Efficient computation of the optimal strikes in the comonotonic upper bound for an arithmetic Asian option

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ABSTRACT

In this paper, an efficient method is proposed which accelerates the computation of the optimal strikes in the comonotonic upper bound for the value of an arithmetic Asian option. Numerical applications are carried out in the setting of Heston's model, in which the distribution function of the underlying asset price is not available in closed form. These numerical results highlight the efficiency of the proposed method.

Keywords: arithmetic Asian option, upper bound, characteristic function, super-hedging strategy

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1 Introduction

Let $S(t)$ be the price of an asset at time t and $r > 0$ the risk-free interest rate, which is assumed to be constant. The pay-off of an arithmetic Asian option maturing at T with strike K is then given by

$$\left(\xi \sum_{i=1}^{N} w_i S(t_i) - \xi K\right)^{+}, \tag{1.1}$$

where $(x)^{+} = \max(x, 0)$, $\sum\limits_{i=1}^{N} w_i = 1$ and $t_1, t_2, \ldots, t_N = T$ are the discrete monitoring times. When $\xi = 1$, it is a call option; when $\xi = -1$ it is a put option. The price of such option is difficult or impossible to be determined in closed form even in the Black–Scholes market model. Although the Monte Carlo method or PDE-based method can be used to numerically calculate the Asian option price, see, e.g., (Broadie and Glasserman, 1996; Večeř, 2001), both approaches are rather time consuming. An alternative is to calculate sharp bounds of the option value. The comonotonicity-based upper bound is an accurate approximation for the Asian option value under different parametric models (Albrecher, Dhaene, Goovaerts and Schoutens, 2005; Chen and Ewald, 2017). It also corresponds to the cheapest static super-hedging strategy with European options; see, e.g., (Albrecher et al., 2005; Chen, Deelstra, Dhaene and Vanmaele, 2008; Chen, Deelstra, Dhaene, Linders and Vanmaele, 2015).

(Simon, Goovaerts and Dhaene, 2000) and (Dhaene, Denuit, Goovaerts, Kaas and Vyncke, 2002a) pioneered the comonotonicity-based approach for calculating an upper bound of this type of Asian options, and showed its application in the Black–Scholes setting. Then, (Albrecher et al., 2005; Chen and Ewald, 2017) applied this approach in the setting of exponential Lévy models and stochastic volatility models, respectively. Essentially, a comonotonicity-based value bound is a weighted sum of European option prices matured on each monitoring time with optimal strikes. To calculate the optimal strikes of the hedging instruments, (Albrecher et al., 2005) numerically built up the distribution function from the density function of the underlying asset price, and its inverse is then found by a bisection method. This algorithm can be computationally intensive, especially when the density function is complicated and not available in closed form. By simulating the stochastic differential system for the underlying asset price process, (Chen and Ewald, 2017) approximated the distribution function of the underlying asset by its empirical distribution. In this paper, we propose to recover the distribution function of the underlying asset price from its characteristic function by using Fourier-cosine series. This method is inspired by the COS method of (Fang and Oosterlee, 2008).

The paper is organized as follows. The problem is formulated in Section 2. In Section 3, an efficient method is derived to calculate the optimal strikes associated with an upper value bound for an arithmetic Asian option. Numerical applications are carried out in Section 4 to show the efficiency of the proposed method. Section 5 concludes.

2 Comonotonic upper bounds and static super-hedging strategies

Consider a finite time horizon $T > 0$. The financial market is described via a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{0\leq t\leq T}, \mathbb{P})$, which satisfies the usual technical conditions of completeness and right-continuity, and where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of Ω. Price processes of traded financial instruments are modelled as stochastic processes on that probability space which are adapted to the filtration $(\mathcal{F}(t))_{0\leq t\leq T}$.

Market participants are assumed to have access to a number of European options with maturities t_i, $i = 1, \dots N$, with $0 = t_0 < t_1 < \cdots < t_N = T$. More precisely, they can trade in European calls and puts on the individual stocks. In particular, consider an asset with a non-negative stochastic price process denoted by $(S(t))_{0\leq t\leq T}$, or in short by S, that pays dividends continuously over time at a constant rate q per unit time. Further, we consider a discretely monitored arithmetic Asian option of European-style maturing at T with strike price $K \geq 0$.

The pay-off at T is given by (1.1) and depends on the underlying process S at the times t_i, $i = 1, \ldots N$, weighted by corresponding positive weights w_i, $i = 1, \ldots, N$, which sum up to one. In practice, an equally weighted sum is often used, i.e., all weights are chosen to be equal to $1/N$.

It is assumed that the financial market is arbitrage-free. For a given stochastic model for the underlying asset price, we further assume that there exists a pricing measure $\mathbb{Q}$, equivalent to the physical probability measure $\mathbb{P}$, such that the current price of any pay-off at time t_i, $i \in \{1, \ldots, N\}$, can be represented as the expectation of the discounted pay-off. In this price-recipe, we discount by means of a continuously compounded time-0 risk-free interest rate r, and we take expectations with respect to $\mathbb{Q}$. For simplicity in notation and terminology, we assume deterministic interest rates. The case of a stochastic interest rate is covered in (Chen et al., 2015).

The price of an Asian call or put at time $t \in [0, T]$ with maturity T and strike K is given by:

$$A(t, T, K; \xi) = \mathrm{e}^{-r(T-t)} \mathbb{E}\left[\left(\xi \sum_{i=1}^{N} w_i S(t_i) - \xi K \right)^+ \middle| \mathcal{F}(t) \right]. \tag{2.1}$$

Note that without loss of generality we will assume that $t < t_1$ in what follows. When $t_1 < t < T$, we absorb the known asset prices in the strike.

Let us further introduce the following notation for the weighted sum given the information $\mathcal{F}(t)$, or in a Markovian setting given $S(t)$:

$$\mathcal{S} = \sum_{i=1}^{N} w_i S(t_i) \mid \mathcal{F}(t), \tag{2.2}$$

and for its corresponding comonotonic counterpart, see, e.g., (Dhaene, Denuit, Goovaerts, Kaas and Vyncke, 2002b)

$$\mathcal{S}^c = \sum_{i=1}^{N} w_i (F^t_{S(t_i)})^{-1}(U), \tag{2.3}$$

with U a uniform $(0, 1)$-random variable and $F^t_{S(t_i)}$ the conditional cumulative distribution function (cdf) of $S(t_i)$ given the information $\mathcal{F}(t)$, or in a Markovian setting given $S(t)$, under the martingale measure $\mathbb{Q}$:

$$F^t_{S(t_i)}(x) = \mathbb{Q}(S(t_i) \leq x \mid \mathcal{F}(t)).$$

The conditional cdf of $\mathcal{S}^c$ given the information $\mathcal{F}(t)$ is analogously denoted by $F^t_{\mathcal{S}^c}$. When $t = 0$ we omit the superscript.

It is well known that $\mathcal{S}^c$ precedes $\mathcal{S}$ in the convex order sense, see, e.g., (Dhaene et al., 2002b),

$$\mathbb{E}[\mathcal{S}] = \mathbb{E}[\mathcal{S}^c], \quad \mathbb{E}[(\mathcal{S} - d)^+] \leq \mathbb{E}[(\mathcal{S}^c - d)^+], \quad \forall d \in \mathbb{R}. \tag{2.4}$$

Consider the function $f(d) = \mathbb{E}[(\mathcal{S}^c - d)^+] - \mathbb{E}[(\mathcal{S} - d)^+]$ of d, which first increases and then decreases (from some c on), but remains non-negative (Dhaene et al., 2002b). Therefore, only for in-the-money options, it is accurate to approximate the option value by the corresponding comonotonic upper bound. For out-of-the-money options or at-the-money options this bound is a real upper bound. Thus in the numerical experiments, we will focus on in-the-money options.

From the theory of comonotonic risks, see, e.g., (Kaas, Dhaene and Goovaerts, 2000; Dhaene et al., 2002b), applied to Asian options, see, e.g., (Simon et al., 2000; Dhaene et al., 2002a; Chen et al., 2008), we have the following result.

Theorem 2.1. *The Asian option price at* $t \in [0,T]$ *defined by* (2.1) *with a strike* $K \in \big((F^t_{\mathcal{S}^c})^{-1+}(0), (F^t_{\mathcal{S}^c})^{-1}(1)\big)$ *is bounded above as follows*

$$\begin{aligned} A(t,T,K;\xi) &\le \mathrm{e}^{-r(T-t)}\mathbb{E}\left[(\xi\mathcal{S}^c-\xi K)^+\middle|\mathcal{F}(t)\right] \\ &= \mathrm{e}^{-rT}\sum_{i=1}^N w_i\mathrm{e}^{rt_i}\mathrm{e}^{-r(t_i-t)}\mathbb{E}\left[(\xi S(t_i)-\xi K_i)^+\middle|\mathcal{F}(t)\right] \\ &=: \mathrm{e}^{-rT}\sum_{i=1}^N w_i\mathrm{e}^{rt_i}E(t,t_i,K_i;\xi), \end{aligned} \tag{2.5}$$

where the strikes $K_i \ge 0$ *are given by*

$$K_i = (F^t_{S(t_i)})^{-1(\alpha_i)}(F^t_{\mathcal{S}^c}(K)), \quad i=1,\dots,N \tag{2.6}$$

with the α_i*,* $i=1,\dots,N$*, chosen in* $[0,1]$ *such that*

$$\sum_{i=1}^N w_iK_i = K.$$

Moreover, the comonotonic upper bound (2.5) *is optimal in the sense that for any set of strikes* k_i*,* $i=1,\dots,N$*, such that* $\sum_{i=1}^N w_ik_i = K$*, it holds that*

$$A(t,T,K;\xi) \le \sum_{i=1}^N w_i\mathrm{e}^{-r(T-t_i)}E(t,t_i,K_i;\xi) \le \sum_{i=1}^N w_i\mathrm{e}^{-r(T-t_i)}E(t,t_i,k_i;\xi). \tag{2.7}$$

This upper bound can be determined in the *infinite market case* (Hobson, Laurence and Wang, 2005; Chen et al., 2015), where it is assumed that all European option prices $E(t,t_i,K;\xi)$ for any maturity t_i, $i=1,\dots,N$, and any strike $K \ge 0$ are known. These European option prices can be calculated analytically or numerically for a given model of the underlying asset price, such as the Black-Scholes model, the Heston model (Heston, 1993). To calculate this upper bound (2.5), one of the key steps is to calculate K_i defined in (2.6). The next section focuses on how to efficiently calculate these K_i for a given model of the underlying asset price.

3 Acceleration of the computation of the optimal upper bound

The distribution function of the underlying asset at time t_i and its inverse play an important role in calculating the optimal strikes (2.6). (Albrecher et al., 2005) numerically built up the distribution function from the density function of the underlying asset price, and its inverse is then found by a bisection method. It is computationally intensive to build up the distribution function from its density function, especially when the density function is very complicated or singular at some points, such as in the Heston model (Heston, 1993), see, e.g., (Drăgulescu and Yakovenko, 2002). To accelerate the procedure proposed by (Albrecher et al., 2005), we recover the distribution function from the characteristic function of the log-asset price, which is available in a large class of stochastic models, such as Lévy models (Albrecher et al., 2005) and Heston's model.

We approximate the distribution function $F_{\log(S_t)}$ by $\hat{F}_{\log(S_t)}$:

$$\hat{F}_{\log(S_t)}(x) = \begin{cases} 1 & \text{if } x \geq b, \\ \frac{x-a}{b-a} + \sum_{k=1}^{n} F_{k,t} \cdot \frac{b-a}{k\pi} \cdot \sin(k\pi \frac{x-a}{b-a}) & \text{if } x \in (a,b), \\ 0 & \text{if } x \leq a, \end{cases} \tag{3.1}$$

where $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ $(-\infty < a < b < +\infty)$ are chosen for a given error tolerance of the approximation (3.1), and

$$F_{k,t} = \frac{2}{b-a} \text{Re} \left\{ \phi \left(\frac{k\pi}{b-a}, t \right) \cdot \exp \left(-\mathrm{i} \frac{ka\pi}{b-a} \right) \right\}, \tag{3.2}$$

where $\phi(\cdot, t)$ is the characteristic function of $\log(S_t)$. In practice, we can set a to be sufficiently small while b is chosen sufficiently large. Correspondingly, the distribution function F_{S_t} can be approximated by $\hat{F}_{S_t}$, given by

$$\hat{F}_{S_t}(x) = \hat{F}_{\log(S_t)} (\log(x)) . \tag{3.3}$$

Since $\hat{F}_{\log(S_t)}$ is given in closed form in terms of sine functions, its inverse can be easily calculated with the bisection method. Hence, the computational cost of the optimal strikes can be significantly reduced in this case.

The approximation formula (3.1) for a general random variable X can be derived as follows:

1. Truncate the support of the density function into a finite interval.

   Let f be the density function of a random variable X whose characteristic function ϕ is defined by

   $$\phi(\omega) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega x} f(x) \mathrm{d}x. \tag{3.4}$$

   As a density function, f, decays to zero at $\pm\infty$, the integration range in (3.4) can be truncated in an interval $[a, b] \subset \mathbb{R}$, which is large enough, such that

   $$\phi_1(\omega) = \int_a^b \mathrm{e}^{\mathrm{i}\omega x} f(x) \mathrm{d}x \approx \phi(\omega). \tag{3.5}$$

2. Approximate the density function f by an auxiliary function f_1, defined by $f_1 = f \cdot \mathbf{1}_{[a,b]}$, where $\mathbf{1}_{[a,b]}$ is an indicator function.

   Since the function f_1 has the compact support $[a, b]$, its Fourier-cosine series expansion is

   $$f_1(x) = \sum_{k=0}^{\prime\infty} A_k \cdot \cos \left(k\pi \frac{x-a}{b-a} \right), \tag{3.6}$$

   where $\sum'$ indicates that the first term in the summation is weighted by one-half, and where the cosine series coefficients A_k are given by

   $$A_k = \frac{2}{b-a} \int_a^b f(x) \cos \left(k\pi \frac{x-a}{b-a} \right) \mathrm{d}x \tag{3.7}$$

   $$\equiv \frac{2}{b-a} \text{Re} \left\{ \phi_1 \left(\frac{k\pi}{b-a} \right) \cdot \exp \left(-\mathrm{i} \frac{ka\pi}{b-a} \right) \right\} \tag{3.8}$$

where $\mathrm{Re}\{\cdot\}$ denotes the real part of the argument. Due to (3.5), A_k can be approximated by F_k, with

$$F_k = \frac{2}{b-a}\mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-\mathrm{i}\frac{ka\pi}{b-a}\right)\right\}. \tag{3.9}$$

Hence, f_1 can be approximated by f_2, with

$$f_2(x) = {\sum_{k=0}^{\infty}}' F_k \cdot \cos\left(k\pi\frac{x-a}{b-a}\right). \tag{3.10}$$

Then, f_2 can be approximated by a truncated summation f_3, with

$$f_3(x) = {\sum_{k=0}^{n}}' F_k \cdot \cos\left(k\pi\frac{x-a}{b-a}\right). \tag{3.11}$$

One may refer to (Fang and Oosterlee, 2008) for an error analysis on the Fourier-cosine series expansion.

3. Approximate the distribution function

   Given the approximations in the previous steps, we approximate the distribution function F_X on the finite interval $[a,b]$ in the following way,

$$\begin{aligned} F_X(x) = \int_{-\infty}^{x} f(s)\mathrm{d}s &\approx \int_a^x f_3(s)\mathrm{d}s \\ &= \int_a^x {\sum_{k=0}^{n}}' F_k \cdot \cos\left(k\pi\frac{s-a}{b-a}\right)\mathrm{d}s \\ &= {\sum_{k=0}^{n}}' F_k \int_a^x \cos\left(k\pi\frac{s-a}{b-a}\right)\mathrm{d}s \\ &= \frac{x-a}{b-a} + \sum_{k=1}^{n} F_k \cdot \frac{b-a}{k\pi}\cdot\sin(k\pi\frac{x-a}{b-a}) \end{aligned} \tag{3.12}$$

   where F_k is given in (3.9).

Note that expressions (3.1) and (3.2) follow from (3.12) and (3.9), respectively, by putting $X := \log(S_t)$.
To conclude this section, we summarize the key steps to calculate an optimal upper bond for an arithmetic Asian option.

Algorithm 1.

1. Calibrate the chosen stochastic model for the asset price to the market data; see, e.g., (Guillaume and Schoutens, 2012).
2. Approximate the distribution function of the underlying asset price at time t, according to (3.1)-(3.3).
3. Calculate the optimal strikes (2.6) and the prices of the corresponding European options.
4. Calculate the upper bound (2.5).

4 Numerical applications

In this section, we consider the Asian option price (2.1) at time zero with equal weights w_i, i.e.,

$$A(0,T,K;\xi) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{N}\xi\sum_{i=1}^{N}S_{t_i} - \xi K\right)^+ \middle| S_0 = s_0\right], \tag{4.1}$$

where S is the underlying asset price process. Rather than calculating $A(0,T,K;\xi)$, we approximate it with the upper bound (2.5).

In the Heston model (Heston, 1993), the asset price process (S) and the variance process (v) under a risk-neutral probability measure $\mathbb{Q}$ evolve according to the following system of stochastic differential equations:

$$\begin{cases} \mathrm{d}S_t = (r-q)S_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t, & S_0 > 0, \\ \mathrm{d}v_t = \kappa(\eta - v_t)\,\mathrm{d}t + \lambda\sqrt{v_t}\,\mathrm{d}\widetilde{W}_t, & v_0 = \sigma_0^2 > 0, \end{cases} \tag{4.2}$$

where $(W_t)_{0\le t\le T}$ and $(\widetilde{W}_t)_{0\le t\le T}$ are correlated Brownian motions satisfying $\mathrm{d}W_t\,\mathrm{d}\widetilde{W}_t = \rho\,\mathrm{d}t$. The parameter η is the long-term average variance, while κ is the speed of the mean-reversion of the variance. The parameter λ is referred to as the volatility of variance since it scales the diffusion term of the variance process.

Different from the Lévy models used in (Albrecher et al., 2005), the density function of the underlying is not immediately available in closed form in the Heston model (Drăgulescu and Yakovenko, 2002). It is time-consuming to build up the distribution from its density function. However, the characteristic function of the log-asset price in the Heston model is available and reads

$$\phi(\omega,t) := \mathbb{E}\left[\exp\left(\mathrm{i}\,\omega\log(S_t)\right) \mid S_0, v_0\right] = \exp(A+B+C), \tag{4.3}$$

where

$$\begin{aligned} A &= \mathrm{i}\,\omega\left(\log(S_0) + (r-q)t\right), \\ B &= \eta\kappa\lambda^{-2}\Big((\kappa - \rho\lambda\omega\mathrm{i} + d)\,t - 2\log\left((1-g\mathrm{e}^{dt})/(1-g)\right)\Big), \\ C &= v_0\lambda^{-2}(\kappa - \rho\lambda\omega\mathrm{i} + d)(1-\mathrm{e}^{dt})/(1-g\mathrm{e}^{dt}), \\ d &= \sqrt{(\rho\lambda\omega\mathrm{i} - \kappa)^2 + \lambda^2(\omega\mathrm{i} + \omega^2)}, \\ g &= (\kappa - \rho\lambda\omega\mathrm{i} + d)/(\kappa - \rho\lambda\omega\mathrm{i} - d). \end{aligned}$$

Hence, we can use the method proposed in Section 3 to approximate the distribution function and calculate its inverse with a bisection method. The parameters in (3.1) are set to be $n = 2^{14}$, $a = -10$, $b = 2500$. The implementation is performed in MATLAB (2014b) (Processor: Intel Core(TM) i7-3770 CPU @ 3.4GHz, RAM: 8GB). The computational cost to calculate the approximation of the distribution function (3.1) is less than $0.01s$. Following the lines of (Albrecher et al., 2005), we also approximated the distribution function by evaluating the integral of the density function with the *integral* function of MATLAB. Its computational cost is about $9.5s$. Hence, our method can efficiently accelerate the computation of the optimal upper bound in

Table 1: Calibrated Heston's model (Guillaume and Schoutens, 2012).

Calibration methods	κ	η	λ	v_0
RMSE full	0.5527	0.1271	0.3748	0.2403
ARPE full	3.1022	0.0923	0.3285	0.2513

the infinite market case, compared to the numerical integral method proposed by (Albrecher et al., 2005).

Based on the simulated paths from the underlying asset price (4.2), (Chen and Ewald, 2017) proposed to approximate its distribution function by the empirical one. We refer to (Sun, Gan and Vanmaele, 2015) for a comparison of the efficiency between this simulation-based method and the characteristic function-based method of the present paper.

With the well-calibrated Heston models (Table 1) from (Guillaume and Schoutens, 2012)[1], we calculate the option prices and the value bounds of the in-the-money Asian options monitored on the fourteen expiration dates of the benchmark SPX options. In the present case, we use the Asian option value (4.1) with $T = 737$ days, $N = 14$ and $t_i \in \{1, 9, 20, 37, 72, 100, 110, 191, 201, 282, 293, 373, 555, 737\}$. The continuously compounded interest rate r is set to be a constant 0.0153. The other model parameters are $S_0 = 873.59$ and $q = 0.0088$. The strikes of the Asian call options range from 500 to 860, while they range from 900 to 1200 for Asian put options. The difference between two consecutive strikes is 20. The Asian option prices are simulated using the Monte Carlo (MC) method with a control variate and $1 \cdot 10^7$ paths. The calculation of the optimal upper bound (2.5) involves the calculation of European option prices with the optimal strikes K_i from (2.6), which can be calculated with Algorithm 1. The European option prices are calculated with the COS method (Fang and Oosterlee, 2008).

The results for Asian call and put options are presented in Table 2 and Table 3, respectively. Comparing the Monte Carlo(MC)-based option price with the corresponding upper bound, show that, the upper bound is a good approximation for the price of an arithmetic Asian option. Note that the upper bound in the infinite market case depends on the model parameters. Hence, this upper bound can suffer from calibration risk as in the case when calculating the option price (Guillaume and Schoutens, 2012). When the monitoring dates of an arithmetic Asian option and the optimal strikes coincide with a set of European options quoted in the market, we can use the market prices of these European options to calculate the upper bound (2.5), rather than using a calibrated model. This is one method to avoid calibration risk. One may refer to (Chen et al., 2015) for more details on the aforementioned methods using European option market prices.

[1](Guillaume and Schoutens, 2012) proposed 18 calibration methods, and calibrated the Heston model to the market prices of SPX options quoted on 2008/12/11. Although the resulting 18 calibrated models fit the market data, we took, as illustrating examples, two of the calibrated models, RMSE and APE with full market data. The maturities of the benchmark options range from 1 day to 737 days.

Table 2: 737-day Asian call option price with the model parameters given in Table 1 and $S_0 = 873.59$, $r = 0.0153$, $q = 0.0088$.

	Heston MC		optimal upper bound	
K	RMSE full	ARPE full	RMSE full	ARPE full
500	367.4518	366.7792	369.0387	366.8906
520	348.5431	347.6028	350.8762	348.1456
540	329.8178	328.5379	332.9937	329.6096
560	311.3189	309.6240	315.4228	311.3229
580	293.0906	290.9071	298.1945	293.3283
600	275.1790	272.4383	281.3387	275.6693
620	257.6328	254.2744	264.8838	258.3901
640	240.4985	236.4761	248.8565	241.5343
660	223.8214	219.1070	233.2817	225.1443
680	207.6446	202.2300	218.1826	209.2607
700	192.0092	185.9056	203.5799	193.9215
720	176.9539	170.1928	189.4926	179.1618
740	162.5132	155.1474	175.9371	165.0132
760	148.7144	140.8173	162.9279	151.5034
780	135.5832	127.2431	150.4768	138.6554
800	123.1359	114.4563	138.5931	126.4878
820	111.3872	102.4799	127.2833	115.0126
840	100.3455	91.3270	116.5512	104.2373
860	90.0111	81.0021	106.3977	94.1619

Table 3: 737-day Asian call option price with the model parameters given in Table 1 and $S_0 = 873.59$, $r = 0.0153$, $q = 0.0088$.

	Heston MC		optimal upper bound	
K	RMSE full	ARPE full	RMSE full	ARPE full
900	93.9196	85.2710	110.1975	98.4652
920	105.0721	96.7637	121.1472	109.8240
940	116.8996	109.0194	132.6505	121.8267
960	129.3802	122.0031	144.6951	134.4479
980	142.4921	135.6745	157.2668	147.6597
1000	156.2090	149.9936	170.3496	161.4318
1020	170.5015	164.9148	183.9262	175.7329
1040	185.3406	180.3935	197.9783	190.5310
1060	200.6961	196.3825	212.4864	205.7936
1080	216.5342	212.8325	227.4307	221.4886
1100	232.8195	229.6980	242.7904	237.5840
1120	249.5156	246.9353	258.5444	254.0486
1140	266.5878	264.5003	274.6713	270.8519
1160	284.0014	282.3541	291.1490	287.9645
1180	301.7213	300.4598	307.9555	305.3581
1200	319.7158	318.7826	325.0685	323.0059

5 Conclusion

In this paper, we proposed an efficient method to accelerate the calculation of the static super-hedging portfolio (2.5)-(2.6) for an arithmetic Asian option for the case when the characteristic function of the underlying asset price is available in closed form. Firstly, this method recovers the distribution function of the underlying asset price from its characteristic function. Then, this approximation formula can be used to efficiently calculate the optimal strikes of European options associated with the comonotonic optimal upper bound for a discrete arithmetic Asian option value.

Numerical applications on the value bounds of an arithmetic Asian option were carried out with well-calibrated Heston models (Table 1). The proposed method was used to approximate the distribution function of the underlying asset, which is not available in closed form. Comparison between the time costs of the proposed method with that of the numerical integration method originally used by (Albrecher et al., 2005), highlights the efficiency of our method. Note that different calibrated models lead to different upper value bounds for the same arithmetic Asian option. This issue is related to calibration risk, which can be eliminated if the upper bound is calculated in a model-free framework; see, e.g., the finite market case in (Chen et al., 2015).

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