)_+\right] \end{aligned} \tag{62}$$

The inverse of $F(x,\theta)$ as given in Lemma 3.1 equals

$$F^{-1}(p,\theta) = \mu\theta + \sigma\sqrt{\theta}\,\Phi^{-1}(p) \qquad (p \in [0,1]) \tag{63}$$

and thus

$$\begin{aligned} &E\left[(V_{u2}-k)_+\right] \\ &= \int_0^1 du \left(\sum_{j=1}^{n} \alpha_j e^{\min\left[(n-j)\mu\Delta+\sigma\sqrt{(n-j)\Delta}\,\Phi^{-1}(u),(n-j)\beta\right]}\ -\ k\right)_+ . \end{aligned} \tag{64}$$

Defining u_k as in (59), we get the result of (55).

(b). The improved upper bound in Proposition 3.5 leads to

$$\begin{aligned} &E\left[(V_{u3}-k)_+\right] \\ &= E_{X(t)}\ E_U\left[\left(\sum_{j=1}^{n} \alpha_j e^{\min\left[F_{CO}^{-1}(U,(n-j)\Delta|X(t)),(n-j)\beta\right]}\ -\ k\right)_+\right] . \end{aligned} \tag{65}$$

We now need the inverse of the function $F_c(x,t_j)$ from Lemma 3.4. This inverse function equals

$$F_c^{-1}(p,t_j) = \frac{jc}{n} + \sigma\sqrt{\frac{j(n-j)}{n}\Delta}\,\Phi^{-1}(p) \qquad (p \in [0,1]) \tag{66}$$

and thus

$$\begin{aligned} E\left[(V_{u3}-k)_+\right] &= \int_{-\infty}^{+\infty} dc\, f(c,t) \int_0^1 du \qquad (67) \\ & \left(\sum_{j=1}^{n} \alpha_j e^{\min\left[\frac{(n-j)c}{n}+\sigma\sqrt{\frac{j(n-j)}{n}\Delta}\,\Phi^{-1}(u),(n-j)\beta\right]} - k \right)_+ . \end{aligned}$$

Defining the function $u_k(c)$ as in (60), we get the result of (56).

(c). For the lower bound, the same arguments as for the first upper bound can be used, resulting in the stop-loss premium above.

Q.E.D.

For these stop-loss premiums, the obvious ordering holds, as is summarized in the following proposition.

Proposition 3.7 *Consider the stop-loss premiums for the boundary values as mentioned in Proposition 3.6, and the stop-loss premium for the original risk* $E\left[(V-k)_+\right]$*. The following relation then holds for each value of* k*:*

$$E\left[(V_l-k)_+\right] \le E\left[(V-k)_+\right] \le E\left[(V_{u3}-k)_+\right] \le E\left[(V_{u2}-k)_+\right]. \qquad (68)$$

Proof. This immediately follows from Proposition 3.5 and Lemma 2.2.

Q.E.D.

3.4 Distributions of the bounds

The importance of the stop-loss premiums of Proposition 3.6 is not only the result of the fact that they give upper and lower bounds for the stop-loss premium of the original variable V. As it happens, they are also very useful when looking for expressions for the distribution functions for the upper and lower bounds, due to the following lemma:

Lemma 3.8 *Consider an arbitrary variable V_a with distribution function*

$$F_a(k) = Prob[V_a \leq k] \ . \tag{69}$$

Provided the expectations exist, the relation between the stop-loss premiums and the distribution function is given by

$$\frac{d}{dk} E\left[(V_a - k)_+\right] = F_a(k) - 1 \ . \tag{70}$$

Due to this most useful property, we now arrive at results for the distribution functions of the upper and lower bounds for V. It turns out that these distributions, which can be considered as the main result of this contribution, are rather easy to compute. We will denote the distribution functions of the upper and lower bounds in the same way as mentioned in (69).

Proposition 3.9 *Consider the stochastic quantities V_{u2}, V_{u3} and V_l as mentioned in Proposition 3.5. The cumulative distribution functions for these variables can be found to be*

$$F_{u2}(k) = Prob[V_{u2} \leq k] = u_k \tag{71}$$

$$F_{u3}(k) = Prob[V_{u3} \leq k] = \int_{-\infty}^{+\infty} dc\ f(c,t)\ u_k(c) \tag{72}$$

$$F_l(k) = Prob[V_l \leq k] = v_k \ , \tag{73}$$

with u_k, v_k and $u_k(c)$ as defined in (59)-(61).

Proof. This immediately follows when applying Lemma 3.8 to the results of Proposition 3.6.

Q.E.D.

4 Numerical illustration

In this section we assess the accuracy of the bounds by considering three quite different cash-flows. The first cash-flow consists of $n = 10$ equal payments $\alpha_j = 1$ at points in time $t_j = j$. For the stochastic accumulation factor, we choose $\mu = 0.07$ and $\sigma = 0.1$, while the dividend paying treshold β equals 0.2. The distribution functions of the bounds are depicted in Figure 1, together with an empirical distribution function of V obtained by Monte-Carlo simulation. The lower bound V_l appears to perform not so well, which may be explained by the conditioning on $Z(t_n)$ instead of $\sum_{i=1}^{n} \min[X(t_i) - X(t_{i-1}), \beta]$. However, the conditional distribution in the latter case is hard to obtain, due to the dependency structure of the terms. The graph also indicates that V_{u2} is indeed a “more dangerous” variable and that V_{u3} slightly improves this bound.

Next, in order to compare the upper bounds with some previous results, see e.g. [5], we should increase β to a relatively large value, say $\beta = 10$. The upper bounds in Figure 2 are rather sharp, especially in the right tail, which is in accordance with [5].

In Figures 3 and 4, we changed the cash-flow to $\alpha_j = j$ and $\alpha_j = 11 - j$ ($j = 1, \ldots, 10$) respectively. In case the cash-flow is decreasing (see Figure 4), both upper bounds show a slightly higher accuracy than in case the cash-flow is increasing (see Figure 3). This could have been expected, taking into account the approximate comonotonicity of the accumulation factors in the beginning of the period.

5 Conclusion

In the present contribution, we considered stochastic cash-flows, in the situation where a dividend is paid out for large increments in the rate of return. We arrived at upper and lower bounds for the cash-flow, the stop-loss premium and the distribution, when the logarithm of the stock price is modelled by means of a Wiener process. In some forth-coming papers, we will extend these calculations to other classes of stochastic processes, and we will try to numerically and graphically compare the different results.

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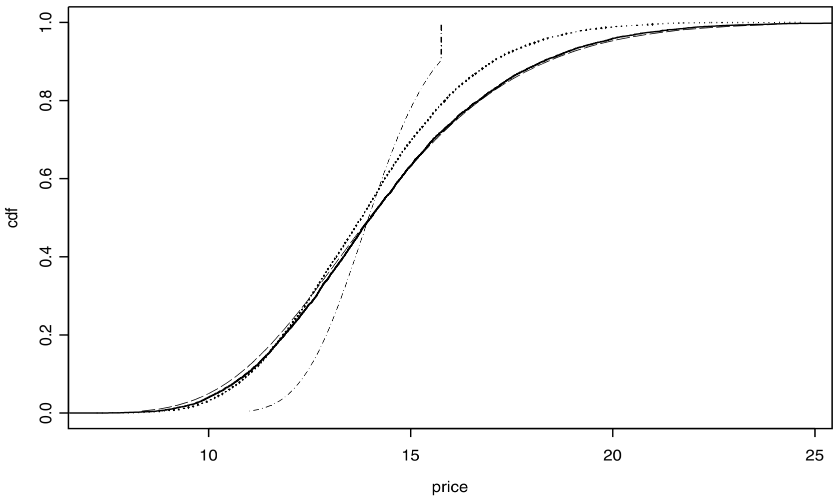

Figure 1: Distribution functions of V_l $(-\cdot-)$, V_{u2} $(--)$ and V_{u3} (—) for $\alpha_j = 1$ $(j = 1, \ldots, 10)$ and $\beta = 0.2$, compared to a simulated version of V $(\cdots)$.

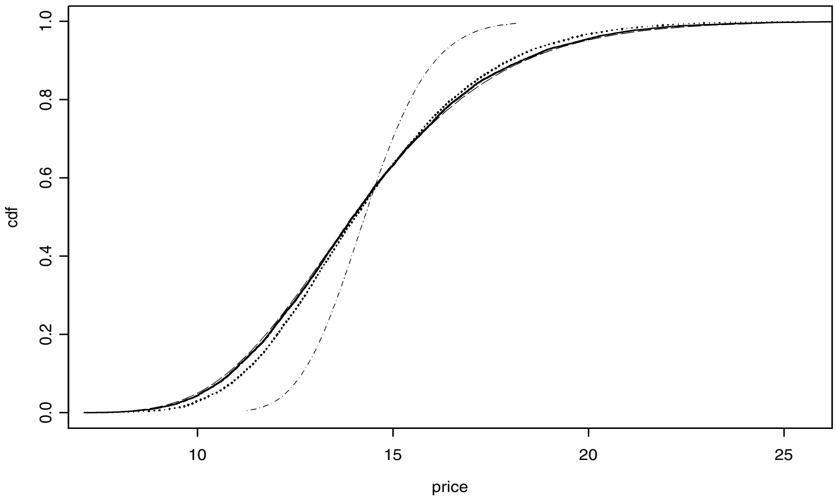

Figure 2: Distribution functions of V_l $(-\cdot-)$, V_{u2} $(--)$ and V_{u3} (—) for $\alpha_j = 1$ $(j = 1, \ldots, 10)$ and $\beta = 10$, compared to a simulated version of V $(\cdots)$.

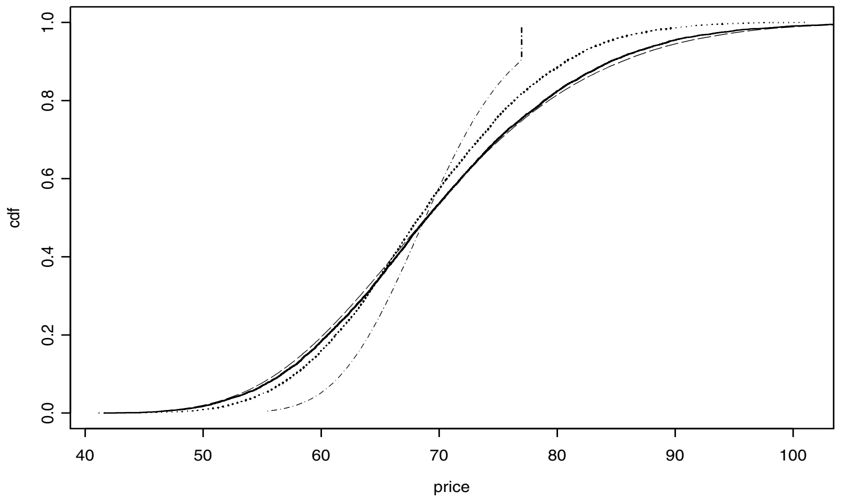

Figure 3: Distribution functions of V_l $(- \cdot -)$, V_{u2} $(- -)$ and V_{u3} (—) for $\alpha_j = j$ $(j = 1, \ldots, 10)$ and $\beta = 0.2$, compared to a simulated version of V $(\cdots)$.

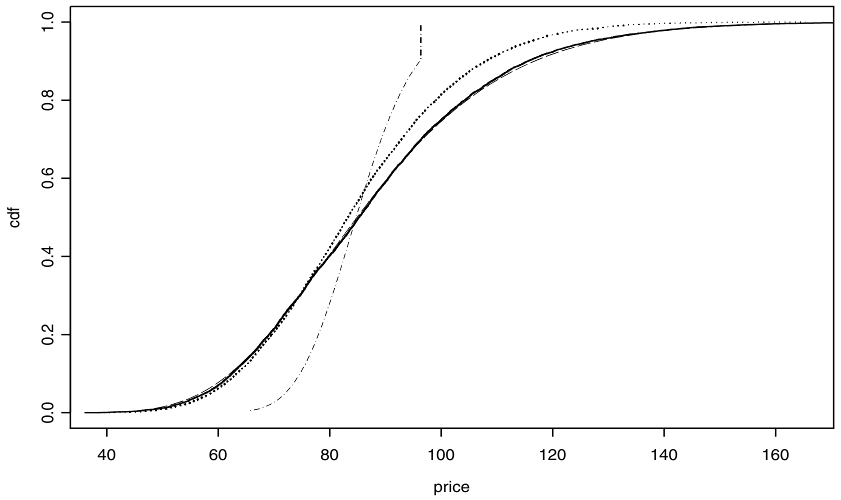

Figure 4: Distribution functions of V_l $(- \cdot -)$, V_{u2} $(- -)$ and V_{u3} (—) for $\alpha_j = 11 - j$ $(j = 1, \ldots, 10)$ and $\beta = 0.2$, compared to a simulated version of V $(\cdots)$.