The individual risk model

Jan Dhaene and David Vyncke*

Abstract

In the individual risk model, the total claims on a portfolio of insurance contracts is the random variable of interest. The total claims is modelled as the sum of all claims on the individual policies, which are assumed independent. We present several techniques, such as convolution and recursions, to obtain results in this model.

Keywords: aggregate claims, total claims, convolution, transform, approximation, recursion.

1 Introduction

In the individual risk model, the total claims on a portfolio of insurance contracts is the random variable of interest. We want to compute, for instance, the probability that a certain capital will be sufficient to pay these claims, or the value-at-risk at level 95% associated with the portfolio, being the 95% quantile of its cumulative distribution function (cdf). The total claims is modelled as the sum of all claims on the individual policies, which are assumed independent. We study other techniques than convolution to obtain results in this model. Using transforms like the moment generating function helps in some special cases. Also, we present approximations based on fitting moments of the distribution. The Central Limit Theorem, which involves fitting two moments, is not sufficiently accurate in the important right-hand tail of the distribution. Hence, we also present two more refined methods using three moments: the translated gamma approximation and the normal power approximation.

2 Convolution

In the individual risk model we are interested in the distribution of the total amount S of claims on a fixed number of n policies:

$$S = X_1 + X_2 + \cdots + X_n, \tag{1}$$

*K.U.Leuven, Department of Applied Economics, Naamsestraat 69, B-3000 Leuven, Belgium, e-mail: jan.dhaene@econ.kuleuven.ac.be, david.vyncke@econ.kuleuven.ac.be

where X_i, $i = 1, 2, \ldots, n$, denotes the claim payments on policy i. Assuming that the risks X_i are mutually independent random variables, the distribution of their sum can be calculated by making use of convolution.

The operation 'convolution' calculates the distribution function of $X + Y$ from those of two independent random variables X and Y, as follows:

$$\begin{aligned} F_{X+Y}(s) &= \Pr[X + Y \leq s] \\ &= \int_{-\infty}^{\infty} F_Y(s - x) dF_X(x) =: F_X * F_Y(s). \end{aligned} \tag{2}$$

The cdf $F_X * F_Y(\cdot)$ is called the convolution of the cdf's $F_X(\cdot)$ and $F_Y(\cdot)$. For the cdf of $X + Y + Z$, it does not matter in which order we perform the convolutions, hence we have

$$(F_X * F_Y) * F_Z \equiv F_X * (F_Y * F_Z) \equiv F_X * F_Y * F_Z. \tag{3}$$

For the sum of n independent and identically distributed random variables with marginal cdf F, the cdf is the n-fold convolution of F, which we write as

$$F * F * \cdots * F =: F^{*n}. \tag{4}$$

3 Transforms

Determining the distribution of the sum of independent random variables can often be made easier by using transforms of the cdf. The moment generating function (mgf) is defined as

$$m_X(t) = \mathrm{E}\left[e^{tX}\right]. \tag{5}$$

If X and Y are independent, the convolution of cdf's corresponds to simply multiplying the mgf's. Sometimes it is possible to recognize the mgf of a convolution and consequently identify the distribution function.

For random variables with a heavy tail, such as the Cauchy distribution, the mgf does not exist. The characteristic function, however, always exists. It is defined as follows:

$$\phi_X(t) = \mathrm{E}\left[e^{itX}\right], \quad -\infty < t < \infty. \tag{6}$$

Note that the characteristic function is one-to-one, so every characteristic function corresponds to exactly one cdf.

As their name indicates, moment generating functions can be used to generate moments of random variables. The k-th moment of X equals

$$\mathrm{E}\left[X^k\right] = \left.\frac{d^k}{dt^k} m_X(t)\right|_{t=0}. \tag{7}$$

A similar technique can be used for the characteristic function.

The probability generating function (pgf) is used exclusively for random variables with natural numbers as values:

$$g_X(t) = \mathrm{E}\left[t^X\right] = \sum_{k=0}^{\infty} t^k \Pr[X = k]. \tag{8}$$

So, the probabilities $\Pr[X = k]$ in (8) serve as coefficients in the series expansion of the pgf.

The cumulant generating function (cgf) is convenient for calculating the third central moment; it is defined as:

$$\kappa_X(t) = \log m_X(t). \tag{9}$$

The coefficients of $\frac{t^k}{k!}$ for $k = 1, 2, 3$ are $\mathrm{E}[X]$, $\mathrm{Var}[X]$ and $\mathrm{E}[(X - \mathrm{E}[X])^3]$. The quantities generated this way are the cumulants of X, and they are denoted by κ_k, $k = 1, 2, \ldots$. The *skewness* of a random variable X is defined as the following dimension-free quantity:

$$\gamma_X = \frac{\kappa_3}{\sigma^3} = \frac{\mathrm{E}[(X-\mu)^3]}{\sigma^3}, \tag{10}$$

with $\mu = \mathrm{E}[X]$ and $\sigma^2 = \mathrm{Var}[X]$. If $\gamma_X > 0$, large values of $X - \mu$ are likely to occur, hence the probability density function (pdf) is skewed to the right. A negative skewness $\gamma_X < 0$ indicates skewness to the left. If X is symmetrical then $\gamma_X = 0$, but having zero skewness is not sufficient for symmetry. For some counterexamples, see [17].

The cumulant generating function, the probability generating function, the characteristic function and the moment generating function are related to each other through the formal relationships

$$\kappa_X(t) = \log m_X(t); \quad g_X(t) = m_X(\log t); \quad \phi_X(t) = m_X(it). \tag{11}$$

4 Approximations

A totally different approach is to approximate the distribution of S. If we consider S as the sum of a 'large' number of random variables, we could, by virtue of the Central Limit Theorem, approximate its distribution by a normal distribution with the same mean and variance as S. It is difficult however to define 'large' formally and moreover this approximation is usually not satisfactory for the insurance practice, where especially in the tails, there is a need for more refined approximations which explicitly recognize the substantial probability of large claims. More technically, the third central moment of S is usually greater than 0, while for the normal distribution it equals 0.

As an alternative for the CLT, we give two more refined approximations: the translated gamma approximation and the normal power approximation (NP). In numerical examples, these approximations turn out to be much more accurate than the CLT approximation, while their respective inaccuracies are comparable, and are minor compared with the errors that result from the lack of precision in the estimates of the first three moments that are involved.

Translated gamma approximation

Most total claim distributions have roughly the same shape as the gamma distribution: skewed to the right ($\gamma > 0$), a non-negative range and unimodal. Besides the usual parameters α and β, we add a third degree of freedom by allowing a shift over a distance x_0. Hence, we approximate the cdf of S by the cdf of $Z + x_0$, where $Z \sim$ gamma(α, β). We choose α, β and x_0 in such a way that the approximating random variable has the same first three moments as S.

The translated gamma approximation can then be formulated as follows:

$$
\begin{aligned}
&F_S(s) \approx G(s - x_0; \alpha, \beta), \text{ where} \\
&G(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} \beta^\alpha e^{-\beta y} dy, \quad x \geq 0.
\end{aligned} \tag{12}
$$

Here $G(x; \alpha, \beta)$ is the gamma cdf. To ensure that α, β and x_0 are chosen such that the first three moments agree, hence $\mu = x_0 + \frac{\alpha}{\beta}$, $\sigma^2 = \frac{\alpha}{\beta^2}$ and $\gamma = \frac{2}{\sqrt{\alpha}}$, they must satisfy

$$
\alpha = \frac{4}{\gamma^2}, \quad \beta = \frac{2}{\gamma\sigma} \quad \text{and} \quad x_0 = \mu - \frac{2\sigma}{\gamma}. \tag{13}
$$

For this approximation to work, the skewness γ has to be strictly positive. In the limit $\gamma \downarrow 0$, the normal approximation appears. Note that if the first three moments of the cdf $F(\cdot)$ are the same as those of $G(\cdot)$, by partial integration it can be shown that the same holds for $\int_0^\infty x^j [1 - F(x)]dx$, $j = 0, 1, 2$. This leaves little room for these cdf's to be very different from each other.

Note that if $Y \sim$ gamma(α, β) with $\alpha \geq \frac{1}{4}$, then roughly $\sqrt{4\beta Y} - \sqrt{4\alpha - 1} \sim$ N$(0, 1)$. For the translated gamma approximation for S, this yields

$$
\Pr\left[\frac{S - \mu}{\sigma} \leq y + \frac{\gamma}{8}(y^2 - 1) - y\left(1 - \sqrt{1 - \frac{\gamma^2}{16}}\right)\right] \approx \Phi(y). \tag{14}
$$

The right hand side of the inequality is written as y plus a correction to compensate for the skewness of S. If the skewness tends to zero, both correction terms in (14) vanish.

NP approximation

The following approximation is very similar to (14). The correction term has a simpler form, and it is slightly larger. It can be obtained by the use of certain expansions for the cdf, but we will not reproduce that derivation here.

If $\mathrm{E}[S] = \mu$, $\mathrm{Var}[S] = \sigma^2$ and $\gamma_S = \gamma$, then, for $s \geq 1$,

$$
\Pr\left[\frac{S - \mu}{\sigma} \leq s + \frac{\gamma}{6}(s^2 - 1)\right] \approx \Phi(s) \tag{15}
$$

or, equivalently, for $x \geq 1$,

$$
\Pr\left[\frac{S - \mu}{\sigma} \leq x\right] \approx \Phi\left(\sqrt{\frac{9}{\gamma^2} + \frac{6x}{\gamma} + 1} - \frac{3}{\gamma}\right). \tag{16}
$$

The latter formula can be used to approximate the cdf of S, the former produces approximate quantiles. If $s < 1$ (or $x < 1$), the correction term is negative, which implies that the CLT gives more conservative results.

5 Recursions

Another alternative to the technique of convolution are recursions. Consider a portfolio of n policies. Let X_i be the claim amount of policy i, $i = 1, \dots, n$ and let the claim probability of policy i be given by $\Pr[X_i > 0] = q_i = 1 - p_i$. It is assumed that for each i, $0 < q_i < 1$ and that the claim amounts of the individual policies are integral multiples of some convenient monetary unit, so that for each i, the severity distribution $g_i(x) = \Pr[X_i = x \mid X_i > 0]$ is defined for $x = 1, 2, \dots$.

The probability that the aggregate claims S equal s, i.e. $\Pr[S = s]$, is denoted by $p(s)$. We assume that the claim amounts of the policies are mutually independent. An exact recursion for the individual risk model is derived in [12]:

$$p(s) = \frac{1}{s} \sum_{i=1}^{n} v_i(s), \qquad s = 1, 2, \dots \tag{17}$$

with initial value given by $p(0) = \prod_{i=1}^{n} p_i$ and where the coefficients $v_i(s)$ are determined by

$$v_i(s) = \frac{q_i}{p_i} \sum_{x=1}^{s} g_i(x) \left[x p(s-x) - v_i(s-x) \right], \qquad s = 1, 2, \dots \tag{18}$$

and $v_i(s) = 0$ otherwise. In case the individual claim amounts have a two-point distribution, this recursion reduces to the recursion in [22].

Other exact and approximate recursions have been derived for the individual risk model, see [8], or [ENCYCLOPEDIA: "De Pril recursions and approximations"].

A common approximation for the individual risk model is to replace the distribution of the claim amounts of each policy by a compound Poisson distribution with parameter λ_i and severity distribution h_i. From the independence assumption, it follows that the aggregate claims S is then approximated by a compound Poisson distribution with parameter

$$\lambda = \sum_{i=1}^{n} \lambda_i \tag{19}$$

and severity distribution h given by

$$h(y) = \frac{\sum_{i=1}^{n} \lambda_i h_i(y)}{\lambda}, \qquad y = 1, 2, \dots, \tag{20}$$

see e.g. [5, 14, 16]. Denoting the approximation for $f(x)$ by $g^{cP}(x)$ in this particular case, we find from Panjer's [21] formula that the approximated probabilities can be computed from the recursion

$$g^{cP}(x) = \frac{1}{x} \sum_{y=1}^{x} y \sum_{i=1}^{n} \lambda_i h_i(y) g^{cp}(x-y) \text{ for } x = 1, 2, \dots, \tag{21}$$

with starting value $g^{cP}(0) = e^{-\lambda}$.

The most common choice for the parameters is $\lambda_i = q_i$ which guarantees that the exact and the approximate distribution have the same expectation. This approximation is often referred to as *the* compound Poisson approximation.

6 Errors

Kaas [19] states that several kinds of error have to be considered when computing the aggregate claims distribution. A first type of error results when the possible claim amounts of the policies are rounded to some monetary unit, e.g. 1000 euro. Computing the aggregate claims distribution of this portfolio generates a second type of error if this computation is done approximately (e.g. moment matching approximation, compound Poisson approximation, De Pril's r-th order approximation, ...). Both types of errors can be reduced at the cost of extra computing time. It is of course useless to apply an algorithm that computes the distribution function exactly if the monetary unit is large.

Bounds for the different types of errors are helpful in fixing the monetary unit and choosing between the algorithms for the rounded model. Bounds for the first type of error can be found e.g. in [13] and [18]. Bounds for the second type of error are considered e.g. in [4, 5, 8, 11, 14].

A third type of error that may arise when computing aggregate claims follows from the fact that the assumption of mutual independency of the individual claim amounts may be violated in practice. Papers considering the individual risk model in case the aggregate claims are a sum of non-independent random variables are [1, 2, 3, 9, 10, 15, 20]. Approximations for sums of non-independent random variables based on the concept of comonotonicity are considered in [6, 7].

References

[1] Bäuerle, N.; Müller, A. (1998). Modeling and comparing dependencies in multivariate risk portfolios, *ASTIN Bulletin* 28, 59-76.

[2] Cossette, H.; Marceau, E. (2000). The discrete-time risk model with correlated classes of business, *Insurance: Mathematics & Economics* 26, 133-149.

[3] Denuit, M.; Dhaene, R.; Ribas, C. (2001). Does positive dependence between individual risks increase stop-loss premiums?, *Insurance: Mathematics & Economics* 28, 305-308.

[4] De Pril, N. (1989). The aggregate claims distribution in the individual life model with arbitrary positive claims, *ASTIN Bulletin* 19, 9-24.

[5] De Pril, N. ; Dhaene, J. (1992). Error bounds for compound Poisson approximations of the individual risk model, *ASTIN Bulletin* 19(1), 135-148.

[6] Dhaene, J.; Denuit M.; Goovaerts, M.J.; Kaas R.; Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and finance: theory, *Insurance: Mathematics & Economics*, 31, 3-33.

[7] Dhaene, J.; Denuit, M.; Goovaerts, M.J.; Kaas, R.; Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and finance: applications", *Insurance: Mathematics & Economics*, 31, 133-161.

[8] Dhaene, J.; De Pril, N. (1994). On a class of approximative computation methods in the individual risk model, *Insurance: Mathematics & Economics* 14, 181-196.

[9] Dhaene, J.; Goovaerts, M.J. (1996). Dependency of risks and stop-loss order *ASTIN Bulletin* 26, 201-212.

[10] Dhaene, J.; Goovaerts, M.J. (1997). On the dependency of risks in the individual life model, *Insurance: Mathematics & Economics* 19, 243-253.

[11] Dhaene, J.; Sundt, B. (1997). On error bounds for approximations to aggregate claims distributions, *ASTIN Bulletin* 27, 243-262.

[12] Dhaene, J.; Vandebroek, M. (1995). Recursions for the individual model, *Insurance: Mathematics & Economics* 16, 31-38.

[13] Gerber, H.U. (1982) On the numerical evaluation of the distribution of aggregate claims and its stop-loss premiums, *Insurance: Mathematics & Economics* 1, 13-18.

[14] Gerber, H.U. (1984). Error bounds for the compound Poisson approximation, *Insurance: Mathematics & Economics* 3, 191-194.

[15] Goovaerts, M.J.; Dhaene, J. (1996). The compound Poisson approximation for a portfolio of dependent risks, *Insurance: Mathematics & Economics* 18, 81-85.

[16] Hipp, C. (1985). Approximation of aggregate claims distributions by compound Poisson distributions, *Insurance: Mathematics & Economics* 4, 227-232. Correction note: *Insurance: Mathematics & Economics* 6, 165.

[17] Kaas, R.; Goovaerts, M.J.; Dhaene, J.; Denuit, M. (2001). *Modern Actuarial Risk Theory*, Kluwer Academic Publishers, Boston.

[18] Kaas, R.; Van Heerwaarden, A.E.; Goovaerts, M.J. (1988). Between individual and collective model for the total claims, *ASTIN Bulletin* 18, 169-174.

[19] Kaas, R. (1993). How to (and how not to) compute stop-loss premiums in practice, *Insurance: Mathematics & Economics* 13, 241-254.

[20] Müller, A. (1997). Stop-loss order for portfolios of dependent risks, *Insurance: Mathematics & Economics* 21, 219-224.

[21] Panjer, H.H. (1981) Recursive evaluation of a family of compound distributions, *ASTIN Bulletin* 12, 22-26.

[22] Waldmann, K.-H. (1994) On the exact calculation of the aggregate claims distribution in the individual life model, *ASTIN Bulletin* 24, 89-96.