Tail Variance Premiums for Log-Elliptical Distributions

Zinoviy Landsman*
Department of Statistics,University of Haifa

Nika Pat
Department of Statistics, University of Haifa

Jan Dhaene
Actuarial Research Group, K.U.Leuven

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Abstract

In this paper we derive expressions for the Tail Variance and the Tail Variance Premium of risks in a multivariate log-elliptical setting. The theoretical results are illustrated by considering lognormal and log-Laplace distributions. We also derive approximate expressions for a Tail Variance - based allocation rule in a multivariate lognormal setting. A numerical example ilustrates the accureteness of the proposed approximations.

1 Introduction

Consider the random variable (r.v.) X representing the claims related to an insurance policy or portfolio over a given insurance period. The cumulative distribution function and the probability density function of X are denoted by $F_X(x)$ and $f_X(x)$, respectively. A premium principle transforms the r.v. X

*Corresponding author: landsman@stat.haifa.ac.il

into a real number. It expresses the amount to be paid by the policyholder (or insurer) as compensation for the transfer of his risk to the insurer (or reinsurer).

Suppose that the insurer is concerned about the claims related to X exceeding a certain threshold, e.g. the quantile $F_X^{-1}(p)$, which is defined by

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in (0,1).$$

It is well-known that the Tail Conditional Expectation (TCE), which is defined by

$$\text{TCE}_p[X] = \mathbb{E}\left[X \mid X > F_X^{-1}(p)\right], \qquad p \in (0,1),$$

is useful in measure the right-tail risk in this case.

The TCE has been studied thoroughly by various authors, see e.g. Dhaene et al. (2006) and the references therein. Furman & Landsman (2006) observe that in many cases the TCE does not provide adequate information about the risks on the right tail. They illustrate their opinion by numerical examples. In particular, TCE is a conditional expectation and hence, does not includes information about deviation of the risk from its expectation in the upper tail. In order to overcome this problem, Furman & Landsman (2006) introduce two new risk measures, the (Conditional) Tail Variance and the (Conditional) Tail Variance Premium. The first is defined by

$$\text{TV}_p[X] = \text{Var}\left[X|X > F_X^{-1}(p)\right], \qquad p \in (0,1).$$

The latter combines the CTE and TV risk measures and is defined by

$$\text{TVP}_p[X] = \text{CTE}_p[X] + \alpha \ \text{TV}_p[X], \qquad \alpha \geq 0,\, p \in (0,1).$$

Furman & Landsman (2006) derive expressions for these new risk measures for the class of elliptical distributions. In this paper, we will consider results related to the TV and the TVP risk measures within the class of logelliptical distributions.

The remainder of this paper is structured as follows. In Section 2, we recapitulate some results on (log-)elliptical distributions. Expressions for the TV of univariate log-elliptical distributions are derived in Section 3. The general results of Section 3 are illustrated in Section 4 where the TV for lognormal and log-Laplace distribution functions are considered. A TV-based allocation rule for comonotonic risks is considered in Section 5. Based on the theoretical results of Section 5, we derive approximate expressions for the TV-based allocation rule for general sums of lognormal r.v.'s. in Section 6. A numerical illustration of our results is presented in Section 7.

2 Elliptical distributions

The random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ is said to have an elliptical distribution, written as $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\psi})$, if its characteristic function can be expressed as

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T\boldsymbol{\mu})\psi\left(\tfrac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right) \tag{1}$$

for some n-dimensional colum-vector $\boldsymbol{\mu}$, some $n \times n$ positive-definite matrix $\boldsymbol{\Sigma}$ and scalar function $\psi(t)$, which is called the *characteristic generator*. An ellipticial distributed random vector $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu},\Sigma,\psi)$ does not necessarily have a probability density. However, if $\mathbf{X}$ has a density $f_{\mathbf{X}}(\mathbf{x})$, then it has the following form:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} g_n\left[\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right], \tag{2}$$

for some function $g_n(\cdot)$, which is called the *density generator*. The condition

$$\int_0^\infty x^{n/2-1} g_n(x)dx < \infty \tag{3}$$

guarantees that $g_n(x)$ is a density generator (Fang, et al. 1987, Ch 2.2). The normalizing constant c_n in (2) is given by

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} g_n(x)dx\right]^{-1}, \tag{4}$$

which is assumed to be finite. From (1) it follows that if $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$, A is a $m \times n$ matrix of rank $m \leq n$ and b is an m-dimensional column-vector, then

$$A\mathbf{X} + b \sim E_m\left(A\boldsymbol{\mu} + b, A\boldsymbol{\Sigma}A^T, g_m\right). \tag{5}$$

In other words, any linear combination of elliptical distributions is again an elliptical distribution with the same characteristic generator ψ, or with same sequence of density generators $g_1, ...g_n$, corresponding to ψ.

Notice that condition (3) does not require the existence of the mean and the covariance of vector $\mathbf{X}$. It can be shown by a simple transformation in the integral for the mean that the condition

$$\int_0^\infty g_1(x)dx < \infty, \tag{6}$$

guarantees the existence of the mean. In this case, the mean vector of $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ is given by $E(\mathbf{X}) = \boldsymbol{\mu}$. On the other hand, the condition

$$|\psi'(0)| < \infty, \tag{7}$$

guarantees that the covariance matrix exists and is given by

$$Cov(\mathbf{X}) = -\psi'(0)\,\boldsymbol{\Sigma}. \tag{8}$$

Chosing the characteristic generator ψ such that

$$\psi'(0) = -1, \tag{9}$$

implies that the covariance matrix is given by

$$Cov(\mathbf{X}) = \boldsymbol{\Sigma}.$$

More details on the elliptical family of distributions can be found in Kelker (1970), Fang et al. (1987), Embrechts et al. (1999) and Landsman & Valdez (2003), amongst others.

Recall that random vector $\mathbf{X}$ has a multivariate log elliptical distribution, written as $\mathbf{X} \backsim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma},\psi)$, if $\log(\mathbf{X}) \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma},\psi)$ with expectations $\boldsymbol{\mu}$, generalized covariance matrix $\boldsymbol{\Sigma}$ and characteristic generator ψ.
As special cases, consider the lognormal distribution $LN_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and log Laplace distribution $LN_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Their characteristic generators are given by

$$\psi(u) = \exp(-u) \tag{10}$$

and

$$\psi(u) = \frac{1}{1+u}, \tag{11}$$

respectively.

3 The Tail Variance of univariate log-elliptical distributions.

Throughout this paper we will only consider elliptical distributions which have a probability density, and hence have a continuous cumulative distribution function.

We will use the notations $F_{\mathbf{X}}$ and $\overline{F}_{\mathbf{X}}$ to denote the cumulative and the decumulative distribution function of the random vector $\mathbf{X}$.

Theorem 1 *Let* $X \sim LE_1(\mu, \sigma, \psi)$*, with* $\sigma > 0$*. Suppose that the related spherical distributed r.v.* $Z \sim E_1(0, 1, \psi)$ *is such that* $\psi(-2\sigma^2)$ *is finite. The Tail Variance* $TV_p[X]$ *of* X *is then given by*

$$\begin{aligned} TV_p[X] &= \frac{e^{2\mu}}{1-p}\psi\left(-2\sigma^2\right)\bar{F}_{Z'}\left(F_Z^{-1}(p)\right) - \left[\frac{1}{1-p}\psi\left(-\frac{\sigma^2}{2}\right)\bar{F}_{Z^*}\left(F_Z^{-1}(p)\right)\right]^2 \quad (12) \\ p &\in (0,1), \end{aligned}$$

where Z^* *and* Z' *are r.v.'s with respective probability densities given by*

$$f_{Z^*}(z) = \frac{e^{\sigma z}}{\psi\left(-\frac{\sigma^2}{2}\right)} f_Z(z) \qquad \text{and} \qquad f_{Z'}(z) = \frac{e^{2\sigma z}}{\psi\left(-2\sigma^2\right)} f_Z(z). \tag{13}$$

Proof. From $Z \stackrel{d}{=} \frac{\ln X - \mu}{\sigma}$ it follows that

$$f_X(x) = \frac{1}{\sigma x} f_Z\left(\frac{\ln x - \mu}{\sigma}\right). \tag{14}$$

Substituting x by $z = \frac{\ln x - \mu}{\sigma}$ in the following integral expression, we find that

$$\begin{aligned} TCE_p[X] &= \frac{1}{1-p}\int_{F_X^{-1}(p)}^{\infty} x f_X(x)dx = \frac{1}{1-p}\int_{F_Z^{-1}(p)}^{\infty} e^{\mu+\sigma z} f_Z(z)dz \\ &= \frac{e^{\mu}\psi\left(-\frac{\sigma^2}{2}\right)}{1-p}\overline{F}_{Z^*}(F_Z^{-1}(p)), \end{aligned} \tag{15}$$

This expression for $TCE_p[X]$ can be found in Valdez et al. (2009).

Again substituting x by $z = \frac{\ln x - \mu}{\sigma}$, we find that $E\left[X^2 | X > F_X^{-1}(p)\right]$ can be expressed as

$$\begin{aligned} E\left[X^2 | X > F_X^{-1}(p)\right] &= \frac{1}{1-p}\int_{F_X^{-1}(p)}^{\infty} x^2 f_X(x)dx = \frac{1}{1-p}\int_{F_Z^{-1}(p)}^{\infty} e^{2(\mu+\sigma z)} f_Z(z)dz \\ &= \frac{e^{2\mu}}{1-p}\psi\left(-2\sigma^2\right)\bar{F}_{Z'}(F_Z^{-1}(p)) \end{aligned} \tag{16}$$

The expression (12) for the Tail Variance of X follows then from substituting (14) and (12) in

$$TV_p[X] = E\left[X^2 | X > F_X^{-1}(p)\right] - \left[TCE_p(x)\right]^2. \tag{17}$$

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4 Examples

In this section, we will illustrate Theorem 1 by applying it to the case of lognormal and log-Laplace distributions.

Example 1 (TV for lognormal distributions) *Suppose that $X \sim LE_1(\mu, \sigma, \psi)$ with characteristic generator*

$$\psi(u) = e^{-u}.$$

Then X is said to be lognormal distributed with parameters μ and σ, notation $X \sim LN(\mu, \sigma)$.
In this case, $Z \sim N(0,1)$ has probability density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

Furthermore,

$$\psi\left(-\frac{t^2}{2}\right) = e^{\frac{t^2}{2}}.$$

From the definition of Z^* in Theorem 1, we obtain that

$$f_{Z^*}(z) = \frac{e^{\sigma z}}{e^{\frac{\sigma^2}{2}}} f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(z-\sigma)^2),$$

which means that

$$Z^* \sim N(\sigma, 1),$$

and also

$$\bar{F}_{Z^*}(F_Z^{-1}(p)) = \Phi(\sigma - \Phi^{-1}(p)),$$

where $\Phi(x)$ is the standard normal cdf. Substituting these expressions in (15), we find

$$TCE_p[X] = \frac{e^{\mu + \frac{\sigma^2}{2}}}{1-p} \Phi\left(\sigma - \Phi^{-1}(p)\right). \tag{18}$$

This expression for $TCE_p[X]$ of $X \sim LN(\mu, \sigma)$ can be found in Dhaene et al. (2008).
Similarly, we find that

$$f_{Z'}(z) = \frac{e^{2\sigma z}}{e^{2\sigma^2}} f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z-2\sigma)^2\right),$$

which means that $Z' \sim N(2\sigma, 1)$. *Furthermore,*

$$\bar{F}_{Z'}(F_Z^{-1}(p)) = \Phi\left(2\sigma - \Phi^{-1}(p)\right)$$

Substituting these expressions in (12), we find

$$TV_p[X] = \frac{e^{2\left(\mu+\sigma^2\right)}}{1-p}\Phi\left(2\sigma - \Phi^{-1}(p)\right) - \left[\frac{e^{\mu+\frac{\sigma^2}{2}}}{1-p}\Phi(\sigma - \Phi^{-1}(p))\right]^2 \tag{19}$$

Example 2 (TV for log-Laplace distributions) *The r.v.* $Y \sim E_1(\mu, \sigma, \psi)$ *with* ψ *given by*

$$\psi(u) = \frac{1}{1+u} \tag{20}$$

is said to be Laplace distributed, notation $Y \sim Lp(\mu, \sigma)$. *The r.v.* $Z \sim Lp(0, 1)$ *has probability density*

$$f_Z(z) = \frac{1}{\sqrt{2}}e^{-\sqrt{2}|z|}.$$

Furthermore,

$$\psi\left(-\frac{t^2}{2}\right) = \frac{1}{1-\frac{t^2}{2}}$$

Moreover, in case $Z \sim Lp(0, 1)$, *we have that*

$$F_Z^{-1}(p) = \begin{cases} \frac{1}{\sqrt{2}}\ln(2p) & : 0 < p < \frac{1}{2}, \\ -\frac{1}{\sqrt{2}}\ln(2(1-p)) & : \frac{1}{2} \leq p < 1. \end{cases} \tag{21}$$

Suppose now that $X \sim LE_1(\mu, \sigma, \psi)$ *with characteristic generator* ψ *given by (20). Then* X *is said to be log-Laplace distributed with parameters* μ *and* σ, *notation* $X \sim LLp(\mu, \sigma)$. *Hereafter, we assume that* $\sigma < \frac{\sqrt{2}}{2}$.
From the definition of Z^* *in Theorem 1, we find that*

$$f_{Z^*}(z) = \frac{e^{\sigma z}}{\psi\left(-\frac{\sigma^2}{2}\right)}f_Z(z) = \frac{1}{\sqrt{2}}\left(1 - \frac{\sigma^2}{2}\right)e^{\sigma z - \sqrt{2}|z|},$$

with $Z \sim Lp(0, 1)$. *Substituting this expression in (15), we find*

$$TCE_p[X] = \frac{e^{\mu}\psi\left(-\frac{\sigma^2}{2}\right)}{1-p}\overline{F}_{Z^*}(F_Z^{-1}(p)) = \frac{e^{\mu}}{(1-p)\sqrt{2}}\int_{F_Z^{-1}(p)}^{\infty} e^{\sigma z - \sqrt{2}|z|}dz.$$

Taking into account (21), we find

$$TCE_p[X] = \begin{cases} \frac{e^{\mu}\left(2\sqrt{2}-\left(\sqrt{2}-\sigma\right)(2p)^{\left(\sigma/\sqrt{2}+1\right)}\right)}{\sqrt{2}(1-p)(2-\sigma^2)} & : 0 < p < \frac{1}{2}, \\ \frac{\sqrt{2}\ e^{\mu}}{(2(1-p))^{\frac{\sigma}{\sqrt{2}}}(\sqrt{2}-\sigma)} & : \frac{1}{2} \leq p < 1. \end{cases} \tag{22}$$

This expression of $TCE_p[X]$ for $X \sim LL_1(\mu, \sigma)$ can be found in Dhaene et al. (2008).
Similarly, we find that

$$f_{Z'}(z) = \frac{e^{2\sigma z}}{\psi\left(-2\sigma^2\right)} f_Z(z) = \frac{1-2\sigma^2}{\sqrt{2}} e^{2\sigma z - \sqrt{2}|z|},$$

Taking into account (16), the expressions derived above lead to

$$\begin{aligned} E\left[X^2 | X > F_X^{-1}(p)\right] &= \frac{e^{2\mu}}{1-p}\psi\left(-2\sigma^2\right)\bar{F}_{Z'}(F_Z^{-1}(p)) \\ &= \begin{cases} \frac{e^{2\mu}}{\sqrt{2}(1-p)}\left(\frac{1-(2p)^{\left(\sqrt{2}\sigma+1\right)}}{\left(2\sigma+\sqrt{2}\right)} + \frac{1}{\sqrt{2}-2\sigma}\right) & : 0 < p < \frac{1}{2}, \\ \frac{\sqrt{2}e^{2\mu}}{(2(1-p))^{\sqrt{2}\sigma}(\sqrt{2}-2\sigma)} & : \frac{1}{2} \leq p < 1. \end{cases} \end{aligned} \tag{23}$$

Combining (17), (22) and (23) leads to the following expression for TV:

$$TV_p[X] = \begin{cases} \frac{e^{2\mu}}{\sqrt{2}(1-p)}\left(\frac{1-(2p)^{\left(\sqrt{2}\sigma+1\right)}}{\left(2\sigma+\sqrt{2}\right)} + \frac{1}{\sqrt{2}-2\sigma}\right) - \left(\frac{e^{\mu}\left(2\sqrt{2}-\left(\sqrt{2}-\sigma\right)(2p)^{\left(\sigma/\sqrt{2}+1\right)}\right)}{\sqrt{2}(1-p)(2-\sigma^2)}\right)^2, \\ \qquad 0 < p < \frac{1}{2} \\ \frac{\sqrt{2}e^{2\mu}}{(2(1-p))^{\sqrt{2}\sigma}(\sqrt{2}-2\sigma)} - \left(\frac{\sqrt{2}\ e^{\mu}}{(2(1-p))^{\frac{\sigma}{\sqrt{2}}}(\sqrt{2}-\sigma)}\right)^2 \\ \qquad \frac{1}{2} \leq p < 1 \end{cases} \tag{24}$$

5 A TV-based allocation rule for comonotonic r.v.'s

The capital allocation problem consists of allocating a given aggregate capital K associated with an aggregate loss

$$S = X_1 + X_2 + \ldots + X_n$$

to its different constituents X_i, see e.g. Dhaene et al. (2011) for a general introduction to capital allocation. Tasche (2004) introduced the TCE based allocation rule, where the aggregate capital K is determined as

$$K = TCE_p\left[S\right], \qquad p \in (0,1),$$

whereas the capital K_i that is located to X_i is determined by

$$K_i = E\left[X_i \mid S > F_S^{-1}(p)\right], \qquad i = 1, 2, \ldots, n \text{ and } p \in (0,1).$$

It is clear that K_i can be interpreted as the contribution of X_i to $\text{TCE}_p\left[S\right]$. Landsman & Valdez (2003) derived a closed-form expression for the TCE based allocation rule in case $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\psi})$.

Furman & Landsman (2006) introduced the *tail covariance allocation rule* in order to allocate the tail variance $\text{TV}_p\left[S\right]$ of a portfolio of risks to its different components. In particular, they propose to allocate $Cov\left[X_i, S | S > F_S^{-1}(p)\right]$ to risk i . This rule is additive in the sense that

$$TV_p\left[S\right] = \sum_{i=1}^{n} Cov\left[X_i, S | S > F_S^{-1}(p)\right] = \sum_{i,j=1}^{n} Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]. \tag{25}$$

They derived expressions for $Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]$ in case $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\psi})$.

Let us now have a look at the allocation rule (25) of Furman and Landsman (2006) in the log-elliptical case and hence, suppose that $\mathbf{X} \backsim \mathbf{LE}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\psi})$. Unfortunately, explicit formula for $Cov[X_i, X_j | S > F_S^{-1}(p)]$ are not available in this case. Dhaene et al (2008) proposed an approximate formula for $E[X_i | S > F_S^{-1}(p)]$ based on the theory of comonotonicity. In the remainder of this section we will follow and generalize their approach in order to obtain an approximation for $Cov[X_i, X_j | S > F_S^{-1}(p)]$, $i, j = 1, ..., n$.

Recall that the random vector $\mathbf{X}$ is said to be comonotonic in case there exist non-decreasing functions $f_1, f_2, \ldots, f_n$ and a r.v. Z such that

$$\mathbf{X} \stackrel{d}{=} \left(f_1(Z), f_2(Z), \ldots, f_n(Z)\right). \tag{26}$$

Equivalently, comonotonicity can be characterized as

$$\mathbf{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{27}$$

where U is a r.v. which is uniformly distributed over the unit interval $(0,1)$. For more details on the notion of comonotonicity, we refer to Dhaene et al. (2002a,b). Hereafter, we will restrict to comonotonic random vectors with continuous marginal cdf's.

Theorem 2 *Let* $\mathbf{X}$ *be a comonotonic random vector with continuous marginal cdf's* $F_{X_i}(x)$*. Let* S *be defined by* $S = X_1 + \ldots + X_n$*. Furthermore, let* $g(x_1, ..., x_m)$*,* $m = 1, \ldots, n$ *be a measurable function such that* $E\left[g(X_{i_1}, ..., X_{i_m})\right] < \infty$ *for any* $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, m\}$*. Then one has that*

$$\begin{aligned} & E\left[g(X_{i_1}, \ldots, X_{i_m}) | S > F_S^{-1}(p)\right] \qquad (28) \\ = \; & E\left[g(X_{i_1}, \ldots, X_{i_m}) | X_{i_1} > F_{X_{i_1}}^{-1}(p), \ldots, X_{i_m} > F_{X_{i_m}}^{-1}(p)\right], \qquad p \in (0,1). \end{aligned}$$

Proof. As $\mathbf{X}$ is comonotonic we have that $F_S^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p)$. Furthermore, the continuity of the marginal cdf's implies that each $F_{X_i}^{-1}(p)$ is a strictly increasing function in p, $0 < p < 1$. Combining these results we find that the the following equivalence relations hold for each i:

$$\sum_{j=1}^{n} F_{X_j}^{-1}(U) > \sum_{j=1}^{n} F_{X_j}^{-1}(p) \Leftrightarrow U > p \Longleftrightarrow F_{X_i}^{-1}(U) > F_{X_i}^{-1}(p), \; i = 1, \ldots, n.$$

Taking into account that $\mathbf{X}$ is comonotonic, we have that

$$(g(X_{i_1}, \ldots, X_{i_m}), S) \overset{d}{=} \left(g\left(F_{X_{i_1}}^{-1}(U), \ldots, F_{X_{i_m}}^{-1}(U)\right), \sum_{j=1}^{n} F_{X_j}^{-1}(U) \right).$$

Hence,

$$\begin{aligned} & E\left[g(X_{i_1}, ..., X_{i_m}) | S > F_S^{-1}(p)\right] \\ = \; & E\left[g\left(F_{X_{i_1}}^{-1}(U), \ldots, F_{X_{i_m}}^{-1}(U)\right) \mid \sum_{j=1}^{n} F_{X_j}^{-1}(U) > \sum_{j=1}^{n} F_{X_j}^{-1}(p) \right] \\ = \; & E\left[g\left(F_{X_{i_1}}^{-1}(U), \ldots, F_{X_{i_m}}^{-1}(U)\right) \mid F_{X_{i_1}}^{-1}(U) > F_{X_{i_1}}^{-1}(p), \ldots, F_{X i_m}^{-1}(U) > F_{X i_m}^{-1}(p) \right] \\ = \; & \mathbb{E}\left[g(X_{i_1}, \ldots, X_{i_m}) \mid X_{i_1} > F_{X_{i_1}}^{-1}(p), \ldots, X_{i_m} > F_{X_{i_m}}^{-1}(p)\right]. \end{aligned}$$

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Theorem 2 is a generalisation of Theorem 3.1 in Dhaene et al. (2008), who proved this result for the special case $g(x) = x$.

As a corollary of the previous theorem, we immediately find the following result, by choosing $m = 2$ and $g(x_1, x_2) = (x_1 - TCE_p[X_i])(x_2 - TCE_p[X_j])$.

Corollary 1 *Let* $\mathbf{X}$ *be a comonotonic random vector with continuous marginal cdf's* $F_{X_i}(x)$*. For any* $p \in (0,1)$*, the contribution* $Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]$ *of risks* i *and* j *to the Tail Variance* $TV_p[S]$ *of* S *is given by*

$$\begin{aligned} Cov\left[X_i, X_j | S > F_S^{-1}(p)\right] &= Cov\left(X_i, X_j | X_i > F_{X_i}^{-1}(p), X_j > F_{Xj}^{-1}(p)\right), \\ i,j &= 1, ..., n, \end{aligned}$$

In the sequel we will call $Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]$ the tail-covariance of X_i and X_j, and denote it by $TCov_p[X_i, X_j]$.

6 A TV based allocation rule for sums of lognormal r.v.'s.

Let the random vector $\mathbf{X} = (X_1, \ \ldots, \ X_n)$ be defined by

$$\mathbf{X} = (e^{Y_1}, \ \ldots, \ e^{Y_n}), \tag{29}$$

where $\mathbf{Y} = (Y_1, \ldots, Y_n) \sim N_n(\boldsymbol{\mu}, \Sigma)$ is an n - dimensional normal vector. Furthermore, S is defined by

$$S = \sum_{k=1}^{n} e^{Y_k}. \tag{30}$$

In general, it is not possible to derive an analytical expression for $E[X_k \mid S > F_S^{-1}(p)]$ in this case. Therefore, we follow the idea presented in Kaas et al. (2000) to approximate (the distribution of) the r.v. S by (the distribution of) the r.v. T which is defined by

$$T = E[S \mid \Lambda] = \sum_{k=1}^{n} \mathbb{E}[e^{Y_k} \mid \Lambda] = \sum_{k=1}^{n} T_k, \tag{31}$$

where $T_k = E[X_k \mid \Lambda]$ and the conditioning r.v. Λ is a linear combination of the Y_i:

$$\Lambda = \sum_{k=1}^{n} \beta_k \ Y_k \tag{32}$$

for appropriate constants β_k, $k = 1, \ldots, n$.
Let us denote the elements of the vector $\boldsymbol{\mu}$ by μ_k and the elements of the

positive definite assumed matrix Σ by $\sigma_{kl},\ k,l = 1,2,...,n$. The correlation between Y_k and Λ is denoted by r_k:

$$r_k = corr\left[Y_k, \Lambda\right] = \frac{1}{\sigma_k\ \sigma_\Lambda}\sum_{l=1}^{n}\beta_l\ \sigma_{kl}, \qquad k = 1,2,...,n, \tag{33}$$

with σ_Λ^2 given by

$$\sigma_\Lambda^2 = \sum_{k=1}^{n}\sum_{l=1}^{n}\beta_k\ \beta_l\ \sigma_{kl}. \tag{34}$$

One possible choice for the coefficients β_k is given by

$$\beta_k = \ e^{\mu_k + \frac{1}{2}\sigma_k^2}, \quad k = 1,...,n. \tag{35}$$

This choice of the coefficients is proposed in Vanduffel et al. (2005), as a slight numerical improvement to the original choice proposed in Kaas et al. (2000) which is such that Λ is a linear transformation of a first-order approximation to the sum S. In the Theorem below we will need the assumption that the correlation coefficients r_k defined in (33) are positive. This assumption is fulfilled when the β_k are given by (35).

Dhaene et al. (2008) approximated $E[X_k \mid S > F_S^{-1}(p)]$ as follows:

$$\begin{aligned} E[X_k \quad | \quad & S > F_S^{-1}(p)] \approx E\left[T_k \mid T > F_T^{-1}(p)\right] \\ = \quad & TCE_p\left[T_k\right], \qquad k = 1,2,\ldots,n. \end{aligned} \tag{36}$$

Based on this observation, we propose to approximate the tail-covariance of X_k and X_j in the following way:

$$TCov_p\left[X_k, X_j\right] \approx E[E(e^{Y_k+Y_j}|\Lambda)|T > F_T^{-1}(p)] - TCE_p\left(T_k\right)TCE_p\left(T_j\right). \tag{37}$$

In the following theorem we derive an expression for this approximation.

Theorem 3 *Let the random vector* $\mathbf{X}$ *be defined by (29). Furthermore, let* T *be defined by (31), where* Λ *is such that all correlations* r_k *defined in (33) are positive. The approximation (37) for* $TCov_p\left[X_k, X_j\right]$ *can be expressed as follows:*

$$\begin{aligned} & \frac{1}{1-p}\exp\left(\mu_k + \mu_j + \frac{\sigma_k^2+\sigma_j^2}{2}\right) \\ & \times\left[\exp(\sigma_{kj})\Phi\left(\sigma_k r_k + \sigma_j r_j - \Phi^{-1}\left(p\right)\right) - \frac{1}{1-p}\Phi\left(\sigma_k r_k - \Phi^{-1}\left(p\right)\right)\Phi\left(\sigma_j r_j - \Phi^{-1}\left(p\right)\right)\right] \end{aligned} \tag{38}$$

Proof. Denote by $Y_{kj} = Y_k + Y_j$. It is clear that

$$Y_{kj} \backsim N\left(\mu_k + \mu_j,\ \delta_{kj}^2\right) \quad \text{with } \delta_{kj}^2 = \sigma_k^2 + 2\sigma_{kj} + \sigma_j^2. \tag{39}$$

We easily find that

$$Y_{kj}|\Lambda = \lambda \backsim N\left(\mu_k + \mu_j + \frac{Cov(Y_{kj}, \Lambda)}{\sigma_\Lambda^2}(\lambda - \mu_\Lambda),\ \delta_{kj}^2 - \frac{Cov^2(Y_{kj},\Lambda)}{\sigma_\Lambda^2}\right).$$

Furthermore, one has that

$$Cov(Y_{kj}, \Lambda) = (r_k\sigma_k + r_j\sigma_j)\,\sigma_\Lambda.$$

Hence,

$$Y_{kj}|\Lambda = \lambda \backsim N\left((\mu_k + \mu_j + (r_k\sigma_k + r_j\sigma_j)\frac{(\lambda - \mu_\Lambda)}{\sigma_\Lambda},\ \delta_{kj}^2 - (r_k\sigma_k + r_j\sigma_j)^2\right).$$

Now we define the r.v.'s W_{kj} as follows:

$$W_{kj} = E\left(e^{Y_{kj}}|\Lambda\right) = \exp\left(\mu_k + \mu_j + \frac{1}{2}\left(\delta_{kj}^2 - (r_k\sigma_k + r_j\sigma_j)^2\right) + (r_k\sigma_k + r_j\sigma_j)Z\right),$$

where

$$Z = \frac{\Lambda - \mu_\Lambda}{\sigma_\Lambda} \backsim N(0,1).$$

Taking into account (37) we find that

$$TCov_p[X_k, X_j] \approx TCE_p(W_{kj}) - TCE_p(T_k)\,TCE_p(T_j). \tag{40}$$

To determine $TCE_p(W_{kj})$ we notice that

$$W_{kj} \backsim LN\left(\mu_k + \mu_j + \frac{1}{2}(\delta_{kj}^2 - (r_k\sigma_k + r_j\sigma_j)^2),\ (r_k\sigma_k + r_j\sigma_j)^2\right).$$

Now using the formula (18) for the TCE of a lognormal distribution, we obtain

$$TCE_p(W_{kj}) = \frac{1}{1-p}\exp\left(\mu_k + \mu_j + \frac{1}{2}\delta_{kj}^2\right)\Phi\left(\sigma_k r_k + \sigma_j r_j - \Phi^{-1}(p)\right). \tag{41}$$

Combining (41) and (18) we finally obtain the following expression from (40):

$$\begin{aligned} & TCov_p\left[X_k, X_j\right] \\ \approx \; & \frac{1}{1-p}\exp\left(\mu_k+\mu_j+\frac{1}{2}\delta_{kj}^2\right)\Phi\left(\sigma_k r_k+\sigma_j r_j-\Phi^{-1}(p)\right) \\ & -\frac{1}{(1-p)^2}\exp\left(\mu_k+\mu_j+\frac{\sigma_k^2+\sigma_j^2}{2}\right)\Phi\left(\sigma_k r_k-\Phi^{-1}(p)\right)\Phi\left(\sigma_j r_j-\Phi^{-1}(p)\right). \end{aligned}$$

Taking into account the expression for δ_{kj}^2 given in (39), we find the approximation (38) for the tail-covariance $TCov_p\left[X_k, X_j\right]$. ■

From this Theorem we can immediately derive the following approximate expression for $Var(X_k|S > F_S^{-1}(p))$ in the multivariate lognormal case:

$$\frac{1}{1-p}\exp\left(2\mu_k+\sigma_k^2\right)\left[\exp(\sigma_k^2)\Phi\left(2\sigma_k r_k-\Phi^{-1}(p)\right)-\frac{1}{1-p}\left(\Phi\left(\sigma_k r_k-\Phi^{-1}(p)\right)\right)^2\right]$$

From this Theorem, we also find the following approximation for the tail-variance $TV_p(S)$:

$$\begin{aligned} & \frac{1}{1-p}\sum_{k,j=1}^{n}\exp\left(\mu_k+\mu_j+\frac{\sigma_k^2+\sigma_j^2}{2}\right) \\ & \times\left[\exp(\sigma_{kj})\Phi\left(\sigma_k r_k+\sigma_j r_j-\Phi^{-1}(p)\right)-\frac{1}{1-p}\Phi\left(\sigma_k r_k-\Phi^{-1}(p)\right)\Phi\left(\sigma_j r_j-\Phi^{-1}(p)\right)\right] \end{aligned} \tag{42}$$

7 Numerical illustration

In this section we give a numerical illustration of our theoretical results. We use the example of a company with 4 business lines that was presented in Dhaene et al. (2008). We illustrate the appropriateness of the proposed approximation for the TV - based allocation rule in a multivariate lognormal setting.

Let us assume that the multivariate risk $\mathbf{X} = (X_1, ..., X_4)$ faced by the company has a multivariate lognormal distribution, $\mathbf{X} \sim LN_4(\mu, \Sigma)$, with marginal expectations and variances given by

$$\left(E\left[X_1\right], E\left[X_2\right], E\left[X_3\right], E\left[X_4\right]\right) = (20, 40, 10, 5)$$

and

$$\left(Var\left[X_1\right], Var\left[X_2\right], Var\left[X_3\right], Var\left[X_4\right]\right) = \left(5^2, 15^2, 2^2, 2^2\right),$$

respectively. Furthermore, the correlation between any pair (X_i, X_j) is given by

$$corr\left[X_i, X_j\right] = 0.75, \qquad i \neq j,$$

which means that the different business lines are rather strongly positive dependent.

It is our goal to calculate the tail variance

$$TV_p(S) = \sum_{i,j=1}^{n} Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]$$

for the probability levels $p = 0$, 0.6, 0.9 and 0.95, respectively.

We first determine $TV_p(S)$ by Monte-Carlo simulation. We use a large sample of 10^6 realizations in order to minimize the standard deviation of the sampling error. The numerical values for the conditional covariances $Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]$ and for the tail variance $TV_p(S)$ obtained by simulation are given in Table 1.

Next, we determine the conditional covariances $Cov\left[X_i, X_j | S > F_S^{-1}(p)\right]$ and the conditional variance $TV_p(S)$ approximately by formulae (38) and (42), respectively. The coefficients β_k are chosen as in (35). The numerical values for these quantities are presented in Table 2.

Table 1				**Table 2**			
Monte Carlo Estimation of				**Comonotonic Approximation of**			
$Cov\left[X_k, X_j \| S > F_S^{-1}(p)\right]$				$Cov\left[X_k, X_j \| S > F_S^{-1}(p)\right]$			
$p = 0.95$				$p = 0.95$			
22.621	10.409	3.356	4.414	20.909	9.186	2.924	3.957
10.409	170.686	5.700	7.736	9.186	172.575	5.413	7.710
3.356	5.700	3.262	1.814	2.924	5.413	3.153	1.669
4.414	7.736	1.814	5.730	3.957	7.710	1.669	5.577
		$TV_p(S)$	269.158			$TV_p(S)$	263.931
$p = 0.9$				$p = 0.9$			
20.626	13.563	3.323	4.225	19.727	13.516	3.077	3.986
13.563	162.453	6.629	8.593	13.516	165.018	6.660	8.965
3.323	6.629	3.098	1.737	3.077	6.660	3.019	1.659
4.225	8.593	1.737	5.001	3.986	8.965	1.659	4.895
		$TV_p(S)$	267.317			$TV_p(S)$	268.383
$p = 0.6$				$p = 0.6$			
18.769	25.501	3.903	4.525	18.656	25.810	3.901	4.523
25.501	163.507	10.654	12.498	25.810	164.318	10.702	12.647
3.903	10.654	2.924	1.820	3.901	10.702	2.929	1.826
4.525	12.498	1.820	3.831	4.523	12.647	1.826	3.837
		$TV_p(S)$	306.835			$TV_p(S)$	308.559
$p = 0$				$p = 0$			
25.000	55.423	7.450	7.373	25.000	55.423	7.450	7.373
55.423	225.000	22.142	22.100	55.423	225.000	22.142	22.100
7.450	22.142	4.000	2.945	7.450	22.142	4.000	2.945
7.373	22.100	2.945	4.000	7.373	22.100	2.945	4.000
	$TV_p(S)$	$= V(S)$	492.865		$TV_p(S)$	$= V(S)$	492.865

Table 3	
Level (p)	The relative difference
0.95	1.942%
0.9	−0.399%
0.6	−0.562%
0	0%

In Table 3 the relative difference between the results of the Monte-Carlo simulation and the results obtained via the comonotonic approximations are given. From Tables 1, 2 and 3 one can conclude that the proposed approximations based on the theory of comonotonicity perform well. This is particularly true because the comonotonic approximations reduce the n -dimensional randomness of the problem to a univariate randomness via the introduction of the condition r.v. Λ defined in (32).

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