The Minimal Entropy Martingale Measure in a market of traded financial and actuarial risks

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Abstract

In arbitrage-free but incomplete markets, the equivalent martingale measure $\mathbb{Q}$ for pricing traded assets is not uniquely determined. A possible approach when it comes to choosing a particular pricing measure is to consider the one that is 'closest' to the physical probability measure $\mathbb{P}$, where closeness is measured in terms of relative entropy.

In this paper, we determine the minimal entropy martingale measure in a market where securities are traded with payoffs depending on two types of risks, which we will call financial and actuarial risks, respectively. In case only purely financial and purely actuarial securities are traded, we prove that financial and actuarial risks are independent under the physical measure if and only if these risks are independent under the entropy measure. Moreover, in such a market the entropy measure of the combined financial-actuarial world is the product measure of the entropy measures of the financial and the actuarial subworlds, respectively.

Keywords: Minimal entropy martingale measure, relative entropy, financial risks, actuarial risks, independence, incomplete markets.

1 Introduction

Despite the never-ending stream of innovations concerning traded assets with payoffs contingent on financial and/or actuarial quantities, most corresponding markets remain incomplete. An obvious question that arises in an arbitrage-free but incomplete market is which pricing measure can be considered as the 'most natural' choice. A possible approach to answer this question consists of searching for the element in the set of all feasible

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martingale measures that is 'closest' to the physical or real-world probability measure $\mathbb{P}$, where closeness is expressed in terms of relative entropy, see Frittelli (1995) and Frittelli (2000). The corresponding pricing measure is usually called the Minimal Entropy Martingale Measure (hereafter often referred to as the *entropy measure*). It is well-known that in a one-period setting, an entropy measure can be interpreted in terms of an Esscher transform of $\mathbb{P}$. These transforms, which were introduced in Esscher (1932), have a long history in actuarial pricing. They have also been used by several authors to define pricing measures in incomplete markets, see e.g. Bühlmann et al. (1996) and Gerber and Shiu (1994).

The Minimal Entropy Martingale Measure is also related to the Esscher-Girsanov transform introduced by Goovaerts and Laeven (2008). In a one-period setting, the (two-parameter) Esscher-Girsanov transform may agree with the (one-parameter) so-called Wang transform (distortion), which has gained some popularity among actuarial practitioners (Labuschagne and Offwood (2010)). However, the two transforms are not equivalent, a fact that becomes most apparent in a dynamic setting, in which the two parameters of the Esscher-Girsanov transform start to play a distinct role: while the two-parameter Esscher-Girsanov transform can generate arbitrage-free prices for financial derivatives driven by general diffusion processes, as shown by Goovaerts and Laeven (2008) and emphasized by Badescu et al. (2009), this is not true for the one-parameter Wang transform; see also Pelsser (2008). Goovaerts and Laeven (2008) also show that independence under the real-world probability measure $\mathbb{P}$ naturally translates into comonotonicity of the Esscher transform with random parameter, thanks to the independent additivity property of the Esscher transform.

Determining the Minimal Entropy Martingale Measure boils down to a relative entropy minimisation under linear constraints. Such a minimisation problem arises in various disciplines, see e.g. Cherny and Maslov (2003). In Kullback and Leibler (1951), relative entropy is interpreted in terms of the expected amount of information given by a set of observations for distinguishing between two potential probability distributions, known as the Kullback-Leibler divergence measure. In the insurance literature, this interpretation is considered e.g. in Brockett (1991). In a financial context, there exists a duality relationship between maximization of expected exponential utility and minimization of entropy, see Frittelli (2000).

The assumption of independence between financial and actuarial risks under the real-world measure $\mathbb{P}$ may be quite reasonable in many situations. The conditions under which it is possible (or not) to transfer the independence assumption from $\mathbb{P}$ to $\mathbb{Q}$, have been investigated in Dhaene et al. (2013). In the current paper, we go one step further by exploring whether a $\mathbb{P}$-world (in-)dependence between financial and actuarial risks is maintained or not under the entropy measure. As far as we are aware, in the literature no attention has been given to this problem.

Hereafter, we will confine ourselves to a one-period, finite state market model. From a technical point of view, such an approach is simple and hence, allows us to concentrate on the key message, without being distracted by analytical details. In order to make this paper sufficiently self-contained, we will repeat some known results on relative entropy.

2 The combined financial-actuarial world and its subworlds

In this section, we introduce a single period world, which is home to a market of traded assets. The payoffs of these assets can be described in terms of random variables (r.v.'s), defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. Here, the universe Ω is given by

$$\Omega = \{(i, j) \mid i = 1, ..., I \text{ and } j = 1, ..., J\},$$

where any (i, j) corresponds to a possible state of the combined financial-actuarial world at the end of the observation period $[0, 1]$. The financial substate is given by $i \in \{1, ..., I\}$ and indicates a possible scenario concerning the evolution of the financial subworld over the time interval under consideration. As an example, each i could represent a set of possible outcomes of the prices at time 1 of a given number of stocks. The actuarial substate is characterized by $j \in \{1, ..., J\}$, where j describes a possible scenario of the actuarial subworld. Each j could identify e.g. a possible number of survivors at time 1 from a given closed population observed at time 0. The σ-algebra Σ is the set of all subsets of Ω and represents all events that may or may not occur in the coming year. Probabilities for these events follow from the real-world probability measure $\mathbb{P}$, which is characterized by

$$\mathbb{P}[(i, j)] = p_{ij} \geq 0, \qquad \text{for } i = 1, ..., I \text{ and } j = 1, ..., J.$$

Remark that we allow some probabilities p_{ij} to be equal to 0, in order to be able to include e.g. the combined scenario (i, j) with strictly positive probability $p_{ij} > 0$, whereas the combined scenario (i, j') with $j' \neq j$ has related probability $p_{ij'} = 0$.

We assume that the combined financial-actuarial world $(\Omega, \Sigma, \mathbb{P})$ is home to a market of $M + 1$ traded assets, denoted by $0, 1, \ldots, M$. The price (or the payoff) at time 1 of each traded asset is given by a r.v. defined on (Ω, Σ). We will consider assets of which the payoff at time 1 depends on both the financial and the actuarial scenario that will unfold. The current price of asset $m \in \{0, 1, 2, \ldots, M\}$, is denoted by $s^{(m)}(0) > 0$, whereas its payoff at time 1 is denoted by $S^{(m)}(1)$. The possible outcomes of $S^{(m)}(1)$ are denoted by $s_{ij}^{(m)} \geq 0$, for $i = 1, \ldots, I$; $j = 1, \ldots, J$, where $s_{ij}^{(m)}$ is the outcome in case (i, j) is the financial-actuarial scenario that unfolds. Notice that we allow different scenarios to lead to the same value of $S^{(m)}(1)$ at time 1, which implies that $\mathbb{P}\left[S^{(m)}(1) = s_{ij}^{(m)}\right] \geq p_{ij}$. Each asset m is characterized by the stochastic process $\left(s^{(m)}(0),\ S^{(m)}(1)\right)$ defined on (Ω, Σ).

Throughout the paper, we will assume that the market of traded assets is perfectly liquid and frictionless (no transaction costs, no trading constraints). We will also assume that the $M + 1$ assets are non-redundant, which means that there exists no vector $\left(a^{(0)}, a^{(1)}, \ldots, a^{(M)}\right)$ of real numbers such that

$$\mathbb{P}\left[\sum_{m=0}^{M} a^{(m)}\ S^{(m)}(1) = 0\right] = 1.$$

Equivalently, the non-redundancy assumption can be stated as follows: there exists no vector $\left(a^{(0)}, a^{(1)}, \ldots, a^{(M)}\right)$ such that

$$\sum_{m=0}^{M} a^{(m)} \; s_{ij}^{(m)} = 0, \qquad \text{for all } \; (i,j) \; \text{ with } p_{ij} > 0.$$

The combined world is assumed to be home to a single bank account with (continuously compounded) deterministic and constant interest rate $r \geq 0$. By convention, asset 0 is the corresponding risk-free zero coupon bond with $s^{(0)}(0) = 1$ and $S^{(0)}(1) = e^r$.

A particular asset $m \in \{0, 1, 2, \ldots, M\}$, is called a *financial asset* in case the following condition holds:

$$s_{ij}^{(m)} = s_{ij'}^{(m)} \text{ for all } j \text{ and } j' \text{ in } \{1, \ldots, J\}.$$

This means that the payoff at time 1 of a financial asset does not depend on the actuarial scenario that unfolds. Hereafter, the possible outcomes of the payoff of financial asset m will be denoted by $s_{i\cdot}^{(m)}$, for $i = 1, \ldots, I$.

Similarly, an asset $m \in \{0, 1, 2, \ldots, M\}$, is said to be an *actuarial asset* in case

$$s_{ij}^{(m)} = s_{i'j}^{(m)} \text{ for all } i \text{ and } i' \text{ in } \{1, \ldots, I\},$$

which means that its payoff does not depend on the financial scenario that will unfold. The possible outcomes of the payoff of actuarial asset m are denoted by $s_{\cdot j}^{(m)}$, for $j = 1, \ldots, J$.

Remark that the risk-free bond (asset 0) is the only asset that can be considered as a financial asset as well as an actuarial asset.

Starting from the combined financial-actuarial world $(\Omega, \Sigma, \mathbb{P})$, we define the financial subworld $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma), \mathcal{F}(\mathbb{P}))$. The financial universe $\mathcal{F}(\Omega)$ is given by

$$\mathcal{F}(\Omega) = \{i \mid i = 1, \ldots, I\},$$

where each i indicates a possible scenario concerning the evolution of the financial world over the coming year. The σ-algebra $\mathcal{F}(\Sigma)$, which is defined as the set of all subsets of $\mathcal{F}(\Omega)$, represents all financial events that may or may not occur in the coming year. Probabilities for these financial events follow from the real-world probability measure $\mathcal{F}(\mathbb{P})$, which is the *projection* of the combined real-world probability measure $\mathbb{P}$ to the financial subworld:

$$\mathcal{F}(\mathbb{P})[i] = \sum_{j=1}^{J} p_{ij} = p_{i\cdot} \geq 0, \qquad \text{for } i = 1, ..., I. \tag{1}$$

Similar to the financial subworld, we describe the actuarial subworld by the probability space $(\mathcal{A}(\Omega), \mathcal{A}(\Sigma), \mathcal{A}(\mathbb{P}))$. The actuarial universe $\mathcal{A}(\Omega)$ is given by

$$\mathcal{A}(\Omega) = \{j \mid j = 1, \ldots, J\},$$

and the probability measure $\mathcal{A}(\mathbb{P})$, which is the *projection* of $\mathbb{P}$ to the actuarial subworld, attaches a real-world probability to each event in the actuarial subworld:

$$\mathcal{A}(\mathbb{P})[j] = \sum_{i=1}^{I} p_{ij} = p_{\cdot j} \geq 0, \qquad \text{for } j = 1, ..., J. \tag{2}$$

Until here, we described the price processes of the $M+1$ traded assets via stochastic processes in the combined world $(\Omega, \Sigma, \mathbb{P})$. The price process of a financial asset $m \in \{0, \ldots, M\}$ can as well be described by the stochastic process $\left(s^{(m)}(0),\ S^{(m)}(1)\right)$ defined on the financial subworld $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma))$. Here, $S^{(m)}(1)$ is a random variable on $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma), \mathcal{F}(\mathbb{P}))$ with an outcome given by $s_{i\cdot}^{(m)} \geq 0$ in case $i \in \{1, \ldots, I\}$ is the financial scenario that unfolds. Observe that different financial scenarios may eventually lead to the same outcome $S^{(m)}(1)$ of the financial asset, implying that $\mathcal{F}(\mathbb{P})\left[S^{(m)}(1) = s_{i\cdot}^{(m)}\right] \geq p_{i\cdot}$.

Similarly, the price process of an actuarial asset $m \in \{0, \ldots, M\}$ can be described by the stochastic process $\left(s^{(m)}(0),\ S^{(m)}(1)\right)$ which is defined on the actuarial subworld $(\mathcal{A}(\Omega), \mathcal{A}(\Sigma))$.

Hereafter, we will often (but not always) assume that financial and actuarial risks are independent under the real-world probability measure $\mathbb{P}$, in the sense that

$$\mathbb{P} \equiv \mathcal{F}(\mathbb{P}) \times \mathcal{A}(\mathbb{P}). \tag{3}$$

This assumption can also be expressed as

$$p_{ij} = p_{i\cdot} \times p_{\cdot j}, \qquad \text{for all } i = 1, ..., I \text{ and } j = 1, ..., J,$$

where the marginal probabilities $p_{i\cdot}$ and $p_{\cdot j}$ are the financial and actuarial real-world probabilities introduced in (1) and (2), respectively.

3 Pricing traded assets

Consider the combined world $(\Omega, \Sigma, \mathbb{P})$ which is home to a market of $M+1$ traded assets as defined above. A probability measure $\mathbb{Q}$ defined on (Ω, Σ) is said to be an *equivalent martingale measure* (or a risk-neutral measure) for this market if it fulfills the following conditions:

(1) $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures.

(2) The future payoff of any traded asset in the combined world, discounted at the risk-free interest rate, is a martingale with respect to $\mathbb{Q}$.

The equivalence condition means that $\mathbb{P}$ and $\mathbb{Q}$ agree on zero-probability events or, equivalently, they agree on the elements (i, j) of Ω with a strictly positive probability. The

$\mathbb{Q}$-martingale condition states that the current price of any traded asset in the combined market is equal to the expected value of the discounted payoff of this asset at time 1, where discounting is performed at the risk-free interest rate r and expectations are taken with respect to the measure $\mathbb{Q}$.

A probability measure $\mathbb{Q}$ defined on (Ω, Σ) is said to be $\mathbb{P}$-absolutely continuous in case $p_{ij} = 0$ implies $q_{ij} = 0$, for all (i, j) of Ω. A $\mathbb{P}$-*absolutely continuous martingale measure* is defined as a measure satisfying the conditions (1') and (2), with

(1') $\mathbb{Q}$ is $\mathbb{P}$-absolutely continuous.

It is well-known that in our discrete setting, the no-arbitrage condition is equivalent to the existence of a (not necessarily unique) equivalent martingale measure, whereas completeness of the arbitrage-free market is equivalent to the existence of a unique equivalent martingale measure, see e.g. Shiryaev et al. (1994). Hereafter, we will always assume that the market of traded assets in the combined world (Ω, Σ) is arbitrage-free, implying that there exists at least one equivalent martingale measure.

For a given equivalent martingale measure $\mathbb{Q}$ in the combined world, we introduce the following notation:

$$\mathbb{Q}\left[(i,j)\right] = q_{ij} \geq 0, \qquad \text{for } i = 1, ..., I \text{ and } j = 1, ..., J.$$

Notice that $q_{ij} = 0$ if and only if $p_{ij} = 0$. The equivalent martingale measure $\mathbb{Q}$ gives rise to the following probability measures for the financial and the actuarial subworld, respectively:

$$\mathcal{F}\left(\mathbb{Q}\right)[i] = \sum_{j=1}^{J} q_{ij} = q_{i\cdot} \geq 0, \qquad \text{for } i = 1, ..., I,$$

and

$$\mathcal{A}\left(\mathbb{Q}\right)[j] = \sum_{i=1}^{I} q_{ij} = q_{\cdot j} \geq 0, \qquad \text{for } j = 1, ..., J.$$

The measures $\mathcal{F}\left(\mathbb{Q}\right)$ and $\mathcal{A}\left(\mathbb{Q}\right)$ are called the *projections* of $\mathbb{Q}$ to the financial and the actuarial subworld, respectively. Based on these projections, we introduce the probability measure $\mathcal{F}\left(\mathbb{Q}\right) \times \mathcal{A}\left(\mathbb{Q}\right)$ on the combined measurable space (Ω, Σ). In terms of the notations introduced above, it is defined by

$$\left(\mathcal{F}\left(\mathbb{Q}\right) \times \mathcal{A}\left(\mathbb{Q}\right)\right)\left[(i,j)\right] = q_{i\cdot} \times q_{\cdot j}, \qquad \text{for } i = 1, ..., I \text{ and } j = 1, ..., J.$$

Financial and actuarial risks are said to be independent under the measure $\mathbb{Q}$ if the following condition holds:

$$\mathbb{Q} \equiv \mathcal{F}\left(\mathbb{Q}\right) \times \mathcal{A}\left(\mathbb{Q}\right), \tag{4}$$

or equivalently,

$$q_{ij} = q_{i\cdot} \times q_{\cdot j}, \qquad \text{for all } i = 1, ..., I \text{ and } j = 1, ..., J.$$

Until here, we considered equivalent martingale measures in the combined world $(\Omega, \Sigma, \mathbb{P})$, which is home to a market of assets with financial and/or actuarial payoffs. We can as well restrict to the financial subworld $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma), \mathcal{F}(\mathbb{P}))$ and the corresponding submarket of financial assets, and define equivalent martingale measures in this subworld. Similarly, the notion of equivalent martingale measure can be defined in the actuarial subworld $(\mathcal{A}(\Omega), \mathcal{A}(\Sigma), \mathcal{A}(\mathbb{P}))$ and the corresponding actuarial submarket of actuarial assets.

Consider a combined world $(\Omega, \Sigma, \mathbb{P})$ with a corresponding market of traded assets and let $\mathbb{Q}$ be an equivalent martingale measure in this world. The projection $\mathcal{F}(\mathbb{Q})$ of $\mathbb{Q}$ is an equivalent martingale measure in the financial subworld with the corresponding submarket of traded financial assets. A similar remark holds for the projection $\mathcal{A}(\mathbb{Q})$ of $\mathbb{Q}$ in the actuarial subworld. This means that our assumption about an arbitrage-free pricing framework in the combined market implies that also the financial and actuarial submarkets are arbitrage-free. In general, $\mathbb{P}$ and $\mathcal{F}(\mathbb{Q}) \times \mathcal{A}(\mathbb{Q})$ do not necessarily agree on sure events and moreover, $\mathcal{F}(\mathbb{Q}) \times \mathcal{A}(\mathbb{Q})$ is not necessarily a martingale measure in the combined world. In the special case that $\mathbb{P}$ fulfills the independence assumption (3), we have that $\mathbb{P}$ and $\mathcal{F}(\mathbb{Q}) \times \mathcal{A}(\mathbb{Q})$ are equivalent measures, but the latter measure is still not necessarily a martingale measure in the combined world. For details and examples, we refer to Dhaene, Kukush, Luciano, Schoutens & Stassen (2013).

4 The minimal entropy martingale measures of the combined market and its submarkets

Due to the presence of unhedgeable actuarial and financial risk, the market of traded contingent claims in the combined financial-actuarial world is in general incomplete, implying the existence of more than one equivalent martingale measure for pricing purposes. The non-uniqueness of the pricing measure means that there is no unique arbitrage-free price for non-replicable contingent claims. Hereafter, we investigate the problem of finding the martingale measure that is 'closest' to the real-world probability measure $\mathbb{P}$, where the distance between probability measures is defined in terms of their relative entropy, also called the Kullback-Leibler information. In the remainder of this section, we first determine the Minimal Entropy Martingale Measure $\widehat{\mathbb{Q}}$ of the combined market. Next, we determine the Minimal Entropy Martingale Measures $\widehat{\mathbb{Q}}^f$ and $\widehat{\mathbb{Q}}^a$ corresponding to the financial and the actuarial submarket, respectively. Finally, we investigate the relationship that exists between these measures.

4.1 The entropy measure of the combined market

Consider the combined world $(\Omega, \Sigma, \mathbb{P})$ with the market of $M+1$ traded assets as described above. In this section, we determine the Minimal Entropy Martingale Measure $\widehat{\mathbb{Q}}$ in the most general case, which means that we consider a market where financial, actuarial as well as combined assets may be traded. First, we define the relative entropy of an absolutely continuous probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$.

Definition 1 *Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures defined on the combined financial-actuarial world (Ω, Σ). Furthermore, $\mathbb{Q}$ is $\mathbb{P}$-absolutely continuous. The relative entropy $E(\mathbb{Q}, \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ is then defined by*

$$E(\mathbb{Q}, \mathbb{P}) = \sum_{i,j} q_{ij} \ln \left(\frac{q_{ij}}{p_{ij}} \right),$$

where the sum is taken over all $(i, j) \in \Omega$ with $p_{ij} > 0$, and where $0 \ln 0 = 0$, by convention.

Loosely speaking, the value of $E(\mathbb{Q}, \mathbb{P})$ increases if $\mathbb{Q}$ and $\mathbb{P}$ 'diverge'. Therefore, $E(\mathbb{Q}, \mathbb{P})$ measures the 'similarity' or 'closeness' of the respective probability measures and hence, it can be thought of as a kind of 'distance'. Notice however that the relative entropy is not symmetric, i.e. $E(\mathbb{Q}, \mathbb{P}) \neq E(\mathbb{P}, \mathbb{Q})$, implying that it is not a distance in the usual mathematical sense. Relative entropy has many relevant features. It is always non-negative and it equals zero if and only if the two measures are identical, see e.g. Frittelli (2000).

Based on the notion of relative entropy, we now introduce the notion of Minimal Entropy Martingale Measure in the combined financial-actuarial world, as the particular element in the class of equivalent martingale measures for which the relative entropy is minimised.

Definition 2 *Consider the combined financial-actuarial world $(\Omega, \Sigma, \mathbb{P})$ which is home to the market of traded assets $\{0, 1, \ldots, M\}$. Let $\mathcal{M}$ be the class of all equivalent martingale measures in the combined market. Then $\widehat{\mathbb{Q}} \in \mathcal{M}$ is a Minimal Entropy Martingale Measure of the combined market if it satisfies*

$$E\left(\widehat{\mathbb{Q}}, \mathbb{P}\right) = \min_{\mathbb{Q} \in \mathcal{M}} E(\mathbb{Q}, \mathbb{P}) = \min_{\mathbb{Q} \in \mathcal{M}} \sum_{i,j} q_{ij} \ln \left(\frac{q_{ij}}{p_{ij}} \right). \tag{5}$$

Any $\mathbb{Q} \in \mathcal{M}$ can be characterized by an $I \times J$ - matrix with non-negative components q_{ij}, with $q_{ij} = 0$ if and only if $p_{ij} = 0$, and which satisfy the following conditions:

$$e^{-r} \, \mathbb{E}^{\mathbb{Q}} \left[S^{(m)}(1) \right] = s^{(m)}(0), \qquad \text{for } m = 0, 1, \ldots, M,$$

or, equivalently,

$$e^{-r} \sum_{i,j} q_{ij} \, s_{ij}^{(m)} = s^{(m)}(0), \qquad \text{for } m = 0, 1, \ldots, M, \tag{6}$$

where as before, the sum is taken over all $(i, j) \in \Omega$ with $p_{ij} > 0$. Obviously, the condition for $m = 0$ corresponds to the condition that the probabilities q_{ij} sum up to 1.

Notice that restricting the set of available assets to an appropriate subset of non-redundant assets does not change the class $\mathcal{M}$ of all equivalent martingale measures, implying that also the set of solutions of (5) remains unchanged by this operation.

In the next theorem, we prove that the minimal entropy martingale measure always exists and is unique. Hereafter, we will often call this measure the *combined market entropy measure.* The proof is based on Theorem 2.2 of Frittelli (2000).

Theorem 3 *The arbitrage-free combined market is home to a unique minimal entropy martingale measure.*

Proof: Let $\widehat{\mathcal{M}} \supseteq \mathcal{M}$ be the class of all $\mathbb{P}$-absolutely continuous martingale measures and consider the following minimisation problem:

$$\min_{\mathbb{Q}\in\widehat{\mathcal{M}}} \sum_{i,j} q_{ij} \ln\left(\frac{q_{ij}}{p_{ij}}\right). \tag{7}$$

The no-arbitrage assumption implies that the set $\mathcal{M}$, and thus also $\widehat{\mathcal{M}}$, is not empty. Furthermore, $\widehat{\mathcal{M}}$ is a closed and bounded set in $\mathbb{R}^{I\times J}$, while $\sum_{i,j} q_{ij} \ln\left(\frac{q_{ij}}{p_{ij}}\right)$ is continuous on $\widehat{\mathcal{M}}$. Hence, the objective function in (7) reaches a minimum in the set $\widehat{\mathcal{M}}$. The uniqueness of this minimum follows from the fact that $x \to x \ln\left(\frac{x}{p_{ij}}\right)$ is strictly convex on $[0,1]$ for any $p_{ij} > 0$. Let us denote this minimum by $\widehat{\mathbb{Q}}$.
It remains to prove that $\widehat{\mathbb{Q}} \in \mathcal{M}$. Hence, we have to prove that $p_{ij} > 0$ implies $\widehat{q}_{ij} > 0$, for any i and j. The no-arbitrage condition implies that $\mathcal{M}$ contains at least one element $\mathbb{Q}^e$. Consider the convex combination

$$\mathbb{Q}^x = x\ \mathbb{Q}^e + (1-x)\ \widehat{\mathbb{Q}}$$

with $x \in [0,1]$. Obviously, any $\mathbb{Q}^x \in \widehat{\mathcal{M}}$. The probabilities of $\mathbb{Q}^x$ are given by

$$q_{ij}^x = x\ q_{ij}^e + (1-x)\ \widehat{q}_{ij} = \widehat{q}_{ij} + x\ \left(q_{ij}^e - \widehat{q}_{ij}\right).$$

For $x > 0$, the derivative of the relative entropy $E\left(\mathbb{Q}^x, \mathbb{P}\right)$ with respect to x is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} E\left(\mathbb{Q}^x, \mathbb{P}\right) = \sum_{i,j} \left(q_{ij}^e - \widehat{q}_{ij}\right) \ln\left(\frac{q_{ij}^x}{p_{ij}}\right).$$

This leads to

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}x} E\left(\mathbb{Q}^x, \mathbb{P}\right)\right|_{x=0} &= \sum_{i,j} \left(q_{ij}^e - \widehat{q}_{ij}\right) \ln\left(\frac{\widehat{q}_{ij}}{p_{ij}}\right) \\ &= \sum_{i,j} q_{ij}^e \ln\left(\frac{\widehat{q}_{ij}}{p_{ij}}\right) - E\left(\widehat{\mathbb{Q}}, \mathbb{P}\right). \end{aligned}$$

As $\mathbb{Q}^0 \equiv \widehat{\mathbb{Q}}$, which is the unique minimum of optimisation problem (7), we must have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} E\left(\mathbb{Q}^x, \mathbb{P}\right)\right|_{x=0} \geq 0,$$

or equivalently,

$$E\left(\widehat{\mathbb{Q}}, \mathbb{P}\right) \leq \sum_{i,j} q_{ij}^e \ln\left(\frac{\widehat{q}_{ij}}{p_{ij}}\right). \tag{8}$$

Suppose now that $\widehat{\mathbb{Q}}$ is not equivalent to $\mathbb{P}$. In that case, there exists a scenario (i,j) such that $p_{ij} > 0$, while $\widehat{q}_{ij} = 0$. This implies that the right hand side of (8) reaches $-\infty$, which is impossible as relative entropy is always non-negative. Hence, $\widehat{\mathbb{Q}} \in \mathcal{M}$. We conclude that the minimal entropy martingale measure $\widehat{\mathbb{Q}}$ exists and is unique. ■

Hereafter, we will always denote the unique minimal entropy martingale measure by $\widehat{\mathbb{Q}}$. Since for any element of the set $\mathcal{M}$, the q_{ij} sum up to 1, we can replace the minimisation problem (5) by

$$\min_{\mathbb{Q}\in\mathcal{M}} \sum_{i,j} q_{ij} \left(\ln\left(\frac{q_{ij}}{p_{ij}}\right) - 1\right),$$

which leads to the same entropy measure $\widehat{\mathbb{Q}}$ for the combined market.

We solve the adapted optimisation problem under linear constraints by the method of Lagrange multipliers. Remark that we can apply this method on the class of equivalent martingale measures $\mathcal{M}$, which is an open set, provided the minimum exists. The existence of the minimal entropy martingale measure was proven in Theorem 3. The Lagrangian L for this problem is now given by

$$L = \sum_{i,j} q_{ij} \left(\ln\left(\frac{q_{ij}}{p_{ij}}\right) - 1\right) - \sum_{m=0}^{M} \lambda^{(m)} \left(\sum_{i,j} q_{ij}\ s_{ij}^{(m)} - e^r\ s^{(m)}(0)\right).$$

Determining the partial derivatives with respect to the variables q_{ij} and $\lambda^{(m)}$, and setting them equal to zero, leads to the following system of equations:

$$\begin{cases} \ln\left(\dfrac{q_{ij}}{p_{ij}}\right) = \displaystyle\sum_{m=0}^{M} \lambda^{(m)}\ s_{ij}^{(m)}, & \text{for all } (i,j) \in \Omega \text{ with } p_{ij} > 0, \\ \displaystyle\sum_{i,j} q_{ij}\ s_{ij}^{(m)} = e^r\ s^{(m)}(0), & \text{for all } m = 0, 1, \ldots, M. \end{cases} \tag{9}$$

Let us denote the probabilities related to the unique entropy measure $\widehat{\mathbb{Q}}$ by $(\widehat{q}_{ij};\ (i,j) \in \Omega)$. Taking into account the first series of equations in (9), as well as the fact that $\widehat{\mathbb{Q}}$ and $\mathbb{P}$ are equivalent, we find that the probabilities $\widehat{q}_{ij}$ can be expressed as

$$\widehat{q}_{ij} = p_{ij}\ e_{ij}, \quad \text{for any } (i,j) \in \Omega, \tag{10}$$

with the coefficients e_{ij} given by

$$e_{ij} = \exp\left(\sum_{m=0}^{M} \lambda^{(m)}\ s_{ij}^{(m)}\right), \quad \text{for any } (i,j) \in \Omega, \tag{11}$$

where $\left(\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(M)}\right)$ satisfies (9). As the condition for $m = 0$ corresponds to the condition that the probabilities $\widehat{q}_{ij}$ sum up to 1, we can conclude from (10) and (11) that $0 \leq \widehat{q}_{ij} \leq 1$ holds for any $(i,j) \in \Omega$. The projections of the entropy measure $\widehat{\mathbb{Q}}$ to the financial and the actuarial subworld are characterized as follows

$$\widehat{q}_{i\cdot} = \sum_{j=1}^{J} p_{ij}\ e_{ij}, \quad \text{for } i = 1, 2, \ldots, I, \tag{12}$$

and

$$\widehat{q}_{\cdot j} = \sum_{i=1}^{I} p_{ij}\ e_{ij}, \quad \text{for } j = 1, 2, \ldots, J, \tag{13}$$

respectively.

In order to determine the Lagrange multipliers $\lambda^{(m)}$, we combine the martingale conditions in (9) with the expressions (10). We find that the $\lambda^{(m)}$ follow from

$$\sum_{i,j} p_{ij}\ e_{ij}\ s_{ij}^{(m)} = e^r\ s^{(m)}(0), \qquad \text{for all } m = 0, 1, \ldots, M. \tag{14}$$

Our assumption about the non-redundancy of the set of assets implies that these *martingale equations* lead to a unique vector of Lagrange multipliers. Indeed, suppose that (14) admits two different solutions $\left\{\lambda_k^{(m)} \mid m = 0, 1 \ldots, M\right\}$, for $k = 1, 2$. Taking into account (10) and (11), and the fact that the $\widehat{q}_{ij}$ are uniquely determined, we find that

$$\exp\left(\sum_{m=0}^{M} \lambda_1^{(m)}\ s_{ij}^{(m)}\right) = \exp\left(\sum_{m=0}^{M} \lambda_2^{(m)}\ s_{ij}^{(m)}\right), \quad \text{for any } (i,j) \text{ with } p_{ij} > 0,$$

and thus

$$\sum_{m=0}^{M} \left(\lambda_1^{(m)} - \lambda_2^{(m)}\right)\ s_{ij}^{(m)} = 0, \quad \text{for any } (i,j) \text{ with } p_{ij} > 0.$$

Obviously, this contradicts the non-redundancy assumption, so that we can conclude that the martingale equations lead to a unique vector of Lagrange multipliers, which we will hereafter denote by $\left\{\lambda^{(m)} \mid m = 0, 1 \ldots, M\right\}$.

From the equation (14) for $m = 0$, it follows that

$$\exp\left(\lambda^{(0)} e^r\right) \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m=1}^{M} \lambda^{(m)}\ S^{(m)}(1)\right)\right] = 1.$$

The expressions (11) for the e_{ij} can then be rewritten as

$$e_{ij} = \frac{\exp\left(\sum\limits_{m=1}^{M} \lambda^{(m)}\ s_{ij}^{(m)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum\limits_{m=1}^{M} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}, \quad \text{for any } (i,j) \in \Omega. \tag{15}$$

As stated previously, we observe from (10) and (15) that the entropy measure $\widehat{\mathbb{Q}}$ is an *equivalent* martingale measure in the combined financial-actuarial market, which can be interpreted as an Esscher-like transform of $\mathbb{P}$.

Solving (10) for p_{ij}, replacing e_{ij} by (15) and summing over all i and j leads to the following expression:

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\exp\left(-\sum_{m=1}^{M} \lambda^{(m)}\ S^{(m)}(1)\right)\right] \times \mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m=1}^{M} \lambda^{(m)}\ S^{(m)}(1)\right)\right] = 1. \tag{16}$$

So we find that

$$e_{ij}^{-1} = \frac{\exp\left(-\sum\limits_{m=1}^{M} \lambda^{(m)}\ s_{ij}^{(m)}\right)}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\exp\left(-\sum\limits_{m=1}^{M} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}, \quad \text{for any } (i,j) \in \Omega. \tag{17}$$

These expressions will be used hereafter.

4.2 The entropy measure of the financial submarket

In Definition 2, we considered the Minimal Entropy Martingale Measure $\widehat{\mathbb{Q}}$ for the market of $M+1$ traded assets in the combined financial-actuarial world. Similarly, we can define the Minimal Entropy Martingale Measure $\widehat{\mathbb{Q}}^f$ for the submarket of financial assets.

Consider the $\mathcal{F}(\mathbb{P})$-absolutely continuous probability measure $\mathbb{Q} \equiv (q_1, q_2, \ldots, q_I)$ on $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma))$. The relative entropy $E(\mathbb{Q}, \mathcal{F}(\mathbb{P}))$ of $\mathbb{Q}$ with respect to $\mathcal{F}(\mathbb{P})$ is defined as follows:

$$E(\mathbb{Q}, \mathcal{F}(\mathbb{P})) = \sum_i q_i \ln\left(\frac{q_i}{p_{i\cdot}}\right),$$

with summation over all i with $p_{i\cdot} > 0$, and where $0 \ln 0 = 0$, by convention.

We denote the subset of $\{0, 1, \ldots, M\}$ composed of all financial assets, by $\mathcal{N}^f$. The set of all financial assets with exception of the risk-free bond is called the set of *purely financial assets* and will be denoted by $\mathcal{N}_0^f$. Let $\mathcal{M}^f$ be the class of all equivalent martingale measures $\mathbb{Q}$ in the financial submarket. Any $\mathbb{Q} \in \mathcal{M}^f$ can be characterized by a vector $(q_1, q_2, \ldots, q_I)$ with non-negative components and $q_i = 0$ if and only if $p_{i\cdot} = 0$, satisfying the following conditions:

$$e^{-r} \sum_i q_i\ s_{i\cdot}^{(m)} = s^{(m)}(0), \qquad \text{for all } m \in \mathcal{N}^f. \tag{18}$$

The no-arbitrage assumption for the combined world implies that also the financial subworld is arbitrage-free and hence, $\mathcal{M}^f$ is non-empty.

The measure $\widehat{\mathbb{Q}}^f \in \mathcal{M}^f$ is the Minimal Entropy Martingale Measure of this financial submarket if it satisfies

$$E\left(\widehat{\mathbb{Q}}^f, \mathcal{F}(\mathbb{P})\right) = \min_{\mathbb{Q}\in\mathcal{M}^f} E\left(\mathbb{Q}, \mathcal{F}(\mathbb{P})\right) = \min_{\mathbb{Q}\in\mathcal{M}^f} \sum_i q_i \ln\left(\frac{q_i}{p_{i\cdot}}\right). \tag{19}$$

Based on similar arguments as used for the combined market, we have that in the financial submarket, the minimisation problem (19) always leads to a unique solution. Hereafter, we will often call this unique measure the *financial market entropy measure*.

Since $\sum_i q_i = 1$ for all measures of the set $\mathcal{M}^f$, we can replace the minimisation problem (19) by the equivalent minimisation problem

$$\min_{\mathbb{Q}\in\mathcal{M}^f} \sum_i q_i \left(\ln\left(\frac{q_i}{p_{i\cdot}}\right) - 1\right).$$

Proceeding in a similar way as in the combined market case, we solve the adapted optimisation problem under linear constraints by the method of Lagrange multipliers. The Lagrangian L for this problem is given by

$$L = \sum_i q_i \left(\ln\left(\frac{q_i}{p_{i\cdot}}\right) - 1\right) - \sum_{m\in\mathcal{N}^f} \lambda_f^{(m)} \left(\sum_i q_i\ s_{i\cdot}^{(m)} - e^r\ s^{(m)}(0)\right).$$

Determining the partial derivatives with respect to the variables q_i and $\lambda_f^{(m)}$ and setting them equal to zero, leads to the following system of equations:

$$\begin{cases} \ln\left(\dfrac{q_i}{p_{i\cdot}}\right) = \displaystyle\sum_{m\in\mathcal{N}^f} \lambda_f^{(m)}\ s_{i\cdot}^{(m)}, & \text{for all } i \text{ with } p_{i\cdot} > 0, \\ \displaystyle\sum_i q_i\ s_{i\cdot}^{(m)} = e^r\ s^{(m)}(0), & \text{for all } m \in \mathcal{N}^f. \end{cases} \tag{20}$$

Let us denote the probabilities related to the financial market entropy measure $\widehat{\mathbb{Q}}^f$ by $\left(\widehat{q}_1^f, \widehat{q}_2^f, \ldots, \widehat{q}_I^f\right)$. Taking into account the first series of equations in (20), as well as the fact that $\widehat{\mathbb{Q}}^f$ and $\mathbb{P}$ are equivalent, we find that the probabilities $\widehat{q}_i^f$ can be expressed as

$$\widehat{q}_i^f = p_{i\cdot}\ e_i^f, \qquad \text{for } i = 1, 2, \ldots, I, \tag{21}$$

with

$$e_i^f = \exp\left(\sum_{m\in\mathcal{N}^f} \lambda_f^{(m)}\ s_{i\cdot}^{(m)}\right), \qquad \text{for } i = 1, 2, \ldots, I, \tag{22}$$

where the Lagrange coefficients $\lambda_f^{(m)}$ follow from (20). Notice that $0 \leq \widehat{q}_i^f \leq 1$ holds for every i.

Combining the martingale conditions in (20) with the expression (21), we find that the $\lambda_f^{(m)}$ follow from the following system of *martingale equations*:

$$\sum_i p_{i\cdot}\ e_i^f\ s_{i\cdot}^{(m)} = e^r\ s^{(m)}(0)\,, \qquad \text{for all } m \in \mathcal{N}^f. \tag{23}$$

Because the financial submarket is non-redundant as well, a similar argument as in the combined market can be used to prove that these equations admit a unique solution.

From the martingale equation (23) for $m = 0$, we find that

$$\exp\left(\lambda_f^{(0)} e^r\right) \times \mathbb{E}^{\mathcal{F}(\mathbb{P})}\left[\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda_f^{(m)}\ S^{(m)}(1)\right)\right] = 1. \tag{24}$$

Taking into account this relation, we can rewrite the expressions (22) for the factors e_i^f as follows:

$$e_i^f = \frac{\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda_f^{(m)}\ s_{i\cdot}^{(m)}\right)}{\mathbb{E}^{\mathcal{F}(\mathbb{P})}\left[\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda_f^{(m)}\ S^{(m)}(1)\right)\right]}, \qquad \text{for } i = 1, 2, \ldots, I. \tag{25}$$

Again, we observe that the unique entropy measure $\widehat{\mathbb{Q}}^f$, which is an *equivalent* martingale measure in the financial subworld, can be interpreted as an Esscher-like transform of $\mathcal{F}(\mathbb{P})$.

4.3 The entropy measure of the actuarial submarket

Similar as in the financial submarket, we can define the Minimal Entropy Martingale Measure of the actuarial submarket.

Consider the $\mathcal{A}(\mathbb{P})$-absolutely continuous probability measure $\mathbb{Q} \equiv (q_1, q_2, \ldots, q_J)$ on $(\mathcal{A}(\Omega), \mathcal{A}(\Sigma))$. The relative entropy $E(\mathbb{Q}, \mathcal{A}(\mathbb{P}))$ of $\mathbb{Q}$ with respect to $\mathcal{A}(\mathbb{P}) \equiv (p_{\cdot 1}, p_{\cdot 2}, \ldots, p_{\cdot J})$ is defined by

$$E(\mathbb{Q}, \mathcal{A}(\mathbb{P})) = \sum_j q_j \ln\left(\frac{q_j}{p_{\cdot j}}\right),$$

where the sum is taken over all j with $p_{\cdot j} > 0$, and where $0 \ln 0 = 0$, by convention.

We introduce the notation $\mathcal{N}^a$ for the set of all actuarial assets, while $\mathcal{N}_0^a = \mathcal{N}^a \setminus \{0\}$ is the set of all *purely actuarial assets*. Furthermore, $\mathcal{M}^a$ is the non-empty class of all equivalent martingale measures $\mathbb{Q}$ in the actuarial submarket. Any $\mathbb{Q} \in \mathcal{M}^a$ can be characterized by a vector $(q_1, q_2, \ldots, q_J)$ with non-negative components and $q_j = 0$ if and only if $p_{\cdot j} = 0$, satisfying the following conditions:

$$e^{-r} \sum_j q_j\ s_{\cdot j}^{(m)} = s^{(m)}(0), \qquad \text{for all } m \in \mathcal{N}^a. \tag{26}$$

The Minimal Entropy Martingale Measure $\widehat{\mathbb{Q}}^a$ is the unique element of $\mathcal{M}^a$ which satisfies

$$E\left(\widehat{\mathbb{Q}}^a, \mathcal{A}(\mathbb{P})\right) = \min_{\mathbb{Q}\in\mathcal{M}^a} E(\mathbb{Q}, \mathcal{A}(\mathbb{P})) = \min_{\mathbb{Q}\in\mathcal{M}^a} \sum_j q_j \ln\left(\frac{q_j}{p_{\cdot j}}\right). \tag{27}$$

Hereafter, we will often call this measure the *actuarial market entropy measure.*

Denoting the probabilities related to the actuarial market entropy measure $\widehat{\mathbb{Q}}^a$ by $(\widehat{q}_1^a, \widehat{q}_2^a, \ldots, \widehat{q}_J^a)$, we find that the probabilities $\widehat{q}_j^a$ can be expressed as

$$\widehat{q}_j^a = p_{\cdot j}\ e_j^a, \qquad \text{for } j = 1, 2, \ldots, J, \tag{28}$$

with

$$e_j^a = \exp\left(\sum_{m\in\mathcal{N}^a} \lambda_a^{(m)}\ s_{\cdot j}^{(m)}\right), \qquad \text{for } j = 1, 2, \ldots, J. \tag{29}$$

The Lagrange multipliers $\lambda_a^{(m)}$ are derived from the system of *martingale equations*:

$$\sum_j p_{\cdot j}\ e_j^a\ s_{\cdot j}^{(m)} = e^r\ s^{(m)}(0), \qquad \text{for all } m \in \mathcal{N}^a, \tag{30}$$

where the uniqueness of this set of multipliers follows from the non-redundancy assumption of the actuarial submarket.

The expressions (29) can be rewritten as follows:

$$e_j^a = \frac{\exp\left(\sum_{m\in\mathcal{N}_0^a} \lambda_a^{(m)}\ s_{\cdot j}^{(m)}\right)}{\mathbb{E}^{\mathcal{A}(\mathbb{P})}\left[\exp\left(\sum_{m\in\mathcal{N}_0^a} \lambda_a^{(m)}\ S^{(m)}(1)\right)\right]}, \qquad \text{for } j = 1, 2, \ldots, J. \tag{31}$$

We observe that the unique entropy measure $\widehat{\mathbb{Q}}^a$, which is an *equivalent* martingale measure in the actuarial subworld, can be interpreted as an Esscher-like transform of $\mathcal{A}(\mathbb{P})$.

4.4 Some examples

In this subsection, we illustrate the technique of determining the minimal entropy martingale measure by considering two examples of a combined world with a risk-free zero coupon bond, a financial asset, an actuarial asset and a combined financial-actuarial asset traded in the market. For each example, we derive the combined market entropy measure $\widehat{\mathbb{Q}}$, as well as the entropy measures $\widehat{\mathbb{Q}}^f$ and $\widehat{\mathbb{Q}}^a$ of the financial and the actuarial submarket, respectively.

Example 1 Consider a combined financial-actuarial world with three possible scenarios in each subworld, i.e.

$$\Omega = \{(i, j) \mid i, j = 1, 2, 3\}.$$

Suppose that the real-world probabilities p_{ij} are given by

$$\mathbb{P} = (p_{ij})_{i,j} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & 0 \end{pmatrix}.$$

The projections $\mathcal{F}(\mathbb{P})$ and $\mathcal{A}(\mathbb{P})$ of the real-world probability measure $\mathbb{P}$ on the financial and actuarial subworlds are then given by

$$\mathcal{F}(\mathbb{P}) = \begin{pmatrix} p_{1\cdot} \\ p_{2\cdot} \\ p_{3\cdot} \end{pmatrix} = \begin{pmatrix} \frac{3}{6} \\ \frac{2}{6} \\ \frac{1}{6} \end{pmatrix} \quad \text{and} \quad \mathcal{A}(\mathbb{P}) = \begin{pmatrix} p_{\cdot 1} \\ p_{\cdot 2} \\ p_{\cdot 3} \end{pmatrix} = \begin{pmatrix} \frac{3}{6} \\ \frac{2}{6} \\ \frac{1}{6} \end{pmatrix},$$

respectively. Notice that $p_{11} \neq p_{1\cdot} \times p_{\cdot 1}$, from which we conclude that financial and actuarial risks are not independent under the physical measure $\mathbb{P}$.
We assume that the risk-free interest rate r is 0 and that the current price of the risk-free zero coupon bond is 1. In the combined market, a pure financial asset and a pure actuarial asset are traded. Their initial price is $s^{(m)}(0) = \frac{1}{2}$, for $m = 1, 2$, while the possible outcomes of their payoffs $S^{(m)}(1)$ at time 1 are given by

$$\begin{pmatrix} s_{1\cdot}^{(1)} \\ s_{2\cdot}^{(1)} \\ s_{3\cdot}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s_{\cdot 1}^{(2)} \\ s_{\cdot 2}^{(2)} \\ s_{\cdot 3}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

respectively. In this market, also a combined asset is traded, with initial price $s^{(3)}(0)$ and possible outcomes for its payoff $S^{(3)}(1)$ at time 1 given by

$$s_{ij}^{(3)} = \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The martingale equations (14) for the combined market of this example read as follows:

$$\begin{cases} e_{11} + e_{12} + e_{13} + e_{21} + e_{22} + e_{31} = 6 \\ e_{21} + e_{22} + 2e_{31} = 3 \\ e_{12} + 2e_{13} + e_{22} = 3 \\ e_{11} = 6\ s^{(3)}(0) \end{cases}$$

with the e_{ij}, according to (11), given by

$$\begin{cases} e_{11} = \exp\left(\lambda^{(0)} + \lambda^{(3)}\right) \\ e_{12} = \exp\left(\lambda^{(0)} + \lambda^{(2)}\right) \\ e_{13} = \exp\left(\lambda^{(0)} + 2\lambda^{(2)}\right) \\ e_{21} = \exp\left(\lambda^{(0)} + \lambda^{(1)}\right) \\ e_{22} = \exp\left(\lambda^{(0)} + \lambda^{(1)} + \lambda^{(2)}\right) \\ e_{31} = \exp\left(\lambda^{(0)} + 2\lambda^{(1)}\right). \end{cases}$$

The Lagrange multipliers $\lambda^{(m)}$, for $m = 0, 1, 2, 3$, then follow from

$$\begin{cases} \lambda^{(0)} = \ln\left(\frac{\left(3-6\ s^{(3)}(0)\right)^2}{2\ s^{(3)}(0)}\right) \\ \lambda^{(1)} = \ln\left(\frac{2\ s^{(3)}(0)}{3-6\ s^{(3)}(0)}\right) \\ \lambda^{(2)} = \ln\left(\frac{2\ s^{(3)}(0)}{3-6\ s^{(3)}(0)}\right) \\ \lambda^{(3)} = \ln\left(\frac{12\left(s^{(3)}(0)\right)^2}{\left(3-6\ s^{(3)}(0)\right)^2}\right). \end{cases}$$

The probabilities $\widehat{q}_{ij}$, which determine the combined market entropy measure $\widehat{\mathbb{Q}}$, are calculated by equation (10):

$$\widehat{\mathbb{Q}} = \left(\widehat{q}_{ij}\right)_{i,j} = \begin{pmatrix} s^{(3)}(0) & \frac{1}{2} - s^{(3)}(0) & \frac{s^{(3)}(0)}{3} \\ \frac{1}{2} - s^{(3)}(0) & \frac{s^{(3)}(0)}{3} & 0 \\ \frac{s^{(3)}(0)}{3} & 0 & 0 \end{pmatrix}. \tag{32}$$

From this matrix, we see that $s^{(3)}(0) \in \left(0, \frac{1}{2}\right)$ is required in order to guarantee that $\widehat{\mathbb{Q}}$ is a proper equivalent martingale measure in the combined market. This condition on the initial price of the combined asset is a necessary and sufficient condition for the combined market to be arbitrage-free. Therefore, in the remainder of this example we assume that $s^{(3)}(0) \in \left(0, \frac{1}{2}\right)$.

The projections $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ and $\mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ of $\widehat{\mathbb{Q}}$ on the financial and the actuarial subworld can easily be determined from (32):

$$\mathcal{F}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{1\cdot} \\ \widehat{q}_{2\cdot} \\ \widehat{q}_{3\cdot} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{s^{(3)}(0)}{3} \\ \frac{1}{2} - \frac{2s^{(3)}(0)}{3} \\ \frac{s^{(3)}(0)}{3} \end{pmatrix} \text{ and } \mathcal{A}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{\cdot 1} \\ \widehat{q}_{\cdot 2} \\ \widehat{q}_{\cdot 3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{s^{(3)}(0)}{3} \\ \frac{1}{2} - \frac{2s^{(3)}(0)}{3} \\ \frac{s^{(3)}(0)}{3} \end{pmatrix}. \tag{33}$$

The set of all financial assets is given by $\mathcal{N}^f = \{0, 1\}$. In order to obtain the financial market entropy measure $\widehat{\mathbb{Q}}^f$, we first determine the financial market martingale equations (23):

$$\begin{cases} 3e_1^f + 2e_2^f + e_3^f = 6 \\ 2e_2^f + 2e_3^f = 3 \end{cases}$$

where, according to (22), the e_i^f are given by

$$\begin{cases} e_1^f = \exp\left(\lambda_f^{(0)}\right) \\ e_2^f = \exp\left(\lambda_f^{(0)} + \lambda_f^{(1)}\right) \\ e_3^f = \exp\left(\lambda_f^{(0)} + 2\lambda_f^{(1)}\right). \end{cases}$$

This leads us to the following values for the Lagrange multipliers:

$$\lambda_f^{(0)} = \ln\left(\frac{8-\sqrt{10}}{4}\right)$$

and

$$\lambda_f^{(1)} = \ln\left(\frac{-1+\sqrt{10}}{3}\right).$$

From (21), we find that the financial market entropy measure $\widehat{\mathbb{Q}}^f$ is given by

$$\widehat{\mathbb{Q}}^f = \begin{pmatrix} \widehat{q}_1^f \\ \widehat{q}_2^f \\ \widehat{q}_3^f \end{pmatrix} = \begin{pmatrix} \frac{8-\sqrt{10}}{8} \\ \frac{-2+\sqrt{10}}{4} \\ \frac{4-\sqrt{10}}{8} \end{pmatrix}. \tag{34}$$

The set of all actuarial assets is given by $\mathcal{N}^a = \{0, 2\}$. As $\mathcal{A}(\mathbb{P}) = \mathcal{F}(\mathbb{P})$ and moreover, the initial prices as well as the $\mathbb{P}$-world distributions of the payoffs of the purely actuarial asset and the purely financial asset are identical, we immediately find that the entropy measures in both submarkets are equal. Hence,

$$\widehat{\mathbb{Q}}^a = \begin{pmatrix} \widehat{q}_1^a \\ \widehat{q}_2^a \\ \widehat{q}_3^a \end{pmatrix} = \begin{pmatrix} \frac{8-\sqrt{10}}{8} \\ \frac{-2+\sqrt{10}}{4} \\ \frac{4-\sqrt{10}}{8} \end{pmatrix}. \tag{35}$$

Comparing (33) with (34) and (35) leads to the conclusion that $s^{(3)}(0) = \frac{12-3\sqrt{10}}{8}$ is a necessary and sufficient condition for the projections $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ and $\mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ of the combined market entropy measure $\widehat{\mathbb{Q}}$ to be equal to the entropy measures of the financial and the actuarial submarkets, respectively:

$$\mathcal{F}\left(\widehat{\mathbb{Q}}\right) = \widehat{\mathbb{Q}}^f \text{ and } \mathcal{A}\left(\widehat{\mathbb{Q}}\right) = \widehat{\mathbb{Q}}^a \Leftrightarrow s^{(3)}(0) = \frac{12-3\sqrt{10}}{8}.$$

We can conclude that when $s^{(3)}(0) \neq \frac{12-3\sqrt{10}}{8}$, prices of financial assets under the combined market entropy measure $\widehat{\mathbb{Q}}$ differ from the corresponding prices under the financial market entropy measure $\widehat{\mathbb{Q}}^f$. The same conclusion holds for actuarial assets. ∇

In the following example, we consider a combined market where financial and actuarial risks are independent under the real-world probability measure $\mathbb{P}$. It will be shown that in this example the $\mathbb{P}$ - independence will only translate in $\widehat{\mathbb{Q}}$ - independence, when the combined financial-actuarial asset has a specific price.

Example 2 Consider a combined financial-actuarial world with three possible scenarios in the financial subworld and two possible scenarios in the actuarial subworld. The combined financial-actuarial universe Ω is given by

$$\Omega = \{(i,j) \mid i = 1,2,3 \text{ and } j = 1,2\}.$$

Assume that the real-world probability measure $\mathbb{P}$ is characterized by

$$\mathbb{P} = (p_{ij})_{i,j} = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{10} \\ \frac{1}{20} & \frac{3}{20} \end{pmatrix}.$$

Then the projections $\mathcal{F}(\mathbb{P})$ and $\mathcal{A}(\mathbb{P})$ are given by

$$\mathcal{F}(\mathbb{P}) = \begin{pmatrix} p_{1\cdot} \\ p_{2\cdot} \\ p_{3\cdot} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{pmatrix} \quad \text{and} \quad \mathcal{A}(\mathbb{P}) = \begin{pmatrix} p_{\cdot 1} \\ p_{\cdot 2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix},$$

respectively. Furthermore, one can easily verify that $\mathbb{P} = \mathcal{F}(\mathbb{P}) \times \mathcal{A}(\mathbb{P})$.

The risk-free interest rate r is assumed to be equal to 0. Apart from the risk-free bond with initial price $s^{(0)}(0) = 1$, there are 3 assets traded in the combined market: a financial asset, labeled 1, with current price $s^{(1)}(0) = 50$ and possible payoffs at time 1 given by

$$\begin{pmatrix} s^{(1)}_{1\cdot} \\ s^{(1)}_{2\cdot} \\ s^{(1)}_{3\cdot} \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix},$$

an actuarial asset, labeled 2, with current price $s^{(2)}(0) = 70$ and possible payoffs at time 1 given by

$$\begin{pmatrix} s^{(2)}_{\cdot 1} \\ s^{(2)}_{\cdot 2} \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \end{pmatrix},$$

and also a combined asset, labeled 3, with current price $s^{(3)}(0)$ and possible payoffs at time 1 given by

$$s^{(3)}_{ij} = \begin{cases} 100 & \text{if } i = j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The combined market martingale equations (14), for $m = 0, 1, 2, 3$, are equal to

$$\begin{cases} 2e_{11} + 6e_{12} + 2e_{21} + 6e_{22} + e_{31} + 3e_{32} = 20 \\ e_{11} + 3e_{12} = 5 \\ 6e_{12} + 6e_{22} + 3e_{32} = 14 \\ 10e_{11} = s^{(3)}(0) \end{cases}$$

with the e_{ij}, according to (11), given by

$$\begin{cases} e_{11} = \exp\left(\lambda^{(0)} + 100\lambda^{(1)} + 100\lambda^{(3)}\right) \\ e_{12} = \exp\left(\lambda^{(0)} + 100\lambda^{(1)} + 100\lambda^{(2)}\right) \\ e_{21} = \exp\left(\lambda^{(0)}\right) \\ e_{22} = \exp\left(\lambda^{(0)} + 100\lambda^{(2)}\right) \\ e_{31} = \exp\left(\lambda^{(0)}\right) \\ e_{32} = \exp\left(\lambda^{(0)} + 100\lambda^{(2)}\right). \end{cases}$$

From these systems of equations, we obtain the following values for the parameters e_{ij}:

$$\begin{cases} e_{11} = \frac{s^{(3)}(0)}{10} \\ e_{12} = \frac{50-s^{(3)}(0)}{30} \\ e_{21} = e_{31} = \frac{30-s^{(3)}(0)}{15} \\ e_{22} = e_{32} = \frac{20+s^{(3)}(0)}{45}. \end{cases}$$

The probabilities $\widehat{q}_{ij}$ of the combined market entropy measure are now determined by equation (10):

$$\widehat{\mathbb{Q}} = (\widehat{q}_{ij})_{i,j} = \begin{pmatrix} \frac{s^{(3)}(0)}{100} & \frac{50-s^{(3)}(0)}{100} \\ \frac{30-s^{(3)}(0)}{150} & \frac{20+s^{(3)}(0)}{150} \\ \frac{30-s^{(3)}(0)}{300} & \frac{20+s^{(3)}(0)}{300} \end{pmatrix}. \tag{36}$$

This entropy measure $\widehat{\mathbb{Q}}$ gives rise to the following projections:

$$\mathcal{F}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{1\cdot} \\ \widehat{q}_{2\cdot} \\ \widehat{q}_{3\cdot} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix} \quad \text{and} \quad \mathcal{A}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{\cdot 1} \\ \widehat{q}_{\cdot 2} \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{7}{10} \end{pmatrix},$$

which are independent of the current price $s^{(3)}(0)$. The product measure $\mathcal{F}\left(\widehat{\mathbb{Q}}\right) \times \mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ can now easily be determined:

$$\mathcal{F}\left(\widehat{\mathbb{Q}}\right) \times \mathcal{A}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \frac{3}{20} & \frac{7}{20} \\ \frac{1}{10} & \frac{7}{30} \\ \frac{1}{20} & \frac{7}{60} \end{pmatrix}. \tag{37}$$

Comparing (36) and (37), it is easy to prove that

$$\widehat{\mathbb{Q}} = \mathcal{F}\left(\widehat{\mathbb{Q}}\right) \times \mathcal{A}\left(\widehat{\mathbb{Q}}\right) \Leftrightarrow s^{(3)}(0) = 15.$$

Next, we determine the entropy measures of the financial and actuarial submarkets. The set of all financial assets is given by $\mathcal{N}^f = \{0, 1\}$. From (23), it follows that the martingale equations of the financial market are given by

$$\begin{cases} 2e_1^f + 2e_2^f + e_3^f = 5 \\ 4e_1^f = 5, \end{cases}$$

where, according to (22), the e_i^f are determined by

$$\begin{cases} e_1^f = \exp\left(\lambda_f^{(0)} + 100\lambda_f^{(1)}\right) \\ e_2^f = \exp\left(\lambda_f^{(0)}\right) \\ e_3^f = \exp\left(\lambda_f^{(0)}\right). \end{cases}$$

This leads us to the following values for the parameters e_i^f:

$$\begin{cases} e_1^f = \frac{5}{4} \\ e_2^f = e_3^f = \frac{5}{6}. \end{cases}$$

The financial market entropy measure $\widehat{\mathbb{Q}}^f$ then follows from equation (21):

$$\widehat{\mathbb{Q}}^f = \begin{pmatrix} \widehat{q}_1^f \\ \widehat{q}_2^f \\ \widehat{q}_3^f \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}.$$

The set of all actuarial assets is given by $\mathcal{N}^a = \{0, 2\}$. In this case, the actuarial market martingale equations in (30) read as

$$\begin{cases} e_1^a + 3e_2^a = 4 \\ 15e_2^a = 14, \end{cases} \tag{38}$$

where the e_j^a, according to (29), are given by

$$\begin{cases} e_1^a = \exp\left(\lambda_a^{(0)}\right) \\ e_2^a = \exp\left(\lambda_a^{(0)} + 100\lambda_a^{(2)}\right). \end{cases}$$

Solving the system of equations (38), we obtain

$$\begin{cases} e_1^a = \frac{6}{5} \\ e_2^a = \frac{14}{15}. \end{cases}$$

From equation (28), it follows that the actuarial market entropy measure $\widehat{\mathbb{Q}}^a$ is given by

$$\widehat{\mathbb{Q}}^a = \begin{pmatrix} \widehat{q}_1^a \\ \widehat{q}_2^a \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{7}{10} \end{pmatrix}.$$

Contrary to Example 1, the projections $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ and $\mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ of the combined market entropy measure $\widehat{\mathbb{Q}}$ in this example equal the entropy measures of the financial and the actuarial submarkets, respectively, regardless of the current price $s^{(3)}(0)$ of the combined asset. Nevertheless, $s^{(3)}(0)$ influences the dependence structure between financial and actuarial risks under $\widehat{\mathbb{Q}}$. The independence assumption under $\mathbb{P}$ will only translate in $\widehat{\mathbb{Q}}$ - independence, provided $s^{(3)}(0) = 15$. $\triangledown$

The traded assets in previous example have simple payoffs such that the obtained martingale equations can be solved easily. Notice however that the conclusions from this example remain to hold in a more general setting containing a pure financial, a pure actuarial and a combined asset: $\mathbb{P}$ - independence will only translate in $\widehat{\mathbb{Q}}$ - independence in case the combined financial-actuarial asset has a specific initial price.

5 The minimal entropy martingale measure in a combined world where only financial assets are traded

Consider the combined world $(\Omega, \Sigma, \mathbb{P})$ with a market of $M+1$ traded assets as described above. In this section, we assume that only financial assets are traded. In this special case, we have that

$$\mathcal{N}^f = \{0, 1, \ldots, M\} \text{ and } \mathcal{N}^a = \{0\}.$$

For any asset m, the vector of payoffs is given by $\left(s_{1\cdot}^{(m)}, s_{2\cdot}^{(m)}, \ldots, s_{I\cdot}^{(m)}\right)$, where $S^{(m)}(1) = s_{i\cdot}^{(m)} \geq 0$ if the financial scenario that unfolds is given by i.

In the following subsections, we first determine the entropy measures $\widehat{\mathbb{Q}}^f$ and $\widehat{\mathbb{Q}}^a$ corresponding to the financial and the actuarial submarkets, respectively. Then, we determine the entropy measure $\widehat{\mathbb{Q}}$ of the combined market. Finally, we investigate the relationship between these three entropy measures.

5.1 The entropy measures of the submarkets

Consider the *financial subworld* $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma), \mathcal{F}(\mathbb{P}))$ and the market of $M+1$ traded financial assets. The entropy measure $\widehat{\mathbb{Q}}^f$ of this submarket follows from the results in Subsection 4.2, with $\mathcal{N}^f = \{0, 1, \ldots, M\}$. In particular, we find that

$$\widehat{q}_i^f = p_{i\cdot}\ e_i^f, \qquad \text{for } i = 1, 2, \ldots, I, \tag{39}$$

with

$$e_i^f = \exp\left(\sum_{m=0}^{M} \lambda_f^{(m)}\ s_{i\cdot}^{(m)}\right), \qquad \text{for } i = 1, 2, \ldots, I. \tag{40}$$

The Lagrange multipliers $\lambda_f^{(m)}$ can be derived from the following martingale equations:

$$\sum_i p_{i\cdot}\ e_i^f\ s_{i\cdot}^{(m)} = e^r\ s^{(m)}(0), \qquad \text{for all } m \in \{0, 1, \ldots, M\}. \tag{41}$$

Next, we consider the *actuarial subworld* $(\mathcal{A}(\Omega), \mathcal{A}(\Sigma), \mathcal{A}(\mathbb{P}))$ and the actuarial submarket where only the risk-free bond is traded. The actuarial market entropy measure $\widehat{\mathbb{Q}}^a$ follows from the results in Subsection 4.3, with $\mathcal{N}^a = \{0\}$. We have that

$$\widehat{q}_j^a = p_{\cdot j}\ e_j^a, \qquad \text{for } j = 1, 2, \ldots, J,$$

with

$$e_j^a = \exp\left(\lambda_a^{(0)}\ e^r\right), \qquad \text{for } j = 1, 2, \ldots, J.$$

The unique Lagrange multiplier $\lambda_a^{(0)}$ follows from the martingale equation (30) for $m = 0$, i.e.

$$\sum_j p_{\cdot j}\ e_j^a = 1.$$

Taking into account that e_j^a does not depend on j, we find that

$$e_j^a = 1, \qquad \text{for } j = 1, 2, \ldots, J.$$

Hence,

$$\widehat{q}_j^a = p_{\cdot j}, \qquad \text{for } j = 1, 2, \ldots, J, \tag{42}$$

or, equivalently,

$$\widehat{\mathbb{Q}}^a = \mathcal{A}(\mathbb{P}). \tag{43}$$

This means that in a market where only financial risks are traded, the actuarial market entropy measure $\widehat{\mathbb{Q}}^a$ is identical to the projection $\mathcal{A}(\mathbb{P})$ of the physical probability measure on the actuarial subworld. This result was to be expected as there are no actuarial risks traded, which implies that the pricing measure $\widehat{\mathbb{Q}}^a$ that is closest to $\mathcal{A}(\mathbb{P})$ is $\mathcal{A}(\mathbb{P})$ itself.

5.2 The entropy measure of the combined market

The combined market entropy measure $\widehat{\mathbb{Q}}$ follows from the results in Subsection 4.1. In particular, we find that

$$\widehat{q}_{ij} = p_{ij}\ e_{ij}, \quad \text{for any } (i,j) \in \Omega, \tag{44}$$

where the coefficients e_{ij} are defined by

$$e_{ij} = \exp\left(\sum_{m=0}^{M} \lambda^{(m)}\ s_{i\cdot}^{(m)}\right), \quad \text{for any } (i,j) \in \Omega. \tag{45}$$

Obviously, the e_{ij} do not depend on j. Therefore, we will denote them by $e_{i\cdot}$ in this section. The martingale equations (14) can be written as

$$\sum_i p_{i\cdot}\ e_{i\cdot}\ s_{i\cdot}^{(m)} = e^r\ s^{(m)}(0), \qquad \text{for all } m \in \{0, \ldots, M\}. \tag{46}$$

Comparing the martingale equations (41) and (46), while taking into account that the $e_{i\cdot}$ and the e_i^f are uniquely determined and of the form (40) and (45), respectively, we find that

$$e_{i\cdot} = e_i^f \qquad \text{for } i = 1, 2, \ldots, I. \tag{47}$$

and also

$$\widehat{q}_{ij} = p_{ij}\ e_i^f, \quad \text{for any } (i,j) \in \Omega. \tag{48}$$

Taking into account (39), these expressions for the $\widehat{q}_{ij}$ result in

$$\widehat{q}_{i\cdot} = p_{i\cdot}\ e_i^f = \widehat{q}_i^f, \qquad \text{for } i = 1, 2, \ldots, I, \tag{49}$$

which means that the financial market entropy measure $\widehat{\mathbb{Q}}^f$ is identical to the projection $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ of the combined market entropy measure $\widehat{\mathbb{Q}}$, i.e.

$$\widehat{\mathbb{Q}}^f = \mathcal{F}\left(\widehat{\mathbb{Q}}\right). \tag{50}$$

In the following theorem, the relation between the entropy measures of the combined market and the corresponding submarkets is further explored.

Theorem 4 *Consider the combined financial-actuarial world* $(\Omega, \Sigma, \mathbb{P})$ *where only financial assets are traded. Let* $\widehat{\mathbb{Q}}$ *be the combined market entropy measure. The financial and actuarial market entropy measures* $\widehat{\mathbb{Q}}^f$ *and* $\widehat{\mathbb{Q}}^a$ *are then characterized by*

$$\widehat{\mathbb{Q}}^f = \mathcal{F}\left(\widehat{\mathbb{Q}}\right) \text{ and } \widehat{\mathbb{Q}}^a = \mathcal{A}(\mathbb{P}). \tag{51}$$

Moreover, financial and actuarial risks are independent under the $\mathbb{P}$*-measure if and only if they are independent under the* $\widehat{\mathbb{Q}}$*-measure:*

$$\mathbb{P} = \mathcal{F}(\mathbb{P}) \times \mathcal{A}(\mathbb{P}) \Leftrightarrow \widehat{\mathbb{Q}} = \mathcal{F}\left(\widehat{\mathbb{Q}}\right) \times \mathcal{A}\left(\widehat{\mathbb{Q}}\right). \tag{52}$$

In case of $\mathbb{P}$ *- independence between financial and actuarial risks, one has that*

$$\widehat{\mathbb{Q}}^a = \mathcal{A}\left(\widehat{\mathbb{Q}}\right). \tag{53}$$

Proof: The relations (51) have been proven above.

In order to prove the equivalence relation (52), let us first assume that financial and actuarial risks are $\mathbb{P}$ - independent. Taking into account (48) and (49), we find that

$$\widehat{q}_{ij} = \widehat{q}_{i\cdot} \times p_{\cdot j}, \qquad \text{for all } (i,j) \in \Omega.$$

Summing over all i leads to

$$\widehat{q}_{\cdot j} = p_{\cdot j}, \qquad \text{for } j = 1, 2, ..., J. \tag{54}$$

Hence,

$$\widehat{q}_{ij} = \widehat{q}_{i\cdot} \times \widehat{q}_{\cdot j}, \qquad \text{for all } (i,j) \in \Omega,$$

which means that financial and actuarial risks are $\widehat{\mathbb{Q}}$ - independent.

Next, we assume that financial and actuarial risks are $\widehat{\mathbb{Q}}$ - independent. In this case, the relations (48) and (49) lead to

$$p_{ij} = p_{i\cdot} \times \widehat{q}_{\cdot j}, \qquad \text{for all } (i,j) \in \Omega.$$

Summing over all i, we find that

$$p_{\cdot j} = \widehat{q}_{\cdot j}, \qquad \text{for } j = 1, 2, ..., J,$$

and hence,

$$p_{ij} = p_{i\cdot} \times p_{\cdot j}, \qquad \text{for all } (i,j) \in \Omega,$$

which means that financial and actuarial risks are $\mathbb{P}$ - independent.

Finally, in case of independence, the relations (42) and (54) lead to

$$\widehat{q}_{\cdot j} = \widehat{q}_j^a, \qquad \text{for } j = 1, 2, ..., J,$$

which means that (53) holds. ■

Theorem 4 states that in a market where only financial assets are traded, a $\mathbb{P}$-world independence between financial and actuarial risks implies that also under the combined market entropy measure $\widehat{\mathbb{Q}}$, financial and actuarial risks are independent. Important to notice is that this implication does not state that $\mathbb{P}$ - independence translates into independence under any pricing measure $\mathbb{Q}$. Some simple examples of (in-)complete markets with $\mathbb{P}$-world independence but where no equivalent martingale measure exists under which financial and actuarial risks are independent, can be found in Dhaene et al. (2013).

5.3 Illustration

From Theorem 4, we know that in a market where only financial assets are traded, $\widehat{\mathbb{Q}}^f = \mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ holds for any possible dependence structure between financial and actuarial risks under the physical measure $\mathbb{P}$. Moreover, we found that $\widehat{\mathbb{Q}}^a = \mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ holds, provided financial and actuarial risks are independent under $\mathbb{P}$. In the following example, we explore whether this $\mathbb{P}$ - independence is an essential requirement or not for this last statement to hold.

Example 3 Consider the combined financial-actuarial world with three possible scenarios in each subworld and with physical measure $\mathbb{P}$, as described in Example 1. In the corresponding market, we assume now that only 2 financial assets are traded, namely the risk-free zero coupon bond with $r = 0$, and the financial asset with initial price $s^{(1)}(0) = \frac{1}{2}$ and possible payoffs at time 1 given by

$$\begin{pmatrix} s_{1\cdot}^{(1)} \\ s_{2\cdot}^{(1)} \\ s_{3\cdot}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

In order to determine the combined market entropy measure $\widehat{\mathbb{Q}}$, we consider the martingale equations (46), which can be expressed as

$$\begin{cases} 3e_{1\cdot} + 2e_{2\cdot} + e_{3\cdot} = 6 \\ 2e_{2\cdot} + 2e_{3\cdot} = 3, \end{cases}$$

with the $e_{i\cdot}$, according to (45), given by

$$\begin{cases} e_{1\cdot} = \exp\left(\lambda^{(0)}\right) \\ e_{2\cdot} = \exp\left(\lambda^{(0)} + \lambda^{(1)}\right) \\ e_{3\cdot} = \exp\left(\lambda^{(0)} + 2\lambda^{(1)}\right). \end{cases}$$

These systems of equations lead to the following numerical values for the Lagrange multipliers:

$$\lambda^{(0)} = \ln\left(\frac{8-\sqrt{10}}{4}\right) \tag{55}$$

and

$$\lambda^{(1)} = \ln\left(\frac{-1+\sqrt{10}}{3}\right). \tag{56}$$

According to equations (44) and (45), the probabilities $\widehat{q}_{ij}$ are then given by

$$\widehat{\mathbb{Q}} = \begin{pmatrix} \frac{8-\sqrt{10}}{24} & \frac{8-\sqrt{10}}{24} & \frac{8-\sqrt{10}}{24} \\ \frac{-2+\sqrt{10}}{8} & \frac{-2+\sqrt{10}}{8} & 0 \\ \frac{4-\sqrt{10}}{8} & 0 & 0 \end{pmatrix}.$$

The projections $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ and $\mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ of the combined market entropy measure $\widehat{\mathbb{Q}}$ on the financial and the actuarial subworld, respectively, can easily be determined:

$$\mathcal{F}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{1\cdot} \\ \widehat{q}_{2\cdot} \\ \widehat{q}_{3\cdot} \end{pmatrix} = \begin{pmatrix} \frac{8-\sqrt{10}}{8} \\ \frac{-2+\sqrt{10}}{4} \\ \frac{4-\sqrt{10}}{8} \end{pmatrix} \quad \text{and} \quad \mathcal{A}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{\cdot 1} \\ \widehat{q}_{\cdot 2} \\ \widehat{q}_{\cdot 3} \end{pmatrix} = \begin{pmatrix} \frac{14-\sqrt{10}}{24} \\ \frac{1+\sqrt{10}}{12} \\ \frac{8-\sqrt{10}}{24} \end{pmatrix}.$$

Let us now determine the entropy measure of the financial submarket. Taking into account that the martingale equations for the combined market and the financial submarket are identical, we find that $\lambda_f^{(0)}$ and $\lambda_f^{(1)}$ are given by (55) and (56), respectively. From (39) and (40), it follows then that the financial market entropy measure is given by

$$\widehat{\mathbb{Q}}^f = \begin{pmatrix} \widehat{q}_1^f \\ \widehat{q}_2^f \\ \widehat{q}_3^f \end{pmatrix} = \begin{pmatrix} \frac{8-\sqrt{10}}{8} \\ \frac{-2+\sqrt{10}}{4} \\ \frac{4-\sqrt{10}}{8} \end{pmatrix}.$$

We observe that the measure $\widehat{\mathbb{Q}}^f$ is identical to the financial projection $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$ of the combined market entropy measure $\widehat{\mathbb{Q}}$, which confirms our earlier derived general result (51). According to (43), the actuarial market entropy measure is given by

$$\widehat{\mathbb{Q}}^a = \mathcal{A}\left(\mathbb{P}\right) = \begin{pmatrix} \frac{3}{6} \\ \frac{2}{6} \\ \frac{1}{6} \end{pmatrix}.$$

We can conclude that the actuarial market entropy measure $\widehat{\mathbb{Q}}^a$ is different from the actuarial projection $\mathcal{A}\left(\widehat{\mathbb{Q}}\right)$ of the combined market entropy measure $\widehat{\mathbb{Q}}$. $\triangledown$

From the preceding example, we conclude that in a combined world where only financial assets are traded, and where financial and actuarial risks are not independent under $\mathbb{P}$, it may happen that the actuarial market entropy measure $\widehat{\mathbb{Q}}^a$ is different from the actuarial projection of the combined market entropy measure.

6 The minimal entropy martingale measure in a combined world without traded combined assets

In this section, we investigate a second special case of the general combined financial-actuarial world described in Section 4. We suppose now that no combined assets are available in the market. Hence, apart from the risk-free zero coupon bond, only purely financial and purely actuarial assets are traded. In terms of the earlier introduced notations $\mathcal{N}^f$ and $\mathcal{N}^a$ for the sets of financial and actuarial assets, respectively, this means that

$$\mathcal{N}^f \cup \mathcal{N}^a = \{0, 1, \ldots, M\}. \tag{57}$$

Hereafter, we determine the entropy measures of the financial and the actuarial submarkets, as well as the entropy measure of the combined market. Furthermore, we investigate the relationship between these measures.

6.1 The entropy measures of the submarkets and the combined market

The entropy measure $\widehat{\mathbb{Q}}^f$ corresponding to the market $\mathcal{N}^f$ of traded financial assets in the financial subworld $(\mathcal{F}(\Omega), \mathcal{F}(\Sigma), \mathcal{F}(\mathbb{P}))$ follows from the results in Subsection 4.2. In particular, we have that $\widehat{q}_i^f$, e_i^f and the martingale equations are given by (21), (22) and (23), respectively.

Similarly, the entropy measure $\widehat{\mathbb{Q}}^a$ of the market $\mathcal{N}^a$ of traded actuarial assets in the actuarial subworld $(\mathcal{A}(\Omega), \mathcal{A}(\Sigma), \mathcal{A}(\mathbb{P}))$ follows from Subsection 4.3. In particular, we have that $\widehat{q}_j^a$, e_j^a and the corresponding martingale equations are given by (28), (29) and (30), respectively.

Let us now determine the entropy measure $\widehat{\mathbb{Q}}$ of the combined market. From (10) and (11) in Subsection 4.1, we find that

$$\widehat{q}_{ij} = p_{ij}\ e_{ij}, \quad \text{for any } (i,j) \in \Omega, \tag{58}$$

where the coefficients e_{ij} are defined by

$$e_{ij} = \exp\left(\lambda^{(0)} e^r + \sum_{m \in \mathcal{N}_0^f} \lambda^{(m)}\ s_{i\cdot}^{(m)} + \sum_{m \in \mathcal{N}_0^a} \lambda^{(m)}\ s_{\cdot j}^{(m)}\right), \quad \text{for any } (i,j) \in \Omega. \tag{59}$$

From (14), it follows that the martingale equations of the combined market are given by

$$\sum_{i,j} p_{ij}\ e_{ij}\ s_{i\cdot}^{(m)} = e^r\ s^{(m)}(0), \qquad \text{for all } m \in \mathcal{N}^f \tag{60}$$

and

$$\sum_{i,j} p_{ij}\ e_{ij}\ s_{\cdot j}^{(m)} = e^r\ s^{(m)}(0), \qquad \text{for all } m \in \mathcal{N}^a. \tag{61}$$

In the following theorem, we explore the relationship between $\mathbb{P}$ - and $\widehat{\mathbb{Q}}$ - independence of financial and actuarial risks.

Theorem 5 *Consider the combined financial-actuarial world $(\Omega, \Sigma, \mathbb{P})$ where, apart from the risk-free asset, only purely financial and purely actuarial assets are traded. In this case, financial and actuarial risks are independent under the $\mathbb{P}$-measure if and only they are independent under the $\widehat{\mathbb{Q}}$-measure:*

$$\mathbb{P} = \mathcal{F}(\mathbb{P}) \times \mathcal{A}(\mathbb{P}) \Leftrightarrow \widehat{\mathbb{Q}} = \mathcal{F}\left(\widehat{\mathbb{Q}}\right) \times \mathcal{A}\left(\widehat{\mathbb{Q}}\right). \tag{62}$$

Moreover, in case of $\mathbb{P}$ - independence between financial and actuarial risks, one has that

$$\widehat{\mathbb{Q}}^f = \mathcal{F}\left(\widehat{\mathbb{Q}}\right) \text{ and } \widehat{\mathbb{Q}}^a = \mathcal{A}\left(\widehat{\mathbb{Q}}\right). \tag{63}$$

Proof: Let us first prove the $\Rightarrow$ implication in (62). Therefore, suppose that financial and actuarial risks are $\mathbb{P}$ - independent. Taking into account (15) and (57), we can express the coefficients e_{ij} as follows:

$$e_{ij} = \frac{\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ s_{i\cdot}^{(m)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ S^{(m)}(1)\right)\right]} \times \frac{\exp\left(\sum_{m\in\mathcal{N}_0^a} \lambda^{(m)}\ s_{\cdot j}^{(m)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m\in\mathcal{N}_0^a} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}, \tag{64}$$

which holds for any $(i,j)\in\Omega$. From (58) and (64), we find that

$$\widehat{q}_{i\cdot} = p_{i\cdot} \times \sum_{j=1}^{J} p_{\cdot j}\ e_{ij} = p_{i\cdot} \times \frac{\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ s_{i\cdot}^{(m)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}, \qquad \text{for } i=1,2,...,I. \tag{65}$$

A similar expression holds for the actuarial subworld. Hence, we can conclude that

$$\widehat{q}_{ij} = (p_{i\cdot} \times p_{\cdot j})\ \ e_{ij} = \widehat{q}_{i\cdot} \times \widehat{q}_{\cdot j}, \qquad \text{for any } \ (i,j)\in\Omega,$$

which means that financial and actuarial risks are $\widehat{\mathbb{Q}}$ - independent.

Next, we prove the $\Leftarrow$ implication in (62). Suppose that financial and actuarial risks are $\widehat{\mathbb{Q}}$ - independent. Taking into account (17) and (57), we can express the coefficients e_{ij}^{-1} as follows:

$$e_{ij}^{-1} = \frac{\exp\left(-\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ s_{i\cdot}^{(m)}\right)}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\exp\left(-\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ S^{(m)}(1)\right)\right]} \times \frac{\exp\left(-\sum_{m\in\mathcal{N}_0^a} \lambda^{(m)}\ s_{\cdot j}^{(m)}\right)}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\exp\left(-\sum_{m\in\mathcal{N}_0^a} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}, \tag{66}$$

which holds for any $(i,j)\in\Omega$. From (58) and (66), we find that

$$p_{i\cdot} = \widehat{q}_{i\cdot} \times \sum_{j=1}^{J} \widehat{q}_{\cdot j}\ e_{ij}^{-1} = \widehat{q}_{i\cdot} \times \frac{\exp\left(-\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ s_{i\cdot}^{(m)}\right)}{\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\exp\left(-\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}, \qquad \text{for } i=1,2,...,I.$$

A similar expression holds for the actuarial subworld. Hence, we can conclude that

$$p_{ij} = (\widehat{q}_{i\cdot} \times \widehat{q}_{\cdot j})\ \ e_{ij}^{-1}\ = p_{i\cdot} \times p_{\cdot j}, \qquad \text{for any } \ (i,j)\in\Omega,$$

which means that financial and actuarial risks are $\mathbb{P}$ - independent.

It remains to prove that (63) holds in case of $\mathbb{P}$ - independence (or equivalently, $\widehat{\mathbb{Q}}$ - independence) between financial and actuarial risks. Hereafter, we only give the proof for the financial submarket. The actuarial submarket case is proven in a similar way.
By using the expressions (64) for the coefficients e_{ij}, we can simplify the martingale equations (60) for the combined market as follows:

$$\sum_i p_{i\cdot} \frac{\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ s_{i\cdot}^{(m)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m\in\mathcal{N}_0^f} \lambda^{(m)}\ S^{(m)}(1)\right)\right]}\ s_{i\cdot}^{(m)} = e^r\ s^{(m)}(0)\,, \qquad \text{for all } m\in\mathcal{N}^f.$$

Comparing these martingale equations for the combined market with the martingale equations (23) for the financial submarket, while taking into account the expression (25), we find that $\lambda^{(m)} = \lambda_f^{(m)}$ for $m \in \mathcal{N}_0^f$. Similarly, we can prove that $\lambda^{(m)} = \lambda_a^{(m)}$ for $m \in \mathcal{N}_0^a$.

From (25), (31) and (64), it follows then that

$$e_{ij} = e_i^f \times e_j^a$$

holds for any $(i,j)\in\Omega$. Hence, from the $\mathbb{P}$ - independence assumption we find that

$$\widehat{q}_{ij} = p_{ij}\ e_{ij} = \left(p_{i\cdot}\ e_i^f\right)\times\left(p_{\cdot j}\ e_j^a\right) = \widehat{q}_i^f \times \widehat{q}_j^a, \qquad \text{for all } (i,j)\in\Omega.$$

The latter expression immediately leads us to

$$\widehat{q}_{i\cdot} = \widehat{q}_i^f\,, \qquad \text{for } i = 1,2,\ldots,I,$$

and

$$\widehat{q}_{\cdot j} = \widehat{q}_j^a, \qquad \text{for } j = 1,2,\ldots,J,$$

which means that (63) holds, when financial and actuarial risks are independent. ■

6.2 Illustration

In the following example, we show that in a market where only purely financial and purely actuarial assets are traded, the equality (63) between the projections of the combined market entropy measure and the corresponding entropy measures of the submarkets may no longer hold in case financial and actuarial risks are not independent under $\mathbb{P}$.

Example 4 Consider again the combined financial-actuarial world with three possible scenarios in each subworld and with physical measure $\mathbb{P}$, as described in Example 1. Suppose now that, apart from the risk-free zero coupon bond, one purely financial asset (labeled 1) and one purely actuarial asset (labeled 2) are traded. Both assets have an initial price $s^{(m)}(0) = \frac{1}{2}$, while their possible payoffs at time 1 are given by

$$\begin{pmatrix} s_{1\cdot}^{(1)} \\ s_{2\cdot}^{(1)} \\ s_{3\cdot}^{(1)} \end{pmatrix} = \begin{pmatrix} s_{\cdot 1}^{(2)} \\ s_{\cdot 2}^{(2)} \\ s_{\cdot 3}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

In order to determine the combined market entropy measure $\widehat{\mathbb{Q}}$, we first write down the martingale equations from (60) and (61) for $m = 0, 1, 2$:

$$\begin{cases} e_{11} + e_{12} + e_{13} + e_{21} + e_{22} + e_{31} = 6 \\ e_{21} + e_{22} + 2e_{31} = 3 \\ e_{12} + 2e_{13} + e_{22} = 3 \end{cases}$$

where according to (59), the e_{ij} are given by

$$\begin{cases} e_{11} = \exp\left(\lambda^{(0)}\right) \\ e_{12} = \exp\left(\lambda^{(0)} + \lambda^{(2)}\right) \\ e_{13} = \exp\left(\lambda^{(0)} + 2\lambda^{(2)}\right) \\ e_{21} = \exp\left(\lambda^{(0)} + \lambda^{(1)}\right) \\ e_{22} = \exp\left(\lambda^{(0)} + \lambda^{(1)} + \lambda^{(2)}\right) \\ e_{31} = \exp\left(\lambda^{(0)} + 2\lambda^{(1)}\right) \end{cases}$$

These systems of equations result in the following values for the Lagrange parameters:

$$\begin{cases} \lambda^{(0)} = \ln\left(\frac{9-3\sqrt{3}}{2}\right) \\ \lambda^{(1)} = \lambda^{(2)} = \ln\left(\sqrt{\frac{1}{3}}\right). \end{cases}$$

Taking into account previous calculations, we find that the combined market entropy measure $\widehat{\mathbb{Q}}$ is given by

$$\widehat{\mathbb{Q}} = \begin{pmatrix} \frac{3-\sqrt{3}}{4} & \frac{-1+\sqrt{3}}{4} & \frac{3-\sqrt{3}}{12} \\ \frac{-1+\sqrt{3}}{4} & \frac{3-\sqrt{3}}{12} & 0 \\ \frac{3-\sqrt{3}}{12} & 0 & 0 \end{pmatrix}.$$

The projections of $\widehat{\mathbb{Q}}$ on the financial and the actuarial subworld can easily be determined:

$$\mathcal{F}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{1\cdot} \\ \widehat{q}_{2\cdot} \\ \widehat{q}_{3\cdot} \end{pmatrix} = \begin{pmatrix} \frac{9-\sqrt{3}}{12} \\ \frac{\sqrt{3}}{6} \\ \frac{3-\sqrt{3}}{12} \end{pmatrix} \quad \text{and} \quad \mathcal{A}\left(\widehat{\mathbb{Q}}\right) = \begin{pmatrix} \widehat{q}_{\cdot 1} \\ \widehat{q}_{\cdot 2} \\ \widehat{q}_{\cdot 3} \end{pmatrix} = \begin{pmatrix} \frac{9-\sqrt{3}}{12} \\ \frac{\sqrt{3}}{6} \\ \frac{3-\sqrt{3}}{12} \end{pmatrix}.$$

The submarkets in this example are identical to the submarkets considered in Example 1. As a consequence, the submarket entropy measures in the current example are identical to the corresponding entropy measures derived in Example 1. In particular, we find that

$$\widehat{\mathbb{Q}}^f = \begin{pmatrix} \widehat{q}_1^f \\ \widehat{q}_2^f \\ \widehat{q}_3^f \end{pmatrix} = \begin{pmatrix} \frac{8-\sqrt{10}}{8} \\ \frac{-2+\sqrt{10}}{4} \\ \frac{4-\sqrt{10}}{8} \end{pmatrix} \quad \text{and} \quad \widehat{\mathbb{Q}}^a = \begin{pmatrix} \widehat{q}_1^a \\ \widehat{q}_2^a \\ \widehat{q}_3^a \end{pmatrix} = \begin{pmatrix} \frac{8-\sqrt{10}}{8} \\ \frac{-2+\sqrt{10}}{4} \\ \frac{4-\sqrt{10}}{8} \end{pmatrix}.$$

We can conclude that $\mathcal{F}\left(\widehat{\mathbb{Q}}\right) \neq \widehat{\mathbb{Q}}^f$ and $\mathcal{A}\left(\widehat{\mathbb{Q}}\right) \neq \widehat{\mathbb{Q}}^a$, which means that the financial and actuarial projection of the combined market entropy measure differ from the entropy measures of the financial and the actuarial subworld, respectively. ∇

The previous example shows that we have to clearly specify the modeling environment when we want to price financial or actuarial assets under the minimal entropy martingale measure. For a purely financial asset, the price under the combined market entropy measure $\widehat{\mathbb{Q}}$ (or, equivalently, under the projection $\mathcal{F}\left(\widehat{\mathbb{Q}}\right)$) will in general differ from the price under the financial market entropy measure $\widehat{\mathbb{Q}}^f$. Notice however that from Theorem 5, it follows that these prices are equal in case financial and actuarial risks are independent under the physical measure $\mathbb{P}$. Similar conclusions can be formulated concerning prices of actuarial assets.

7 Conclusion

In arbitrage-free but incomplete markets, the equivalent martingale measure for pricing traded assets is not uniquely determined. A possible approach when choosing a particular pricing measure is to look for the one that is 'closest' to the physical probability measure $\mathbb{P}$, where closeness is measured in terms of relative entropy.

In this paper, we considered the problem of determining the minimal entropy martingale measure in a market where securities are traded with payoffs depending on financial as well as actuarial risks. Therefore, we modeled a combined financial-actuarial world with a universe consisting of combined financial-actuarial scenarios. We determined the entropy measure of the combined market consisting of financial, actuarial and combined financial-actuarial assets, as well as the entropy measures corresponding to the financial and the actuarial submarkets.

We proved that in a market where only financial assets are traded, independence of financial and actuarial risks under the real-world probability measure is equivalent to independence under the combined market entropy measure. Moreover, pricing financial assets under the financial market entropy measure is identical to pricing these financial assets under the combined market entropy measure. In such a market, the actuarial market entropy measure coincides with the projection of the real-world probability measure on the actuarial subworld.

In a market where purely financial as well as purely actuarial securities are traded, we proved that financial and actuarial risks are independent under the real-world probability measure if and only if these risks are independent under the combined market entropy measure. Moreover, in case of independence, the entropy measure of the combined financial-actuarial market is the product measure of the entropy measures of the financial and the actuarial submarkets. The latter property does not always hold when financial and actuarial risks are not independent under the real-world probability measure. In this case, the price of a financial asset under the combined market entropy measure will in general differ from the price under the financial market entropy measure. This difference is due to the fact that the available information in the combined world is larger than in the financial subworld which leads to a different set of martingale measures from which we choose the 'closest' one. A similar reasoning holds for actuarial assets.

In the general case, i.e. in a market where apart from financial and actuarial assets, also combined financial-actuarial assets are traded, no general conclusions can be made. In particular, independence of financial and actuarial risks under the physical measure does not always translate into independence under the combined market entropy measure, and vice versa. Moreover, there is no link between the projections of the combined market entropy measure at the one hand and the entropy measures of the submarkets at the other hand, even in case of $\mathbb{P}$ - independence.

In this paper, we considered a one-period, finite state market model. The results in this paper can be extended to a multiple period setting. Similar results can also be derived in a continuous-time market model.

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