RISK-SHARING RULES AND THEIR PROPERTIES, WITH APPLICATIONS TO PEER-TO-PEER INSURANCE

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Abstract

This paper offers a systematic treatment of risk-sharing rules for insurance losses, based on a list of relevant properties. A number of candidate risk-sharing rules are considered, including the conditional mean risk-sharing rule proposed in Denuit and Dhaene (2012) and the newly introduced quantile risk-sharing rule. Their compliance with the proposed properties is established. Then, methods for building new risk-sharing rules are discussed. The results derived in this paper are helpful in the development of peer-to-peer insurance (or crowdsurance), as well as to manage contingent risk funds where a given budget is distributed among claimants.

Keywords: pooling, peer-to-peer (P2P) insurance, crowdsurance, conditional mean risk-sharing rule, quantile risk-sharing rule, comonotonicity.

1 Introduction and motivation

In a risk-sharing pool, each participant is compensated from the pool for his or her individual losses. In return, he or she pays an ex-post contribution to the pool, which is determined so that the sum of all the individual contributions matches the aggregate loss of the pool. The simplest case arises when the pool is homogeneous (i.e. the individual losses are identically distributed) and comprises exchangeable losses. In this case, a natural risk-sharing rule consists of every participant contributing ex-post an equal part of the aggregate loss and the latter is thus distributed uniformly over all pool members. We refer the reader to Albrecht and Huggenberger (2017) for a thorough investigation of the homogeneous case.

The problem to find an appropriate and simple risk-sharing rule in the heterogeneous case appears to be considerably more complex. For insurance purposes, some degree of standardization is needed so that risk sharing cannot integrate all aspects of individual preferences (e.g. individual utility functions) but only general behavioral traits, such as risk aversion. In this setting, Denuit and Dhaene (2012) introduced and investigated the conditional mean risk-sharing rule, where each participant contributes the conditional expectation of the loss brought to the pool, given the aggregrate loss covered by the pool. The properties of this risk-sharing rule have been further explored e.g. by Denuit and Robert (2021a, 2021b).

Recently, risk sharing has been re-visited in the context of peer-to-peer (P2P) insurance or crowdsurance. P2P insurance refers to risk sharing networks where a group of individuals pool their resources together to insure against a given peril. P2P insurance systems so revive the ancestral compensation mechanism consisting in using the contributions of the many to balance the misfortunes of the few. See, e.g., Abdikerimova and Feng (2022) and the references therein. Such risk-sharing mechanisms are also found in mutual insurance arrangements or partnerships among lawyers, farmers, or physicians for instance, who form risk pools to protect themselves against professional risks. Natural catastrophes and major industrial risks (e.g., induced by nuclear plants) are also typically covered by funds or pools where risk sharing operates. There is thus a clear need for a better understanding of risk-sharing rules to support the development of these emerging markets or strengthen the resilience against major risks.

In its pure form, P2P insurance does not transfer any risk to a partnering insurer so that there is no insurer who buffers the random difference between total premiums and total claims with own capital. It thus requires a high level of trust among participants. The reason is that the contributions to be paid ex post are theoretically unlimited and remain unknown until the end of the period, and some participants may be unable, or unwilling to pay their contributions at that time. Also, unless participants are ready to pay expensive contributions, the risk-bearing capacity of the community is limited. This is why in practice P2P insurance commonly includes some transfer to a partnering insurer. To avoid counterparty risk and to be able to deal with larger sums insured, Denuit (2020) considered a system where the upper risk layer is transferred to a partnering insurer whereas the community pools the lower layer, under the conditional mean risk-sharing rule. The risk sharing is then limited to the lower layer of individual risks. By partnering with an established insurance company, the community benefits from the claim settlement expertise developed by the insurer and its risk-bearing capacity. Each participant brings a deposit ex ante, replacing the insurance premium, with the guarantee that the final amount due never exceeds this down payment. If the common fund is insufficient to pay for the claims then the insurance carrier pays the excess. Conversely, if the pool has few claims then the surplus is given back to the participants or to a cause the pool members care about. A cash-back or give-back mechanism thus operates ex post to distribute the surplus. The analysis conducted in this paper also applies to this hybrid solution combining P2P insurance for the lower layer and traditional risk transfer to a partnering insurer for the upper layer, by restricting the losses to be shared to their lower layer.

Inspired by the literature devoted to premium calculation principles and risk measures, we propose a list of desirable properties for risk-sharing rules and we analyze the compliance of the conditional mean risk-sharing rule and the newly introduced quantile risk-sharing rule with this list. We also explain how to generate new risk-sharing rules by combining or adapting existing ones in various ways. This also allows us to bridge seemingly unrelated risk-sharing rules.

Part of the list of properties for risk-sharing rules proposed in this paper are in fact "conservation properties", in the sense that when they hold, they guarantee that certain properties of the stand-alone risk-sharing rule (where everyone keeps his or her own risk, see Example 2.4 for a formal definition) remain valid. The following properties that we consider in this paper belong to this class of "conservation properties": reshuffling, normalization, translativity, positive homogeneity, constancy, no-ripoff, and actuarial fairness. The stand-alone risk-sharing rule satisfies all these properties. Then, there are other properties which in one way or another, guarantee that the risk-sharing rule satisfies some properties that "improve" the situation compared to the stand-alone situation (or at least, they do not worsen the situation compared to the stand-alone situation). These properties could be called as "improvement properties". The following properties that we consider in this paper belong to this class: willingness-to-join (or convex-order improvement) and comonotonicity. The motivation for sharing risk enters via the improvement properties that we impose on the risk-sharing rule. The willingness-to-join property means in fact that when joining the pool, participants replace the original loss by a new loss which is "of the same size" (in the sense that expectations are equal) but which is "less variable" (in the sense that the newly allocated loss is likely to have less very large values, with smaller variance and stop-loss premiums). This corresponds to a particular stochastic dominance relation called convex order. Strict improvements are identified by comparing the variances before and after pooling. The comonotonicity property guarantees that the interests of the different participants are aligned. In practice, the willingness-to-join property should always hold to ensure the success of the risk-sharing scheme, whereas the comonotonicity property may be desirable or not, depending on the goal that participants want to reach. Next, there are some properties which modify the pool by redistributing losses among participants before pooling operates. The idea here is that redistribution operates locally, among pairs or small groups of participants, whereas risk-sharing rules act globally within the pool. The fair-bilateral-redistributing property, the fair-merging property and the fair-splitting property fall in this category. They guarantee that participants in the pool are not affected by local redistribution by others. These properties are referred to as "local redistribution properties". Notice that these properties are not relevant for all risk-sharing rules. For instance, we do not apply them to the stand-alone risk-sharing rule (since some participants redistribute losses among them and thus do not stay alone) nor to rules depending on quantities that do not adjust when losses are redistributed (as it will be the case for some rules used to illustrate this paper). Finally, there are properties corresponding to the behavior of the risk-sharing rule within some specific pools, containing perfectly positively dependent (or comonotonic) losses or exchangeable losses. This leads to the stand-alone property for comonotonic losses and the uniformity property for exchangeable losses that can be considered as "specific-pool properties".

Notice that other authors have proposed relevant properties for risk sharing. For instance, Hieber and Lucas (2020) imposed self-sufficiency, positivity and fairness for the distribution of mortality credits in their modern life-care tontines. In economics, Bourles et al. (2021) studied the implications of altruistic transfers in risk sharing. As far as we are aware, most properties proposed in the present paper have not yet been formalized in the literature devoted to risk sharing and open the door to a more systematic treatment of risk-sharing rules for insurance losses. We do not address the related problem of capital allocation which requires the specification of risk measures for every participant. Considering personal insurance lines, it seems indeed difficult that participants elicit risk measures governing their individual choices. We refer the reader e.g. to Filipovic and Svindland (2008) for a nice exposition and the connection between optimal capital and risk allocations. Let us also mention the contribution by Embrechts et al. (2018) who characterize (Pareto-)optimal risk-sharing rules, where the (Pareto-)optimality is expressed in terms of a sum of quantile-based risk measures applied to the individual losses in the pool. The approach in the present paper is different as we investigate properties that risk-sharing rules may or may not obey, and we determine the properties of some existing as well as newly introduced risk-sharing rules.

Several papers have also been devoted to cost sharing, adopting a game theoretic perspective. See e.g. Chen et al. (2017) for an application of this approach to risk sharing. These authors proposed two properties for an allocation rule: stability and monotonicity employing concepts of core and population monotonicity from cooperative game theory. Stability requires that no participant would face lower risk if he or she were alone rather than participating in the pool. Monotonicity requires that a new entry will not lead to higher risks allocated to existing participants. These concepts are very appealing but they appear to be difficult to transpose in the setting of this paper because they also require introducing risk measures. We come back to this issue in the concluding section.

It is noteworthy that the present paper proposes a list of properties that only depend on the losses or their distribution function and not on contextual elements such as individual or group preferences, risk measures, etc. This is in contrast with the well-developed literature devoted to optimal risk sharing (with potentially different definitions of Pareto optimality) where these contextual elements are required. Here, we propose a criterion based on the convex stochastic order relation which is closely related to several theories for choice under risk and risk measures but only involve the joint distribution of the losses to be shared. In this way, we thus avoid the introduction of any contextual element in our analysis. It may of course be possible to consider other stochastic order relations, but the convex order is widely used in actuarial science and economics so that it seems to be the ideal candidate in our setting.

The remainder of the text is organized as follows. Section 2 precisely defines risk sharing and risk-sharing rules. Some simple examples are given which are further used to illustrate the properties listed in Section 3. Section 4 is devoted to the conditional mean risk-sharing rule. After having recalled its definition, we study its compliance with the properties proposed in Section 3. Section 5 introduces the new quantile risk-sharing rule, where participants all contribute the quantile at the same probability level of their individual losses. The properties of this new risk-sharing rule are thoroughly investigated. Section 6 proposes different modifications to the conditional mean risk-sharing rule, resulting from a distribution change agreed by participants before risk sharing operates. This allows us to relate several existing risk-sharing rules, which offers a better understanding of their respective properties. Section 7 generates new risk-sharing rules from existing ones, by convex combination, network structure or restriction of the risk sharing to claimant participants. The latter approach applies to contingent risk funds where individuals contribute ex ante a fixed amount that is distributed ex post among those participants having experienced a pre-defined event. The final Section 8 discusses the results in connection to commercial insurance and emerging P2P insurance. The proofs of the main results are gathered in the appendix.

2 Risk sharing and risk-sharing rules

Consider n individuals, numbered $i = 1, 2, \ldots, n$. Each of them is exposed to some peril causing a random non-negative monetary loss at the end of a given observation period, taken to be the time interval $(0, 1)$. Let us denote the loss of person i by $X_i \geq 0$. Unless stated otherwise, we assume that the random variables X_i have finite means. These losses are defined on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We assume that this space is rich enough to contain all the random variables mentioned throughout the text. A n-dimensional random vector of losses $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_n)$ will be called a pool (of losses). The joint (cumulative) distribution function of the pool $\boldsymbol{X}_n$ is denoted by $F_{\boldsymbol{X}_n}$, the marginal distribution functions of the individual losses being denoted by $F_1, F_2, \ldots, F_n$, respectively. The total loss faced by the n participants in the pool $\boldsymbol{X}_n$ is denoted by $S_n = \sum_{i=1}^n X_i$. Throughout the paper, all equalities and inequalities between random variables are assumed to hold almost surely. Similarly, (in-)equalities between random vectors are meant to hold componentwise and almost surely. The latter are not to be confused with stochastic inequalities (corresponding to convex order in this paper).

Without any insurance or pooling, any individual bears his or her own loss, which means that individual i suffers the loss x_i, where x_i stands for the realization of X_i, which will be observed at time 1. Throughout this paper, we make the notational convention that a random variable is denoted by an upper-case letter (e.g. X_i), while its realization (observed at time 1) is generally denoted by the corresponding lower-case letter (e.g. x_i). A random vector is denoted by a bold upper-case letter with subscript indicating its dimension (e.g. $\boldsymbol{X}_n$), while its realization (known at time 1) is denoted by the corresponding bold lower-case small letter (e.g. $\boldsymbol{x}_n$).

Instead of each individual bearing his or her own risk, the participants in the pool $\boldsymbol{X}_n$ may decide to share their risks. Following von Bieberstein et al. (2019), we focus in this paper on formal risk pools where losses are observable and risk sharing among members can be specified in an explicit and perfectly enforceable contract, in the sense of the following definitions.

Definition 2.1 (risk sharing)**.** *Risk sharing in a pool* $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_n)$ *is a two-stage process. In the ex-ante step at time 0, the random losses comprised in the pool* $\boldsymbol{X}_n$ *are re-allocated by transforming* $\boldsymbol{X}_n$ *into another random vector* $\boldsymbol{H}(\boldsymbol{X}_n)$ *of the same dimension:*

$$\boldsymbol{H}(\boldsymbol{X}_n) = \big(H_1(\boldsymbol{X}_n), H_2(\boldsymbol{X}_n), \ldots, H_n(\boldsymbol{X}_n)\big), \tag{2.1}$$

where all $H_i(\boldsymbol{X}_n) \geq 0$*, and such that the full allocation property*

$$\sum_{i=1}^{n} H_i(\boldsymbol{X}_n) = \sum_{i=1}^{n} X_i \tag{2.2}$$

is satisfied.

The ex-post step takes place at time 1*, at the moment that the realization* $\boldsymbol{x}_n = (x_1, x_2, \ldots, x_n)$ *of the pool* $\boldsymbol{X}_n$ *is observed. At that time, each participant* i *to the pool* $\boldsymbol{X}_n$ *receives the realization* x_i *of his or her loss* X_i *from the pool. In return, each participant* i *contributes to the pool the realization* $H_i(\boldsymbol{x}_n)$ *of his or her re-allocated loss* $H_i(\boldsymbol{X}_n)$*.*

We are now ready to provide a formal definition of risk-sharing rules.

Definition 2.2 (risk-sharing rule)**.** *A risk-sharing rule* $\boldsymbol{H}$ *is a mapping which transforms any pool* $\boldsymbol{X}_n$ *into another random vector* $\boldsymbol{H}(\boldsymbol{X}_n)$ *of the same dimension, see* (2.1)*, where all* $H_i(\boldsymbol{X}_n) \geq 0$*, and such that the full allocation condition (2.2) is satisfied.*

Some risk-sharing rules $\boldsymbol{H}$ depend on individual losses X_i only through the aggregate loss S_n. For an aggregate risk-sharing rule, the only information that is not known at time 0 is thus the outcome of S_n, while for a general risk-sharing rule, the information not known at time 0 is the outcomes of the individual losses X_i. They are said to be aggregate risk-sharing rules. We refer the reader to Feng et al. (2020) for a discussion of this kind of rule, in comparison with general ones. For an aggregate risk-sharing rule $\boldsymbol{H}$, we will often use the simpler notation $\boldsymbol{H}(S_n)$ for $\boldsymbol{H}(\boldsymbol{X}_n)$.

Depending on the risk-sharing rule under consideration, $\boldsymbol{H}$ may require the knowledge of the joint distribution function $F_{\boldsymbol{X}_n}$ of $\boldsymbol{X}_n$ or just of a set of parameters (such as the mean values or the dimension of $\boldsymbol{X}_n$, for instance). By a slight abuse of notation, we keep the simple notation $H_i(\boldsymbol{X}_n)$ for the amount allocated to participant i, without referring explicitly to the underlying joint distribution function or parameter values.

Remark 2.3. In the hybrid solution combining P2P insurance for the lower layer and traditional risk transfer to a partnering insurer for the upper layer, as described in the introduction, X_i corresponds to the part of the loss that is not transferred to the partnering insurer. In liability insurance for instance, some large losses may happen which exceed the risk-sharing capacities of the P2P insurance community. Considering that loss amount Y_i for participant i is a compound sum of the form $Y_i = \sum_{k=1}^{N_i} C_{ik}$, an excess-of-loss cover may be needed to cap losses caused by a given event to some maximum l_i. In this case, the amount $\sum_{k=1}^{N_i}(C_{ik} - l_i)_+$ is transferred to the insurer and does not enter risk pooling. The corresponding premium must be supported directly by participant i. The random variable X_i then corresponds to the part of the loss not transferred to the insurer through the excess-of-loss cover. In our example,

$$X_i = \sum_{k=1}^{N_i} \min\{C_{ik}, l_i\}$$

when participants retain an amount up to l_i for each per-event loss C_{ik} while the upper layer (l_i, ∞) of each C_{ik} is transferred to the partnering insurer through an individual excess-of-loss cover.

Let us now have a look at some simple examples of risk-sharing rules that will be used to illustrate the properties listed in the next section. The first one corresponds to the case where individuals decide not to pool their risks.

Example 2.4 (stand-alone risk-sharing rule)**.** *The case where each participant in the pool bears his or her own risk can be described by the straightforward risk-sharing rule*

$$\boldsymbol{H}(\boldsymbol{X}_n) = \boldsymbol{X}_n. \tag{2.3}$$

This is just the stand-alone situation where each agent keeps his or her own initial loss. The pool just acts as a register to collect data about individual losses, without re-allocating them among participants.

The most simple and well-known risk-sharing rule consists of equally sharing the aggregate loss S_n. In this case, S_n is uniformly distributed over all participants, leading to the following risk-sharing rule.

Example 2.5 (uniform risk-sharing rule)**.** *The uniform risk-sharing rule* $\boldsymbol{H}^{\text{uni}}$ *is an aggregate risk-sharing rule defined as*

$$H_i^{\text{uni}}(\boldsymbol{X}_n) = H_i^{\text{uni}}(S_n) = \frac{S_n}{n}, \qquad i = 1, 2, \ldots, n. \tag{2.4}$$

Individual contributions are thus equal for all participants.

Proportional rules have been widely used in a risk-sharing context in practice. Participants adopting such rule agree to take a fixed percentage of the total loss S_n, in accordance with the expected values, the variance or the standard deviation of the losses they bring to the pool, for instance. In this paper, we consider the following simple rule.

Example 2.6 (mean proportional risk-sharing rule)**.** *The mean proportional risk-sharing rule* $\boldsymbol{H}^{\text{prop}}$ *is an aggregate risk-sharing rule defined as*

$$H_i^{\text{prop}}(\boldsymbol{X}_n) = H_i^{\text{prop}}(S_n) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n, \qquad i = 1, 2, \ldots, n.$$

Participants adopting $\boldsymbol{H}^{\text{prop}}$ *thus agree to take a fixed percentage of the total loss* S_n*, in accordance with the expected values of the losses they bring to the pool compared to the total expected loss.*

The idea behind the following two risk-sharing rules is that participants can be ranked according to any relevant characteristic reflecting their risk-bearing capacity, and losses are allocated to them according to their magnitude.

Example 2.7 (order statistics risk-sharing rule)**.** *Recall that order statistics, denoted as* $X_{(i)}$*, correspond to the components of* $\boldsymbol{X}_n$ *ranked in ascending order, that is,* $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$*. Under the order statistics risk-sharing rule for a pool* $\boldsymbol{X}_n$*, the re-allocated loss vector is defined by*

$$\boldsymbol{H}^{\text{ord}}\left(\boldsymbol{X}_n\right) = \left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right). \tag{2.5}$$

Considering for instance age as an indicator of financial strength, the largest loss $X_{(n)}$ *could be allocated to the oldest participant, numbered* n*, the second-largest* $X_{(n-1)}$ *to the second-oldest numbered* $n-1$*, and so on. The risk-sharing rule* $\boldsymbol{H}^{\text{ord}}$ *is not an aggregate risk-sharing rule.*

Example 2.8 (multiple layer risk-sharing rule)**.** *Consider increasing limits* $0 < d_1 < d_2 < \ldots < d_{n-1}$ *defining layers for the total loss* S_n*, irrespective of the distribution of* $X_1, X_2, \ldots, X_n$*. Under the multiple layer risk-sharing rule, the re-allocated loss vector for any* $\boldsymbol{X}_n$ *is defined by*

$$\boldsymbol{H}^{\text{layer}}\left(\boldsymbol{X}_n\right) = \boldsymbol{H}^{\text{layer}}\left(S_n\right) = \left(\min\{S_n, d_1\}, (S_n - d_1)_+ - (S_n - d_2)_+, \ldots, (S_n - d_{n-1})_+\right). \tag{2.6}$$

Participant i *thus covers that part of the aggregate loss* S_n *in the layer* (d_{i-1}, d_i)*, setting* $d_0 = 0$ *and* $d_n = \infty$*.*

Notice that another possibility would be to let the limits d_i depend on the (distribution of the) total losses comprised in the pool, for instance by taking for d_i the quantile at level i/n of S_n. In the definition of the multiple layer risk-sharing rule, the limits are defined without reference to the total losses and reflect participants risk-bearing capacities.

For applying the uniform, the order statistics or the multiple layer risk-sharing rules, we do not need to know anything about the joint distribution function $F_{\boldsymbol{X}_n}$ of $\boldsymbol{X}_n$. These rules are thus free of model risk but are not necessarily reasonable choices, depending on the circumstances. The mean proportional risk-sharing rule depends on the joint distribution function $F_{\boldsymbol{X}_n}$ of $\boldsymbol{X}_n$ through expected values, only.

3 Properties of risk-sharing rules

Goovaerts et al. (1984) pioneered the systematic study of the properties that any premium principle should/could possess. Afterwards, numerous authors suggested various requirements that any risk measure should/could satisfy. The interested reader is referred to Denuit et al. (2005) for a general account of the topic. We propose hereafter a non-exhaustive list of reasonable (not necessarily independent) requirements that a risk-sharing rule should/could fulfill and discuss their interpretation. Some requirements are directly inspired from premium calculation principles or risk measures, whereas others are tailored to risk-sharing rules.

3.1 Conservation properties

As explained in the introduction, we begin with properties to be maintained, that is, properties of the stand-alone risk-sharing rule that may also be desirable in general for any rule.

3.1.1 Reshuffling

Let π be a permutation of $\{1, \ldots, n\}$. We say that the pool $\boldsymbol{X}_n$ is reshuffled into the pool $\boldsymbol{X}_n^\pi$ if both random vectors are composed of the same individual losses, but only their places are interchanged, that is, $\boldsymbol{X}_n^\pi = \left(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}\right)$. The reshuffling property expresses the idea that the order in which the individual participants are considered does not change their contribution. This is precisely stated next.

Definition 3.1 (reshuffling property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the reshuffling property if for any pool $\boldsymbol{X}_n$ and any of its reshuffled versions $\boldsymbol{X}_n^\pi$, one has that*

$$\pi(i) = j \Rightarrow H_i\left(\boldsymbol{X}_n\right) = H_j\left(\boldsymbol{X}_n^\pi\right) \text{ for any } i = 1, \ldots, n. \tag{3.1}$$

Example 3.2. *The uniform risk-sharing rule satisfies the reshuffling property since individual contributions are equal for all participants.*

Example 3.3. *The mean proportional rule satisfies the reshuffling property since*

$$\pi(i) = j \Rightarrow H_i^{\text{prop}}\left(\boldsymbol{X}_n\right) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n = H_j^{\text{prop}}\left(\boldsymbol{X}_n^\pi\right) \text{ for any } i = 1, \ldots, n.$$

Example 3.4. *Neither the order statistics rule nor the multiple layer rule satisfy the reshuffling property since the numbering of participants reflects their risk-bearing capacity and the magnitude of losses or layer they assume.*

3.1.2 Normalization

For premium calculation principles, the normalization property means that a zero risk leads to a zero premium. From the full allocation condition (2.2) and the assumed non-negativity $H_i\left(\boldsymbol{X}_n\right) \geq 0$ for all i, we deduce that

$$X_i = 0 \text{ for all } i \Rightarrow H_i(\boldsymbol{X}_n) = 0 \text{ for all } i.$$

It seems reasonable to require that a participant bringing a zero loss should not contribute ex post to S_n. This is in essence the following property, which can be seen as an appropriate normalization property for risk-sharing rules. Its statement requires the following notation. For any pool $\boldsymbol{X}_n$, the reduced pool without participant l, denoted as $\boldsymbol{X}_{n-1}^{(\backslash l)}$, is defined as follows:

$$\boldsymbol{X}_{n-1}^{(\backslash l)} = \left(X_1, X_2, \ldots, X_{l-1}, X_{l+1}, \ldots, X_n\right). \tag{3.2}$$

Notice that we do not change the label of the individual losses when moving from $\boldsymbol{X}_n$ to $\boldsymbol{X}_{n-1}^{(\backslash l)}$. In other words, for any $i \in \{1, 2, \ldots, l-1, l+1, \ldots, n\}$, we call X_i the ith component of $\boldsymbol{X}_{n-1}^{(\backslash l)}$ so that $\boldsymbol{X}_{n-1}^{(\backslash l)}$ has no lth component. This means that in the counting, we skip l and we do not change the "names" of the remaining participants. We make a similar convention for the random vector $\boldsymbol{H}\left(\boldsymbol{X}_{n-1}^{(\backslash l)}\right)$. Precisely, for any $i \in \{1, 2, \ldots, l-1, l+1, \ldots, n\}$, we call $H_i\left(\boldsymbol{X}_{n-1}^{(\backslash l)}\right)$ the ith component of $\boldsymbol{H}\left(\boldsymbol{X}_{n-1}^{(\backslash l)}\right)$. We are now ready to state the following definition.

Definition 3.5 (normalization property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the normalization property if for any pool $\boldsymbol{X}_n$ with $X_j = 0$ for some $j = 1, \ldots, n$, one has that*

$$H_i(\boldsymbol{X}_n) = H_i\left(\boldsymbol{X}_{n-1}^{(\backslash j)}\right) \text{ for any } i \neq j, \tag{3.3}$$

where $\boldsymbol{X}_{n-1}^{(\backslash j)}$ is defined in (3.2).

Taking into account the full allocation condition (2.2), the normalization property (3.3) leads to

$$X_j = 0 \Rightarrow H_j(\boldsymbol{X}_n) = 0. \tag{3.4}$$

Considering (3.3)-(3.4), the normalization property means that individual contributions remain unchanged if a zero risk is added to the pool and that a zero risk has a zero contribution. This reasoning can be iterated so that we see that any subset of losses $X_j = 0$ can be excluded from the risk sharing when participants agree to use a risk-sharing rule satisfying the normalization property.

Remark 3.6. If the risk-sharing rule satisfies the reshuffling property then the normalization property can be defined by considering only a given participant, for instance the last one. Precisely, a risk-sharing rule $\boldsymbol{H}$ satisfying the reshuffling property also satisfies the normalization property if for any pool $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_{n-1}, 0)$, one has that

$$H_i(\boldsymbol{X}_n) = H_i(\boldsymbol{X}_{n-1}) \text{ for all } i \neq n, \tag{3.5}$$

where $\boldsymbol{X}_{n-1} = (X_1, X_2, \ldots, X_{n-1})$.

Example 3.7. *The uniform risk-sharing rule does not satisfy the normalization property. Considering for instance $n = 3$, we see that*

$$H_1^{\text{uni}}\big((X_1, X_2, 0)\big) = \frac{X_1 + X_2}{3} \neq \frac{X_1 + X_2}{2} = H_1^{\text{uni}}\big((X_1, X_2)\big).$$

Participant 3 bringing a zero loss to the pool must also contribute to total losses under the uniform rule.

Example 3.8. *The mean proportional rule satisfies the normalization property, as shown next. Since it satisfies the reshuffling property, it is enough to consider the last participant and establish the validity of* (3.5). *Clearly, $S_n = S_{n-1}$ when $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_{n-1}, 0)$ and $\boldsymbol{X}_{n-1} = (X_1, X_2, \ldots, X_{n-1})$ so that $H_i^{\text{prop}}(\boldsymbol{X}_n) = H_i^{\text{prop}}(\boldsymbol{X}_{n-1})$ for $i = 1, \ldots, n-1$. Moreover, $H_n^{\text{prop}}(\boldsymbol{X}_n) = 0$ since $\mathrm{E}[X_n] = 0$ when $X_n = 0$.*

Example 3.9. *Neither the order statistics rule nor the multiple layer rule satisfy the normalization property since they are both based on a specific ranking for participants to the pool who are attributed a given order statistic or layer, whatever the loss they bring to the pool.*

3.1.3 Translativity

The property of translativity is often imposed on premium calculation principles as well as on risk measures. It basically means that adding a constant to a given loss shifts the premium and capital in a similar way. Hereafter, we define an adapted version of translativity for risk sharing. Let us introduce the notation $\mathbf{1}_{i,n}$ to indicate the n-dimensional vector $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$, with all components equal to 0, except the ith component which equals 1.

Definition 3.10 (translativity property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the translativity property if for any pool $\boldsymbol{X}_n$ and any participant $j = 1, \ldots, n$, one has that*

$$H_i\left(\boldsymbol{X}_n + c\ \mathbf{1}_{j,n}\right) = H_i\left(\boldsymbol{X}_n\right) \text{ for all } i \neq j \text{ and any } c \geq 0. \tag{3.6}$$

Translativity of a risk-sharing rule means that in case a particular participant's loss is increased by a deterministic amount $c \geq 0$, the contributions of the other participants remain unchanged. Applying the full-allocation property (2.2) to $\boldsymbol{X}_n$ and to $\boldsymbol{X}_n + c\ \mathbf{1}_{j,n}$, respectively leads to

$$H_j\left(\boldsymbol{X}_n + c\ \mathbf{1}_{j,n}\right) = H_j\left(\boldsymbol{X}_n\right) + c, \tag{3.7}$$

which means that any increase of X_j by a deterministic amount $c \geq 0$ results in the same increase of the contribution for participant j.

Remark 3.11. If the risk-sharing rule satisfies the reshuffling property, it is enough to consider a given participant to define the translativity property, for instance the last one $j = n$.

Despite its reasonableness, it turns out that translativity is not fulfilled by the simple risk-sharing rules considered so far (except the stand-alone one).

Example 3.12. *The uniform risk-sharing rule does not satisfy the translativity property. Considering for instance $n = 3$, we see that*

$$H_1^{\text{uni}}\big((X_1, X_2, X_3 + c)\big) = \frac{S_3 + c}{3} \neq \frac{S_3}{3} = H_1^{\text{uni}}\big((X_1, X_2, X_3)\big).$$

Participants 1 and 2 must thus also contribute to the deterministic increase in the loss brought to the pool by participant 3.

Example 3.13. *The mean proportional rule does not satisfy the translativity property since, for all $i \neq n$,*

$$H_i^{\text{prop}}\left(\boldsymbol{X}_n + c\ \mathbf{1}_{n,n}\right) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n + c]}\left(S_n + c\right) \neq \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n = H_i^{\text{prop}}\left(\boldsymbol{X}_n\right).$$

The constant addition c is thus also shared among all participants under the mean proportional rule whereas for a risk-sharing rule satisfying the translativity property, it should be assumed by participant n, only.

Example 3.14. *Neither the order statistics rule (since adding a constant c to X_j may modify order statistics) nor the multiple layer rule satisfy the translativity property.*

Some risk-sharing rules satisfying the translativity property will be discussed in the next sections.

3.1.4 Positive homogeneity

Positive homogeneity for premium calculation principles and risk measures means that multiplying the risk under consideration with a positive constant implies multiplying the initial premium or capital value with the same constant. It appears to be reasonable for risk-sharing rules too, and is formally defined as follows.

Definition 3.15 (positive homogeneity property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the positive homogeneity property if for any pool $\boldsymbol{X}_n$, one has that*

$$H_i\left(c\boldsymbol{X}_n\right) = cH_i\left(\boldsymbol{X}_n\right) \text{ for all } i \text{ and any } c \geq 0. \tag{3.8}$$

Example 3.16. *The uniform risk-sharing rule satisfies the positive homogeneity property since*

$$H_i^{\text{uni}}(c\boldsymbol{X}_n) = \frac{cS_n}{n} = cH_i^{\text{uni}}(\boldsymbol{X}_n) \text{ for } i = 1, \dots, n.$$

Example 3.17. *The mean proportional and the order statistics rules both satisfy the positive homogeneity property.*

Example 3.18. *The multiple layer rule does not satisfy the positive homogeneity property since the limits d_j remain unchanged.*

3.1.5 Constancy

Let us now consider the case where the risk brought by one participant to the pool is not random, but equal to some non-negative constant c with probability 1. The case $c = 0$ corresponds to normalization.

Definition 3.19 (constancy property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the constancy property if for any pool $\boldsymbol{X}_n$ with $X_j = c$ for some constant $c \geq 0$ and $j = 1, \dots, n$, one has that* (3.3) *holds true.*

Taking into account the full-allocation condition (2.2), applied to $\boldsymbol{X}_n$ and $\boldsymbol{X}_{n-1}^{(\setminus j)}$, respectively, it follows from Definition 3.19 that

$$X_j = c \Rightarrow H_j\left(\boldsymbol{X}_n\right) = c. \tag{3.9}$$

The constancy property means that in case the risk brought by participant j to the pool is equal to some non-negative constant c, then this participant should only contribute that amount c to the pool, whereas the contributions of the other participants are not depending on whether the constant is added to the pool or not.

Remark 3.20. If the risk-sharing rule satisfies the reshuffling property then the constancy property can be defined by only considering a given participant j in the pool with a constant loss $c \geq 0$, for instance, the last one with $j = n$.

As it was the case for translativity, it turns out that the elementary risk-sharing rules considered so far do not satisfy the constancy property (except the stand-alone one).

Example 3.21. *The uniform risk-sharing rule does not satisfy the constancy property. Considering for instance $n = 3$, we see that*

$$H_1^{\text{uni}}\big((X_1, X_2, c)\big) = \frac{X_1 + X_2 + c}{3} \neq \frac{X_1 + X_2}{2} = H_1^{\text{uni}}\big((X_1, X_2)\big).$$

The deterministic loss $X_3 = c$ is thus also distributed uniformly over the three participants and not only supported by participant 3 under the uniform rule.

Example 3.22. *The mean proportional rule does not satisfy the constancy property. Indeed, let $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_{n-1}, c)$, for some constant $c > 0$, and let $\boldsymbol{X}_{n-1} = (X_1, X_2, \ldots, X_{n-1})$. Then we have that, for any $i \neq n$,*

$$H_i^{\text{prop}}\left(\boldsymbol{X}_n\right) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_{n-1} + c]}\left(S_{n-1} + c\right) \neq \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_{n-1}]} S_{n-1} = H_i^{\text{prop}}\left(\boldsymbol{X}_{n-1}\right).$$

Example 3.23. *Neither the order statistics rule nor the multiple layer rule satisfy the constancy property.*

Examples of risk-sharing rules satisfying the constancy property will be discussed in the next sections.

Remark 3.24 (Normalization and translativity imply constancy)**.** One can easily prove that a risk-sharing rule $\boldsymbol{H}$ satisfying the normalization property (3.3) and the translativity property (3.6), also satisfies the constancy property. Indeed, it follows from the translativity property that for any $i \neq n$, we have that

$$\begin{aligned} H_i\big((X_1, X_2, \ldots, X_{n-1}, c)\big) &= H_i\big(\,(X_1, X_2, \ldots, X_{n-1}, 0) + c\ \mathbf{1}_{n,n}\big) \\ &= H_i\big((X_1, X_2, \ldots, X_{n-1}, 0)\big). \end{aligned}$$

Taking into account the normalization property, we then find that

$$H_i\big((X_1, X_2, \ldots, X_{n-1}, c)\big) = H_i\big((X_1, X_2, \ldots, X_{n-1})\big) \text{ for } i \in \{1, \ldots, n-1\}.$$

A similar proof can be given for the case where $X_j = c$, for any j.

3.1.6 No-ripoff

Define the largest value for the loss X_i as

$$F_i^{-1}(1) = \inf\{x \in \mathbb{R} | F_i(x) = 1\},$$

where $\inf \emptyset = \infty$, by convention. It seems to be unfair to ask participants to contribute more than their maximal loss value $F_i^{-1}(1)$. This is in essence the no-ripoff requirement.

Definition 3.25 (no-ripoff property)**.** *A risk-sharing rule $\boldsymbol{H}$ possesses the no-ripoff property if for any pool $\boldsymbol{X}_n$, one has that*

$$H_i\left(\boldsymbol{X}_n\right) \leq F_i^{-1}(1) \text{ for any } i = 1, 2, \ldots, n. \tag{3.10}$$

Taking into account the fact that the re-allocated losses have to be non-negative, we find that the no-ripoff condition implies that any reallocated loss $H_i(\boldsymbol{X}_n)$ is valued in $\left[0, F_i^{-1}(1)\right]$. This intuitive requirement may nevertheless be violated when participants adopt the simple risk-sharing rules considered so far, as shown next.

Example 3.26. *The uniform risk-sharing rule does not satisfy the no-ripoff property. Considering for instance $n = 2$ and $\boldsymbol{X}_2 = (U, 2U)$ with U uniformly distributed over the interval $[0, 1]$, we have*

$$H_1^{\text{uni}}\big((U, 2U)\big) = \frac{3U}{2} \in [0, 1.5]$$

while $F_1^{-1}(1) = 1$.

Example 3.27. *The mean proportional risk-sharing rule does not necessarily satisfy the no-ripoff property since we might end up with $H_i^{\text{prop}}(S_n) > F_i^{-1}(1)$ when S_n becomes large. This questions the appropriateness of the mean proportional risk-sharing rule in adverse scenarios.*

Example 3.28. *Neither the order statistics rule nor the multiple layer rule satisfy the no-ripoff property.*

Examples of risk-sharing rules satisfying the no-ripoff property will be discussed in the next sections.

3.1.7 Actuarial fairness

Actuarial fairness of a risk-sharing rule means that on average, participants do neither gain nor lose from risk sharing, in the sense that their expected contribution (by joining the pool) is equal to their expected loss (when staying alone).

Definition 3.29 (actuarial fairness property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the actuarial fairness property (or is said to be an actuarially fair risk-sharing rule) if for any pool $\boldsymbol{X}_n$, one has that*

$$\mathrm{E}\left[H_i(\boldsymbol{X}_n)\right] = \mathrm{E}\left[X_i\right] \text{ for any } i = 1, 2, \ldots, n. \tag{3.11}$$

Example 3.30. *The uniform risk-sharing rule does not satisfy the actuarial fairness property. Considering for instance $n = 2$ and $\boldsymbol{X}_2 = (U, 3U)$ with U uniformly distributed over the interval $[0, 1]$, we see that*

$$H_1^{\text{uni}}\big((U, 3U)\big) = 2U$$

and

$$\mathrm{E}\big[H_1^{\text{uni}}\big((U, 3U)\big)\big] = 1 \neq \frac{1}{2} = \mathrm{E}[U].$$

Example 3.31. *The mean proportional rule satisfies the actuarial fairness property since*

$$\mathrm{E}\left[H_i^{\text{prop}}(\boldsymbol{X}_n)\right] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]}\mathrm{E}[S_n] = \mathrm{E}\left[X_i\right], \text{ for any } i = 1, 2, \ldots, n.$$

Example 3.32. *The order statistics rule does not satisfy the actuarial fairness property since the inequalities*

$$\mathrm{E}\left[H_1^{\mathrm{ord}}\left(\boldsymbol{X}_n\right)\right] = \mathrm{E}[X_{(1)}] = \mathrm{E}[\min\{X_1, \ldots, X_n\}] < \mathrm{E}[X_1]$$

and

$$\mathrm{E}\left[H_n^{\mathrm{ord}}\left(\boldsymbol{X}_n\right)\right] = \mathrm{E}[X_{(n)}] = \mathrm{E}[\max\{X_1, \ldots, X_n\}] > \mathrm{E}[X_n]$$

are generally true, except in some trivial cases. This means that participant 1 gains on average by joining the pool whereas participant n looses on average by doing so. Also, the multiple layer rule does not satisfy the actuarial fairness property.

3.2 Improvement properties

3.2.1 Willingness to join or convex-order improvement

Usually, risk-sharing rules are set up such that each person i prefers paying $H_i(\boldsymbol{X}_n)$ over paying X_i corresponding to the stand-alone position. Hence, participants are willing to join the pool. This preference could be expressed in different ways. It is important to realize that we aim at some standardization, as it is commonly the case with traditional insurance products. This is a necessary requirement for managing large insurance pools. Thus, individual preferences are only partially taken into account. First, by selecting the level of risk retention at individual level, for instance within a predefined menu of deductibles. Second, even if the proposed coverage is not optimal for each individual participant, given his or her particular preferences, it must be attractive to all members of a reasonable class (e.g. risk-averse) economic agents.

In this paper, we use the following stochastic dominance relations expressing the common preferences shared by all risk-averse economic agents in the expected utility setting for choice under risk. Given two losses X and Y, X is said to be smaller than Y in the increasing convex order, henceforth denoted by $X \preceq_{\mathrm{ICX}} Y$ if $\mathrm{E}[u(\kappa - X)] \geq \mathrm{E}[u(\kappa - Y)]$ for any non-decreasing concave utility function u and wealth level κ, provided the expectations exist. The increasing convex order is also called stop-loss order in actuarial science since it can be characterized by pointwise comparison of stop-loss transforms: $X \preceq_{\mathrm{ICX}} Y \Leftrightarrow \mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+]$ for all real d. The case where X and Y have the same expected value corresponds to the convex order $\preceq_{\mathrm{CX}}$ which is defined as follows:

$$X \preceq_{\mathrm{CX}} Y \Leftrightarrow X \preceq_{\mathrm{ICX}} Y \text{ and } \mathrm{E}[X] = \mathrm{E}[Y].$$

Thus, $\preceq_{\mathrm{CX}}$ only applies to losses with the same expected value. The term "convex" is used since $X \preceq_{\mathrm{CX}} Y \Leftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all convex functions g for which the expectations exist. The stochastic inequality $X \preceq_{\mathrm{CX}} Y$ intuitively means that X and Y have the same "size" (as $\mathrm{E}[X] = \mathrm{E}[Y]$ holds) but that Y is "more variable" than X. For instance, the variance of Y is larger than the variance of X. For a thorough description of the convex and increasing convex orders and of their applications in insurance studies, we refer the reader to Denuit et al. (2005). A general reference about stochastic order relations is Shaked and Shanthikumar (2007).

Definition 3.33 (willingness-to-join property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the willingness-to-join property if for any pool $\boldsymbol{X}_n$, one has that*

$$H_i(\boldsymbol{X}_n) \preceq_{\text{CX}} X_i \text{ holds for every } i = 1, 2, \ldots, n. \tag{3.12}$$

Remark 3.34. Notice that replacing the convex order $\preceq_{\text{CX}}$ with the increasing convex order $\preceq_{\text{ICX}}$ in (3.12) leads to the same definition. In other words, it suffices to require $\preceq_{\text{ICX}}$ instead of the stronger $\preceq_{\text{CX}}$ in (3.12). Indeed, this is because

$$H_i(\boldsymbol{X}_n) \preceq_{\text{ICX}} X_i \text{ for every } i = 1, 2, \ldots, n \Rightarrow \text{E}[H_i(\boldsymbol{X}_n)] \leq \text{E}[X_i] \text{ for every } i = 1, 2, \ldots, n.$$

But the full allocation condition (2.2) implies that

$$\sum_{i=1}^{n} \text{E}[H_i(\boldsymbol{X}_n)] = \sum_{i=1}^{n} \text{E}[X_i]$$

so that $\text{E}[H_i(\boldsymbol{X}_n)] = \text{E}[X_i]$ must in fact hold for every $i = 1, 2, \ldots, n$, and the stronger stochastic inequality $H_i(\boldsymbol{X}_n) \preceq_{\text{CX}} X_i$ is valid. This reasoning also shows that the willingness-to-join property for all risk-averse economic agents implies actuarial fairness.

Example 3.35. *The uniform risk-sharing rule does not satisfy the willingness-to-join property since it is not actuarially fair.*

Example 3.36. *The mean proportional rule satisfies the willingness-to-join property if*

$$H_i^{\text{prop}}(S_n) = \frac{\text{E}[X_i]}{\text{E}[S_n]} S_n \preceq_{\text{CX}} X_i \text{ holds for every } i = 1, 2, \ldots, n,$$

which is equivalent to

$$\frac{S_n}{\text{E}[S_n]} \preceq_{\text{CX}} \frac{X_i}{\text{E}[X_i]} \text{ for every } i = 1, 2, \ldots, n.$$

The latter $\preceq_{\text{CX}}$-inequality means that S_n and X_i must be ranked in the Lorenz order. Hence, the willingness-to-join property does not hold in general for the mean proportional rule but it does in some particular cases. We refer the reader to Section 2.4.2 in Denuit and Robert (2021c) for more details.

Example 3.37. *Neither the order statistics rule nor the multiple layer one satisfy the willingness-to-join property, because none of them is actuarially fair.*

The inequality in the sense of the convex order appearing in Definition 3.33 is not strict. Equality with respect to the convex order is meant in distribution: if $H_i(\boldsymbol{X}_n)$ is distributed as X_i (as it is the case with the stand-alone risk-sharing rule) then participant i is indifferent between joining the pool or staying alone. Therefore, we consider that joining the pool leads to a strict improvement for participant i if $H_i(\boldsymbol{X}_n)$ is not distributed as X_i, that is, if there exists a convex function g such that $\text{E}[g(H_i(\boldsymbol{X}_n))] < \text{E}[g(X_i)]$. The following simple test can be used to check whether there is a strict improvement, based on the respective variances of $H_i(\boldsymbol{X}_n)$ and X_i.

Proposition 3.38. *For any risk-sharing rule $\boldsymbol{H}$ satisfying the willingness-to-join property and any pool $\boldsymbol{X}_n$, there is a strict improvement for participant i if* $\mathrm{Var}[H_i(\boldsymbol{X}_n)] < \mathrm{Var}[X_i]$ *and this participant is indifferent between joining the pool or staying alone if* $\mathrm{Var}[H_i(\boldsymbol{X}_n)] = \mathrm{Var}[X_i]$*.*

Proof. Recall that given two random variables X and Y,

$$X \preceq_{\mathrm{CX}} Y \text{ and } \mathrm{Var}[X] = \mathrm{Var}[Y] \Rightarrow X \text{ and } Y \text{ are identically distributed.}$$

See e.g. Theorem 3.A.42 in Shaked and Shanthinkumar (2007). Therefore, there is a strict improvement for participant i if $\mathrm{Var}[H_i(\boldsymbol{X}_n)] < \mathrm{Var}[X_i]$ and participant i is indifferent between joining the pool or staying alone when these variances are equal. □

Since variances are often easy to compute, this provides the analyst with an effective test to check whether some risk-sharing rule satisfying the willingness-to-join property leads to a strict improvement when participants pool their respective losses.

Remark 3.39. Notice that $H_i(\boldsymbol{X}_n)$ can be distributed as X_i for other risk-sharing rules than the stand-alone one. Assume for instance that F_{S_n} is continuous and define $\boldsymbol{H}$ as $H_i(\boldsymbol{X}_n) = H_i(S_n) = F_i^{-1}\big(F_{S_n}(S_n)\big)$. Then, $H_i(S_n)$ is distributed as X_i and participant i is indifferent between joining the pool or staying alone according to Definition 3.33. Replacing X_i with $H_i(S_n)$ may nevertheless be considered as attractive by participant i when the latter is less correlated to his or her random end-of-period wealth.

3.2.2 Comonotonicity and aggregate risk-sharing rules

Recall that the left-continuous inverse of the distribution function F_Y of a random variable Y is defined by

$$F_Y^{-1}(p) = \inf\{y \in \mathbb{R} \mid F_Y(y) \geq p\}, \qquad p \in [0,1]. \tag{3.13}$$

As previously stated, $\inf \emptyset = \infty$ by convention in (3.13). A random vector $\boldsymbol{Y}_n = (Y_1, \ldots, Y_n)$ is said to be comonotonic if $\boldsymbol{Y}_n$ is distributed as $\big(F_{Y_1}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\big)$ for some random variable U uniformly distributed over the interval $[0,1]$. Intuitively stated, $\boldsymbol{Y}_n$ is comonotonic if the increase of one of its components implies an increase of all its other components. Comonotonicity is an important dependency structure, with many applications in insurance and finance, see e.g. Dhaene et al. (2002a, 2002b) and Deelstra et al. (2010).

In the context of risk sharing, comonotonicity of the re-allocated loss vector $\boldsymbol{H}(\boldsymbol{X}_n)$ might be a desirable property since it ensures that the interests of all participants are aligned, in the sense that they all have an interest in keeping their losses as small as possible. This leads to the following definition.

Definition 3.40 (comonotonicity)**.** *A risk-sharing rule $\boldsymbol{H}$ is said to be comonotonic if for any pool $\boldsymbol{X}_n$ one has that $\boldsymbol{H}(\boldsymbol{X}_n)$ is a comonotonic random vector.*

Comonotonicity is also referred to as the no-sabotage condition, after Carlier and Dana (2003). It ensures that no individual contribution $H_i(\boldsymbol{X}_n)$ is allowed to increase more than the total losses. It turns out that comonotonic pools play an important role in the study of risk-sharing rules, in the same vein as comonotonic risks do for premium calculation principles and risk measures. The structure of comonotonic pools is further studied in Section 5.3.1.

We know from Denuit and Dhaene (2012) that $\boldsymbol{H}$ is comonotonic if, and only if, there exist non-decreasing functions g_i such that $H_i(\boldsymbol{X}_n) = H_i(S_n) = g_i(S_n)$. Comonotonic risk-sharing rules according to Definition 3.40 are thus necessarily aggregate ones.

Example 3.41. *The uniform risk-sharing rule satisfies the comonotonicity property since the random vector $(S_n/n, \ldots, S_n/n)$ is obviously comonotonic.*

Example 3.42. *The mean proportional risk-sharing rule satisfies the comonotonicity property, since individual contributions are equal to the aggregate losses S_n up to positive coefficients.*

Example 3.43. *The order statistics risk-sharing rule does not satisfy the comonotonicity property, because it is not an aggregate risk-sharing rule.*

Example 3.44. *The multiple layer risk-sharing rule satisfies the comonotonicity property, since the payment in each layer is a non-decreasing function of S_n.*

3.3 Local redistribution properties

In this section, we consider several types of local redistributions of the losses related to two (or more) participants in a pool. Essentially, such an operation involves two (or more) individuals who decide to redistribute the losses they face amongst each other, whereas all the losses belonging to the other participants remain unchanged. In such a situation, it may sound reasonable to assume that for any participant with unchanged losses after the redistribution, also the contribution to the pool remains unchanged by this redistribution. In case this property would not hold, two (or more) participants could decide to redistribute their losses in order to decrease their contributions. Due to the full allocation condition, this would lead to an increase of the contributions of the participants not involved in the redistribution, and this might be considered as "unfair" towards them (hence the names "fair bilateral redistributing", "fair merging", "fair splitting", and "fair redistributing" attributed to the properties considered in this section). As indicated in the introduction, the properties listed in the paper are not necessarily desirable in all situations. The present study establishes the validity of these properties for commonly-used risk-sharing rules and this helps the analyst to select an appropriate rule in the situation under consideration.

3.3.1 Fair bilateral redistributing

Suppose that participants k and l of the pool $\boldsymbol{X}_n$ decide to redistribute their individual losses X_k and X_l among each other before they enter the risk-sharing agreement. In particular, X_k is replaced by $X_k' \geq 0$ and X_l is replaced by $X_l' \geq 0$, such that $X_k' + X_l' = X_k + X_l$. Such a bilateral redistribution does not change the loss of any participant different from k and l. The only changes in the losses happen for participants k and l. These changes are such that the joint losses of participants k and l after the redistribution are equal to their joint losses before the redistribution. A reasonable property for a risk-sharing rule applied to $\boldsymbol{X}_n$ might then be that the contribution paid by any participant different from k and l is not impacted by this mutual redistribution. A risk-sharing rule satisfying this property for any bilateral redistribution of any pool is said to satisfy the fair-bilateral-redistributing property.

Definition 3.45 (fair-bilateral-redistributing property)**.** *A risk-sharing rule* $\boldsymbol{H}$ *satisfies the fair-bilateral-redistributing property if for any pool* $\boldsymbol{X}_n$*, for any different participants* k *and* l *of this pool and any non-negative* X_k' *and* X_l' *such that* $X_k' + X_l' = X_k + X_l$*, one has that*

$$H_i\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right) = H_i\left(\boldsymbol{X}_n\right) \tag{3.14}$$

holds for any i *different from* k *and* l*.*

From the full-allocation condition (2.2) we immediately find that the following additivity property must hold:

$$\begin{aligned} H_k\left(\boldsymbol{X}_n\right) + H_l\left(\boldsymbol{X}_n\right) \quad = \quad & H_k\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right) \\ & + H_l\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right). \end{aligned}$$

This means that for a risk-sharing rule satisfying the fair-bilateral-redistributing property, the bilateral redistribution of X_k and X_l by replacing X_k and X_l by X_k' and X_l' such that $X_k' + X_l' = X_k + X_l$, does not change the contribution of any participant different from k and l. The only changes in the contributions happen for participants k and l. These changes are such that the joint contributions of participants k and l after the bilateral redistribution are equal to their joint allocated losses and contributions before this redistribution.

Example 3.46. *The uniform risk-sharing rule satisfies the fair-bilateral-redistributing property. Indeed, for any* i *different from* k *and* l *we have that*

$$H_i^{\text{uni}}\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right) = \frac{S_n}{n} = H_i^{\text{uni}}\left(\boldsymbol{X}_n\right). \tag{3.15}$$

Notice that also participants k *and* l *have to contribute the same amount* S_n/n*, before as well as after the redistribution. We can conclude that under the uniform risk-sharing rule, not any individual contribution is changed by bilaterally redistributing losses, which means that (3.15) holds for any participant in the pool.*

Example 3.47. *The mean proportional risk-sharing rule satisfies the fair-bilateral-redistributing property. Indeed, for any* i *different from* k *and* l*, we have that*

$$H_i^{\text{prop}}\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right) = \frac{\mathrm{E}\left[X_i\right]}{\mathrm{E}\left[S_n\right]} S_n = H_i^{\text{prop}}\left(\boldsymbol{X}_n\right).$$

Furthermore, one has that

$$H_k\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right) = \frac{\mathrm{E}\left[X_k'\right]}{\mathrm{E}\left[S_n\right]} S_n$$

and

$$H_l\left(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n} + (X_l' - X_l) \times \mathbf{1}_{l,n}\right) = \frac{\mathrm{E}\left[X_l'\right]}{\mathrm{E}\left[S_n\right]} S_n.$$

Example 3.48. *The order statistics risk-sharing rule does not satisfy the fair-bilateral-redistributing property. Indeed, consider the pool (X_1, X_2, X_3) and the redistributed pool $(X_1, X_2 + X_3, 0)$. Then we have that*

$$H_1^{\text{ord}}\big((X_1, X_2, X_3)\big) = \min\{X_1, X_2, X_3\}$$

while

$$H_1^{\text{ord}}\big((X_1, X_2 + X_3, 0)\big) = 0.$$

Obviously, these contributions for the first participant are in general not equal.

Example 3.49. *The multiple layer risk-sharing rule satisfies the fair-bilateral-redistributing property because the contributions of this risk-sharing rule only depend on the aggregate claims S_n and on the limits d_i, $i = 1, 2, \ldots, n-1$, which do not change by a redistribution.*

The following example illustrates the fact that not all aggregate risk-sharing rules satisfy the fair-bilateral-redistributing property.

Example 3.50. *Consider the risk-sharing rule defined by*

$$H_i^{\text{var}}(\boldsymbol{X}_n) = \frac{\operatorname{Var}[X_i]}{\sum_{j=1}^n \operatorname{Var}[X_j]} S_n$$

for any pool $\boldsymbol{X}_n$ of which at least one of the individual losses has a strictly positive variance. As an example, consider the pool (X_1, X_2, X_3) and its redistributed version $(X_1, X_2 + X_3, 0)$. Then we have that

$$H_1^{\text{var}}\big((X_1, X_2, X_3)\big) = \frac{\operatorname{Var}[X_1]}{\sum_{j=1}^3 \operatorname{Var}[X_j]} S_3,$$

whereas

$$H_1^{\text{var}}\big((X_1, X_2 + X_3, 0)\big) = \frac{\operatorname{Var}[X_1]}{\operatorname{Var}[X_1] + \operatorname{Var}[X_2 + X_3]} S_3.$$

Obviously, the contribution for participant 1 may change after the bilateral redistribution between participants 2 and 3 (unless the losses are mutually independent).

Remark 3.51. Consider a risk-sharing rule satisfying the reshuffling property. In this case, it is sufficient to require the condition (3.14) to hold for a single pair (k, l) to define fair bilateral redistributing. One can choose for instance $k = n-1$ and $l = n$.

3.3.2 Fair merging

Let us consider a special case of the bilateral redistributions defined in the preceding section. Suppose that starting from a pool $\boldsymbol{X}_n$, we replace X_k by $X_k' = X_k + X_l$ and X_l by $X_l' = 0$. This means that the redistribution consists of moving the full loss X_l from l to k. In this case, the original pool $\boldsymbol{X}_n$ is transformed into the pool $\boldsymbol{X}_n + X_l \times (\mathbf{1}_{k,n} - \mathbf{1}_{l,n})$. After redistributing, we end up with a pool containing a participant with a zero-loss. It seems reasonable that in such a situation, the losses are shared among the $n-1$ remaining participants, while participant l with zero-loss is free of any contribution and is therefore withdrawn from the pool. This kind of operations is considered in this subsection, where we define the fair-merging property.

Departing from an n-dimensional pool $\boldsymbol{X}_n$, we introduced the $(n-1)$-dimensional pool $\boldsymbol{X}_{n-1}^{(\backslash l)}$; see (3.2). Let us now also introduce the notation $\boldsymbol{X}_{n-1}^{(l\to k)}$ for the $(n-1)$-dimensional pool derived from the original pool $\boldsymbol{X}_n$ in the following way:

$$\boldsymbol{X}_{n-1}^{(l\to k)} = \boldsymbol{Y}_{n-1}^{(\backslash l)} \text{ with } \boldsymbol{Y}_n = \boldsymbol{X}_n + X_l \times (\mathbf{1}_{k,n} - \mathbf{1}_{l,n})\,.$$

This means that $\boldsymbol{X}_{n-1}^{(l\to k)}$ is obtained by first merging X_k and X_l into X_k+X_l and attributing this merged loss completely to participant k, while making participant l loss-free. In other words, in the first step, we replace the pool $\boldsymbol{X}_n$ by the pool $\boldsymbol{Y}_n$ as defined above. In the second step, we remove participant l (with the zero-loss) from the merged pool, that is we replace $\boldsymbol{Y}_n$ by $\boldsymbol{Y}_{n-1}^{(\backslash l)}$.

Having introduced this new notation, we are now ready to define the fair-merging property, which is based on a condition which has to hold for any bilateral redistribution that leads to a zero-claim for one of the participants involved.

Definition 3.52 (fair-merging property)**.** *A risk-sharing rule $\boldsymbol{H}$ satisfies the fair-merging property if for any pool $\boldsymbol{X}_n$, and any different k and l in $\{1,2,\ldots,n\}$, one has that*

$$H_i\left(\boldsymbol{X}_{n-1}^{(l\to k)}\right) = H_i\left(\boldsymbol{X}_n\right) \text{ for any } i \text{ different from } k \text{ and } l. \tag{3.16}$$

Taking into account the full allocation condition (2.2) we find the following relation between the contributions of participants k and l before the merge and the contribution of participant k after the merge:

$$H_k\left(\boldsymbol{X}_{n-1}^{(l\to k)}\right) = H_k\left(\boldsymbol{X}_n\right) + H_l\left(\boldsymbol{X}_n\right).$$

For any pool $\boldsymbol{X}_n$, we have that merging X_k and X_l by replacing X_k by X_k+X_l and X_l by 0 does not change the losses of the participants different from k and l.

Example 3.53. *The uniform risk-sharing rule does not satisfy the fair-merging property. Indeed, for any i different from k and l, we have that*

$$H_i^{\text{uni}}\left(\boldsymbol{X}_{n-1}^{(l\to k)}\right) = \frac{S_n}{n-1}, \text{ while } H_i^{\text{uni}}\left(\boldsymbol{X}_n\right) = \frac{S_n}{n}.$$

Remark 3.54. Any risk-sharing rule satisfying the fair-bilateral-redistributing property and the normalization property, also satisfies the fair-merging property.

Example 3.55. *The mean proportional risk-sharing rule satisfies the fair-merging property. Indeed, for any k and l with $k \neq l$, we have that*

$$H_i^{\text{prop}}\left(\boldsymbol{X}_{n-1}^{(l\to k)}\right) = \frac{\mathrm{E}\left[X_i\right]}{\mathrm{E}\left[S_n\right]} S_n = H_i^{\text{prop}}\left(\boldsymbol{X}_n\right) \text{ for any } i \text{ different from } k \text{ and } l.$$

Notice that this conclusion also follows immediately from the fact that the mean proportional risk-sharing rule satisfies the fair-bilateral-redistributing and normalization properties, as sated in Remark 3.54.

Example 3.56. *The order statistics risk-sharing rule does not satisfy the fair merging property. Indeed, consider the pool* (X_1, X_2, X_3) *and the merged pool* $(X_1, X_2 + X_3)$*. Then we have that*

$$H_1^{\text{ord}}\big((X_1, X_2, X_2)\big) = \min\{X_1, X_2, X_3\}$$

while

$$H_1^{\text{ord}}\big((X_1, X_2 + X_3)\big) = \min\{X_1, X_2 + X_3\}\,.$$

Obviously, the contributions for the first participant before and after the merge are in general not equal.

Remark 3.57. The fair-merging property is not meaningful for all risk-sharing rules. Consider for instance the multiple layer rule. The fair-merging property would require the adaptation of the limits d_i.

The following example illustrates the fact that not all aggregate risk-sharing rules satisfy the fair-merging property.

Example 3.58. *Consider the risk-sharing rule* $\boldsymbol{H}^{\text{var}}$ *introduced in Example 3.50. Consider again the pool* (X_1, X_2, X_3) *and the merged pool* $(X_1, X_2 + X_3)$*. Then we have that*

$$H_1^{\text{var}}\big((X_1, X_2, X_3)\big) = \frac{\text{Var}[X_1]}{\sum_{j=1}^{3} \text{Var}[X_1] + \text{Var}[X_2]} S_3,$$

whereas

$$H_1^{\text{var}}\big((X_1, X_2 + X_3)\big) = \frac{\text{Var}[X_1]}{\text{Var}[X_1] + \text{Var}[X_2 + X_3]} S_3.$$

Obviously, the contribution for participant 1 *may change after the merging operation between participants* 2 *and* 3*.*

Remark 3.59. Consider a risk-sharing rule satisfying the reshuffling property. In this case, it is sufficient to require the condition (3.16) to hold for a single pair (k, l) to define fair merging. For instance, one could take $k = n - 1$ and $l = n$.

3.3.3 Fair splitting

Suppose that participant k of the pool of losses $\boldsymbol{X}_n$ decides to split his or her loss X_k into the sum $X_k' + X_{n+1}$, i.e.

$$X_k = X_k' + X_{n+1} \tag{3.17}$$

where $X_k' \geq 0$ is retained by participant k and $X_{n+1} \geq 0$ is passed to a new participant $n+1$ to be added to the pool. Notice that in general k and $n+1$ will be different participants, but it may also be possible that behind participants k and $n+1$ is a single person, who wants to split his or her risk for some reason. By this split, the original n-dimensional pool $\boldsymbol{X}_n$ is replaced by the $(n+1)$-dimension pool $(\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n}, X_{n+1})$.

Obviously, this splitting operation leaves the losses of all participants in the pool $\boldsymbol{X}_n$ unchanged, except for participant k who moves part of his or her original loss to the new participant $n+1$. This observation gives rise to a reasonable property for a risk-sharing rule, stating that for all participants with unchanged losses after the split, also the contributions after this split remain unchanged.

Definition 3.60 (fair-splitting property)**.** *A risk-sharing rule* $\boldsymbol{H}$ *satisfies the fair-splitting property in case for any pool* $\boldsymbol{X}_n$*, for any* $k \in 1, 2, \ldots, n$*, and any non-negative random variables* X'_k *and* X_{n+1} *satisfying* (3.17)*, one has that*

$$H_i\left(\boldsymbol{X}_n\right) = H_i\big(\left(\boldsymbol{X}_n + \left(X'_k - X_k\right) \times \mathbf{1}_{k,n}, X_{n+1}\right)\big) \text{ if } i \text{ is different from } k \text{ and } n+1. \quad (3.18)$$

Notice that splitting often results in perfectly dependent (or comonotonic) pieces in applications to insurance, for instance $X'_k = \min\{X_k, d\}$ and $X_{n+1} = (X_k - d)_+$ for some non-negative deductible d, or $X'_k = \alpha X_k$ and $X_{n+1} = (1-\alpha)X_k$ for some quota share $\alpha \in (0,1)$. In Definition 3.60, we leave the dependence structure in the pair (X'_k, X_{n+1}) unspecified and only require that (3.17) holds true.

Suppose that the risk-sharing rule $\boldsymbol{H}$ satisfies the fair-splitting property. For any $\boldsymbol{X}_n$ the full allocation condition (2.2) then leads to

$$\sum_{i=1}^{n} H_i\left(\boldsymbol{X}_n\right) = \sum_{i=1}^{n+1} H_i\left(\left(\boldsymbol{X}_n + \left(X'_k - X_k\right) \times \mathbf{1}_{k,n}, X_{n+1}\right)\right). \quad (3.19)$$

Taking into account the fair-splitting condition (3.18), we find that

$$\begin{aligned} H_k\left(\boldsymbol{X}_n\right) &= H_k\big(\left(\boldsymbol{X}_n + \left(X'_k - X_k\right) \times \mathbf{1}_{k,n}, X_{n+1}\right)\big) \\ &\quad + H_{n+1}\big(\left(\boldsymbol{X}_n + \left(X'_k - X_k\right) \times \mathbf{1}_{k,n}, X_{n+1}\right)\big). \end{aligned} \quad (3.20)$$

From (3.18) and (3.19) we thus can conclude that the following holds for any risk-sharing rule satisfying the fair-splitting property: splitting the loss X_k of participant k into a loss X'_k for participant k and a loss $X_{n+1} = X_k - X'_k$ for a new participant $n+1$, leaves the contribution of any participant different from k unchanged, while the contribution of the kth participant before the split is equal to the sum of the contributions of participants k and $n+1$ after the split.

Example 3.61. *The uniform risk-sharing rule does not satisfy the fair-splitting property. Considering for instance the pool* (X_1, X_2) *and the pool* $(X_1, X'_2, X_2 - X'_2)$ *obtained after splitting* X_2*, we see that*

$$H_1^{\text{uni}}\Big(\left(X_1, X'_2, X_2 - X'_2\right)\Big) = \frac{X_1 + X_2}{3} \neq \frac{X_1 + X_2}{2} = H_1^{\text{uni}}\big((X_1, X_2)\big).$$

Example 3.62. *The proportional mean risk-sharing rule satisfies the fair-splitting property since, for any* $k = 1, 2, \ldots, n$*, and any non-negative random variables* X'_k *and* X_{n+1} *satisfying* (3.17)*, one has that*

$$H_i^{\text{prop}}\big(\left(\boldsymbol{X}_n + \left(X'_k - X_k\right) \times \mathbf{1}_{k,n}, X_{n+1}\right)\big) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n = H_i^{\text{prop}}\left(\boldsymbol{X}_n\right)$$

for any i *different from* k *and* $n+1$*.*

Example 3.63. *The order statistics risk-sharing rule does not satisfy the fair-splitting property. Indeed, consider the pool* (X_1, X_2) *and the pool* $(X_1, X'_2, X_2 - X'_2)$ *obtained after splitting* X_2*. Then we have that*

$$H_1^{\text{ord}}\big((X_1, X_2)\big) = \min\left\{X_1, X_2\right\}$$

while

$$H_1^{\text{ord}}\big((X_1, X_2', X_2 - X_2')\big) = \min\{X_1, X_2', X_2 - X_2'\}.$$

Obviously, the contributions for the first participant before and after the split are in general not equal.

Remark 3.64. As it was the case with the fair-merging property, the fair-splitting property is not meaningful for all risk-sharing rules. For instance, applying the fair-splitting property to the multiple layer risk-sharing rule would require the adaptation of the limits d_i. Therefore, we do not discuss the fair-splitting property in that case.

The following example illustrates the fact that not all aggregate risk-sharing rules satisfy the fair-splitting property.

Example 3.65. *Consider the risk-sharing rule* $\boldsymbol{H}^{\text{var}}$ *introduced in Example 3.50. As an example, consider the pool* (X_1, X_2) *and the pool* $(X_1, X_2', X_2 - X_2')$ *obtained after splitting* X_2*. Then we have that*

$$H_1^{\text{var}}\big((X_1, X_2)\big) = \frac{\text{Var}\,[X_1]}{\text{Var}\,[X_1] + \text{Var}\,[X_2]} S_2,$$

whereas

$$H_1^{\text{var}}\big((X_1, X_2', X_2 - X_2')\big) = \frac{\text{Var}\,[X_1]}{\text{Var}\,[X_1] + \text{Var}\,[X_2'] + \text{Var}\,[X_2 - X_2']} S_2.$$

Obviously, the contribution for participant 1 *may change after the splitting operation between participants* 2 *and* 3*.*

Remark 3.66. Consider a risk-sharing rule satisfying the reshuffling property. In this case, it is sufficient to require the condition (3.18) to hold for a single value of k to define fair splitting. For instance, one could take $k = n$.

Remark 3.67. Any risk-sharing rule satisfying the fair-redistributing property and the normalization property, also satisfies the fair-splitting property. The proportional mean risk-sharing rule is an example of a rule which satisfies the fair-distributing property, the normalization property, and as a consequence, also the fair-splitting property.

3.3.4 Fair redistributing

In this subsection, we introduce the fair-redistributing property for risk-sharing rules. As we will see, it is a stronger property than fair bilateral redistributing, fair merging and fair splitting in the sense that the latter three properties are implied by the fair redistribution property.

First, we have to define a redistribution within a pool. Consider the pool $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_n)$. Then we say that the pool $\boldsymbol{Y}_m = (Y_1, Y_2, \ldots, Y_m)$ is a redistribution of the pool $\boldsymbol{X}_n$ if the following condition holds:

$$\sum_{i=1}^{n} X_i = \sum_{j=1}^{m} Y_j.$$

Notice that we do not put any restriction on m, which may be equal to n, smaller than n or larger than n.

Next, we make some notational conventions. A participant i from the original pool $\boldsymbol{X}_n$ who is also present in the redistributed pool $\boldsymbol{Y}_m$, will be labelled by the same index i in both pools. If there is an index i in the original pool which does not appear in the redistributed pool $\boldsymbol{Y}_m$, that means that participant i from the pool $\boldsymbol{X}_n$ is absent in the new pool $\boldsymbol{Y}_m$. Lastly, if index i is present in the redistributed pool $\boldsymbol{Y}_n$ but not in the original pool $\boldsymbol{X}_n$, this means that participant i is not present in the original pool $\boldsymbol{X}_n$ but has been added to the second pool $\boldsymbol{Y}_m$. Let us illustrate these conventions by some examples.

Example 3.68. *Starting from the original pool* $\boldsymbol{X}_3 = (X_1, X_2, X_3)$*,*

- $(Y_1, Y_2, Y_3) = (X_1, X_2 + X_3, 0)$ *is the redistributed pool where participant* 1 *keeps his or her original loss* X_1*, participant* 2 *is allocated the merged loss* $X_2 + X_3$ *and participant* 3 *is allocated the zero-loss.*

- $(Y_1, Y_2) = (X_1, X_2 + X_3)$ *is the redistributed pool where participant* 1 *keeps his or her original loss* X_1*, participant* 2 *is allocated the merged loss* $X_2 + X_3$ *and participant* 3 *is absent in the redistributed pool.*

- $(Y_1, Y_2, Y_3, Y_4) = (X_1, X_2, X_3', X_3 - X_3')$ *is the redistributed pool where participants* 1 *and* 2 *keep their original loss* X_1 *and* X_2*, participant* 3 *is allocated the redistributed loss* X_3'*, while participant* 4 *is a new participant added to the redistributed pool.*

- $(Y_1, Y_2, Y_3) = (X_3, X_2, X_1)$ *is the redistributed pool where participant* 1 *bears loss* X_3*, participant* 2 *keeps his or her original loss* X_1 *and participant* 3 *bears loss* X_1*.*

- $(Y_2, Y_3) = (X_1, X_2 + X_3)$ *is the redistributed pool where participant* 1 *is absent in the redistributed pool, participant* 2 *is allocated the loss* X_1 *and participant* 3 *is allocated the merged loss* $X_2 + X_3$*.*

When $m = n$, the redistribution $\boldsymbol{Y}_m$ can be viewed as a reallocated loss obtained from some risk-sharing rule $\boldsymbol{H}$. The difference we make is that redistribution operates locally, among small groups of participants whereas risk-sharing rules act globally among all participants in general. Thus, redistribution typically leaves many losses unchanged.

Now we are ready to define the fair-redistributing property. The latter guarantees that the contributions of participants in the pool are not affected by redistribution between other participants.

Definition 3.69 (fair-redistributing property)**.** *A risk-sharing rule satisfies the fair-redistributing property if for any pool* $\boldsymbol{X}_n$ *and any redistributed pool* $\boldsymbol{Y}_m$ *of* $\boldsymbol{X}_n$*, one has that*

$$H_i(\boldsymbol{X}_n) = H_i(\boldsymbol{Y}_m) \text{ for any } i \leq n \text{ such that } Y_i = X_i.$$

Remark 3.70. The fair-redistributing property implies the fair-bilateral-redistributing, fair-merging and fair-splitting properties.

Example 3.71. *The uniform risk-sharing rule does not satisfy the fair-merging property, hence it cannot satisfy the fair-redistributing property.*

Example 3.72. *The mean proportional risk-sharing rule satisfies the fair-redistributing property, and hence, it satisfies fair bilateral redistributing, fair merging and fair splitting.*

Example 3.73. *The order statistics risk-sharing rule does not satisfy the fair-splitting property and hence, it cannot satisfy the fair-redistributing property.*

3.4 Specific-pool properties

3.4.1 Stand-alone for comonotonic losses

Within a comonotonic pool, it is reasonable to expect that each participant remains with his or her own loss since no diversification benefit can result from pooling in that case. Hence, the stand-alone risk-sharing rule appears to be reasonable in that case. These considerations lead to the following definition.

Definition 3.74 (stand-alone property for comonotonic losses)**.** *A risk-sharing rule $\boldsymbol{H}$ is said to satisfy the stand-alone property for comonotonic losses if for any comonotonic pool $\boldsymbol{X}_n$ one has that $\boldsymbol{H}\left(\boldsymbol{X}_n\right) = \boldsymbol{X}_n$.*

Example 3.75. *The uniform risk-sharing rule does not satisfy the stand-alone property for comonotonic losses. Considering for instance $n = 2$ and $\boldsymbol{X}_2 = (U, 2U)$ with U uniformly distributed over the interval $[0, 1]$, we see that*

$$H_1^{\text{uni}}\big((U, 2U)\big) = \frac{3U}{2} \neq U.$$

Example 3.76. *The mean proportional risk-sharing rule does not necessarily satisfy the stand-alone property for comonotonic losses since X_i does not necessarily coincide with $\frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n$ when $\boldsymbol{X}_n$ is comonotonic. The same comment applies to the order statistics and the multiple layer rules which do not satisfy the stand-alone property for comonotonic losses.*

The stand-alone property for comonotonic losses will be satisfied by some of the risk-sharing rules discussed in the next sections.

3.4.2 Uniformity for exchangeable losses

Definition 3.74 states that for comonotonic risks, the reallocated loss is equal to the original loss. This requirement considers a special kind of pool and requires that the risk-sharing rule behaves in a particular way in that case. The property defined in the current section considers exchangeable losses. Recall that $\boldsymbol{X}_n$ is called exchangeable if its joint distribution function $F_{\boldsymbol{X}_n}$ is symmetric in its arguments. This means that for any permutation π of $\{1, \ldots, n\}$, the random vector $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$ is distributed as $\boldsymbol{X}_n$. In particular, if $\boldsymbol{X}_n$ is exchangeable then X_1, X_2, …, X_n are identically distributed and the pool is homogeneous. The uniform risk-sharing rule $\boldsymbol{H}^{\text{uni}}$ thus seems to be appropriate under exchangeability. Taking these observation into consideration suggests to consider the following property.

Definition 3.77 (uniformity property for exchangeable losses)**.** *A risk-sharing rule $\boldsymbol{H}$ is said to satisfy the uniformity property for exchangeable losses if for any exchangeable pool $\boldsymbol{X}_n$ one has that $\boldsymbol{H}(\boldsymbol{X}_n) = \boldsymbol{H}^{\text{uni}}(\boldsymbol{X}_n)$.*

Definitions 3.74 and 3.77 are similar in the sense that they set a requirement on the risk-sharing rule only for a special type of pools (comonotonic versus exchangeable losses).

Example 3.78. *The mean proportional risk-sharing rule satisfies the uniformity property for exchangeable losses. It is easy to see that if all expected losses* $\mathrm{E}[X_i]$ *are equal then* $H_i^{\mathrm{prop}}(\boldsymbol{X}_n) = H_i^{\mathrm{uni}}(\boldsymbol{X}_n)$.

Example 3.79. *The order statistics and the multiple layer risk-sharing rules do not satisfy the uniformity property for exchangeable losses.*

4 Conditional mean risk-sharing rule

4.1 Definition

Let us recall the definition of the conditional mean risk-sharing rule introduced by Denuit and Dhaene (2012).

Definition 4.1 (conditional mean risk-sharing rule)**.** *A risk-sharing rule* $\boldsymbol{H}$ *is the conditional mean risk-sharing rule in case for any pool* $\boldsymbol{X}_n$*, the contribution for individual* i *is given by*

$$H_i(\boldsymbol{X}_n) = H_i(S_n) = \mathrm{E}\left[X_i \mid S_n\right], \qquad i = 1, 2, \ldots, n. \tag{4.1}$$

The conditional mean risk-sharing rule is henceforth denoted as $\boldsymbol{H}^{\mathrm{cm}}$.

Under $\boldsymbol{H}^{\mathrm{cm}}$, each participant must contribute the expected value of the loss brought to the pool, given the total loss S_n experienced by the pool. When $\mathrm{Var}[S_n] < \infty$, we have

$$\mathrm{E}\big[\big(X_i - H_i^{\mathrm{cm}}(S_n)\big)^2\big] = \min_f \mathrm{E}\big[\big(X_i - f\left(S_n\right)\big)^2\big],$$

where the minimum is taken over all measurable functions f such that $\mathrm{Var}[f\left(S_n\right)] < \infty$. In words, the contribution $H_i^{\mathrm{cm}}(S_n)$ paid by participant i is the closest to the loss X_i brought to the pool, in the sense that it minimizes the expected squared difference of the risk X_i and any measurable function $f(S_n)$ of the total loss S_n. It then follows that the net pooling effects $X_i - H_i^{\mathrm{cm}}(S_n)$ are uncorrelated among participants.

4.2 Properties

In Denuit and Dhaene (2012), several properties of the conditional mean risk-sharing rule were proven, with special emphasis to Pareto optimality. Let us now establish the validity of the properties presented in the preceding section for this risk-sharing rule.

Proposition 4.2. *The conditional mean risk-sharing rule* $\boldsymbol{H}^{\mathrm{cm}}$ *satisfies the reshuffling property, the normalization property, the translativity property, the positive homogeneity property, the constancy property, the no-ripoff property, the actuarial fairness property, the willingness-to-join property, the fair-bilateral-redistributing property, the fair-merging property, the fair-splitting property, the fair-redistributing property, the stand-alone property for comonotonic losses, and the uniformity property for exchangeable losses. The conditional mean risk-sharing rule does not necessarily satisfy the comonotonicity property.*

The proof of Proposition 4.2 is given in Appendix A. We can see that the conditional mean risk-sharing rule enjoys all the properties listed in Section 3, except the comonotonicity property. This property is nevertheless valid in particular cases, when the conditional expectations $s \mapsto \mathrm{E}[X_i|S_n = s]$ are all non-decreasing. Some sufficient conditions for this property to hold can be provided for pools comprising independent losses, such as the log-concavity of the distributions of the losses X_i or the large-pool case. We refer the reader to Denuit and Robert (2021b, 2021c) for more details.

5 Quantile risk-sharing rule

5.1 Definition

For any random vector $\boldsymbol{X}_n = (X_1, \ldots, X_n)$ with respective marginal distribution functions $F_1, \ldots, F_n$ and a random variable U uniformly distributed over $[0,1]$, we define the comonotonic counterpart $\boldsymbol{X}_n^c$ of $\boldsymbol{X}_n$ as $\boldsymbol{X}_n^c = \left(F_1^{-1}(U), \ldots, F_n^{-1}(U)\right)$, with corresponding comonotonic sum S_n^c given by

$$S_n^c = \sum_{i=1}^{n} F_i^{-1}(U). \tag{5.1}$$

In addition to the left-continuous inverse (3.13), we will also need the right-continuous one. Recall that the right-continuous inverse of the distribution function F_Y of some random variable Y is defined by

$$F_Y^{-1+}(p) = \sup\{y \in \mathbb{R} \mid F_Y(y) \leq p\}, \qquad p \in [0,1], \tag{5.2}$$

where $\sup \emptyset = -\infty$ by convention. For any $\alpha \in [0,1]$, we define the α-inverse $F_Y^{-1(\alpha)}$ of F_Y following Dhaene et al. (2002a) as the following convex combination of the inverses F_Y^{-1} and F_Y^{-1+} of F_Y:

$$F_Y^{-1(\alpha)}(p) = \alpha\, F_Y^{-1}(p) + (1-\alpha)\, F_Y^{-1+}(p), \qquad p \in (0,1). \tag{5.3}$$

For any y in $\left(F_Y^{-1+}(0), F_Y^{-1}(1)\right)$, defining $\alpha_y \in [0,1]$ as

$$\alpha_y = \begin{cases} \frac{F_Y^{-1+}(F_Y(y)) - y}{F_Y^{-1+}(F_Y(y)) - F_Y^{-1}(F_Y(y))} & \text{if } F_Y^{-1+}(F_Y(y)) \neq F_Y^{-1}(F_Y(y)) \\ 1 & \text{otherwise} \end{cases} \tag{5.4}$$

we see that the identity

$$F_Y^{-1(\alpha_y)}(F_Y(y)) = y \tag{5.5}$$

holds true. By convention, we set $\alpha_y = 0$ in case $F_Y(y) = 0$ and $\alpha_y = 1$ in case $F_Y(y) = 1$. We are now ready to state the definition of the quantile risk-sharing rule.

Definition 5.1 (quantile risk-sharing rule)**.** *A risk-sharing rule $\boldsymbol{H}$ is the quantile risk-sharing rule in case for any pool $\boldsymbol{X}_n$, the contribution for individual i is given by*

$$H_i(\boldsymbol{X}_n) = H_i(S_n) = F_i^{-1(\alpha_{S_n})}\left(F_{S_n^c}(S_n)\right), \qquad i = 1, 2, \ldots, n. \tag{5.6}$$

In (5.6), S_n^c *stands for the comononotic sum defined in (5.1), while* α_{S_n} *satisfies*

$$F_{S_n^c}^{-1(\alpha_{S_n})}\left(F_{S_n^c}(S_n)\right) = S_n. \tag{5.7}$$

The quantile risk-sharing rule is henceforth denoted as $\boldsymbol{H}^{\text{quant}}$.

In words, (5.6) means that every participant contributes an amount corresponding to the generalized quantile of his or her individual loss, at the same probability level. Let us comment on the definition (5.7) for α_{S_n}. For any outcome $s \in \left(F_{S_n}^{-1+}(0), F_{S_n}^{-1}(1)\right)$ of S_n, we have that $\alpha_s \in [0,1]$ is determined from

$$F_{S_n^c}^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right) = s. \tag{5.8}$$

The conventions stated above imply that

$$F_{S_n^c}^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right) = \sum_{i=1}^{n} F_i^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right) \text{ for any } s \in \mathbb{R} \tag{5.9}$$

which follows from the well-known additivity property of the generalized quantiles of a comonotonic sum; for more details, see Dhaene et al. (2002a). Taking into account (5.7), (5.9) shows that the full loss allocation condition (2.2) is fulfilled. In case the individual contributions are determined with the quantile risk-sharing rule, each member of the pool contributes the same generalized quantile of his or her own loss X_i and the probability level of the generalized quantile is determined such that the sum of the contributions constitutes the aggregate loss. Notice that the quantile risk-sharing rule does not take into account the dependency structure between the individual losses and only depend on their marginal distributions.

Remark 5.2. If all marginal distribution functions F_i are strictly increasing on $\left(F_{S_n^c}^{-1+}(0), F_{S_n^c}^{-1}(1)\right)$ then the quantile risk-sharing rule simplifies into

$$H_i^{\text{quant}}(S_n) = F_i^{-1}\left(F_{S_n^c}(S_n)\right), \quad i = 1, 2, \ldots, n. \tag{5.10}$$

Also, (5.8) reduces to $F_{S_n^c}^{-1}\left(F_{S_n^c}(s)\right) = s$.

The following simple example compares the quantile risk-sharing rule and the conditional mean risk-sharing one, illustrating their respective differences.

Example 5.3. *Consider the pair of individual losses* $\boldsymbol{X}_2 = (X_1, X_2)$ *with* $X_1 = 2U$ *and* $X_2 = 1 - U$ *where* U *is uniformly distributed over the interval* $[0,1]$. *The aggregate loss is given by* $S_2 = 1 + U$, *which takes values in* $[1,2]$. *It is straightforward to verify that for any* p *in* $[0,1]$ *we have that* $F_1^{-1}(p) = 2p$ *and* $F_2^{-1}(p) = p$. *The comonotonic modification of* $\boldsymbol{X}_2$ *is given by*

$$\boldsymbol{X}_2^c = \left(F_1^{-1}(U), F_2^{-1}(U)\right) = (2U, U).$$

The corresponding comonotonic sum equals $S_2^c = F_1^{-1}(U) + F_2^{-1}(U) = 3U$. *Its distribution function and quantile function are given by*

$$F_{S_2^c}(s) = \frac{s}{3}, \qquad \textit{for any } s \textit{ in } [0,3]$$

and

$$F_{S_2^c}^{-1}(p) = 3p, \qquad \text{for any } p \text{ in } [0,1] \,.$$

The conditional mean risk-sharing rule for $\boldsymbol{X}_2$ *is given by*

$$H_1^{\mathrm{cm}}(S_2) = \mathrm{E}\left[X_1 \mid S_2\right] = X_1 = 2S_2 - 2 = 2U$$

and

$$H_2^{\mathrm{cm}}(S_2) = \mathrm{E}\left[X_2 \mid S_2\right] = X_2 = 2 - S_2 = 1 - U,$$

so that $\boldsymbol{H}^{\mathrm{cm}}(\boldsymbol{X}_2) = \boldsymbol{X}_2$*, which means that each participant pays his or her own individual loss.*

The quantile risk-sharing rule is given by

$$H_1^{\mathrm{quant}}(S_2) = F_1^{-1}\left(F_{S_2^c}\left(S_2\right)\right) = \frac{2}{3}S_2 = \frac{2}{3}(1+U)$$

and

$$H_2^{\mathrm{quant}}(S_2) = F_2^{-1}\left(F_{S_2^c}\left(S_2\right)\right) = \frac{1}{3}S_2 = \frac{1}{3}(1+U),$$

so that $\boldsymbol{H}^{\mathrm{quant}}(\boldsymbol{X}_2) \neq \boldsymbol{X}_2$*. The contributions are comonotonic in this case. Notice that* $F_{S_2^c}\left(S_2\right)$ *is uniformly distributed over the interval* $\left(\frac{1}{3}, \frac{2}{3}\right)$*. The distribution functions*

$$F_{H_1^{\mathrm{cm}}(S_2)}(x) = \frac{x}{2} \text{ for any } x \text{ in } (0,2)\,,$$

and

$$F_{H_1^{\mathrm{quant}}(S_2)}(x) = \frac{3}{2}x - 1 \text{ for any } x \text{ in } \left(\frac{2}{3}, \frac{4}{3}\right),$$

cross exactly once, with the first one having the larger right tail. Since the corresponding expected values are equal, this means that they are ordered in the convex order sense:

$$H_1^{\mathrm{quant}}(S_2) \preceq_{\mathrm{CX}} H_1^{\mathrm{cm}}(S_2) = X_1 = 2U. \tag{5.11}$$

Also, $F_{H_2^{\mathrm{cm}}(S_2)}(x) = x$ *for any* x *in* $(0,1)$*, while* $F_{H_2^{\mathrm{quant}}(S_2)}(x) = 3x - 1$ *for any* x *in* $\left(\frac{1}{3}, \frac{2}{3}\right)$*. Proceeding as above, we find the following convex order relation:*

$$H_2^{\mathrm{quant}}(S_2) \preceq_{\mathrm{CX}} H_2^{\mathrm{cm}}(S_2) = X_2 = 1 - U. \tag{5.12}$$

Assuming that both participants are risk-averse, we can conclude that each of them will prefer the quantile risk-sharing rule over the conditional mean risk-sharing rule (which in this example, comes down to bearing his or her own risk). The quantile risk-sharing rule thus improves the situation for both participants in this example.

Remark 5.4. Notice that quantiles are often not applicable at individual level to determine premiums. This is because of the high probability mass at zero exhibited by individual insurance losses, with $\mathrm{P}[X_i = 0] = F_i(0)$ often in the range 90-99%, so that $F_i^{-1}(p) = 0$ for p up to 90-99%. Hence, if premiums charged by the insurer correspond to quantiles at probability level p, the corresponding premium amounts are 0 if p is smaller than the probability mass at zero. A comparable situation may arise with the quantile risk-sharing rule, as shown next.

Due to the special nature of the comonotonic support, see formula (C.1) in Appendix C, there is a unique point $\boldsymbol{x}_n$ corresponding to any outcome s of S_n^c, of the form (C.2). For any possible outcome of S_n^c, there is thus a unique point $\boldsymbol{x}_n$ in the connected support of S_n^c of which the components sum to s. The components x_i are precisely the values involved in the quantile risk-sharing rule. Hence, the probability that S_n^c is smaller than, or equal to s is equal to the value of the joint distribution function of $\boldsymbol{X}_n^c$ at that point, and we know from Dhaene et al. (2002a) that the joint distribution function of a comonotonic random vector is equal to the mininimum of the respective marginal distribution functions. This allows us to write

$$F_{S_n^c}(s) = \min\left\{F_1\big(F_1^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right)\big), \ldots, F_n\big(F_n^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right)\big)\right\}. \tag{5.13}$$

Now, consider losses X_i obeying zero-augmented distributions, as those encountered in the majority of insurance applications (compound Poisson distributions with continuous severities, or zero-augmented Gamma or LogNormal distributions, for instance). This means that $F_i(0) > 0$ and F_i is continuously increasing over $(0, \infty)$. Then, formula (5.13) shows that the identity $F_{S_n^c}(0) = \min\{F_1(0), \ldots, F_n(0)\}$ is valid. Once the value of S_n is known to be equal to s, two situations may occur. Either $F_{S_n^c}(s) > \max\{F_1(0), \ldots, F_n(0)\}$ and every participant contributes to the total loss, that is, $H_i^{\text{quant}}(s) > 0$ for all i. Or $F_{S_n^c}(s) \leq \max\{F_1(0), \ldots, F_n(0)\}$ and participants with larger probabilities masses at zero, or no-claim probabilities, i.e. those participants i for which $F_{S_n^c}(s) \leq F_i(0)$ do not have to contribute ex post. Assuming that participants are numbered so that $F_1(0) < F_2(0) < \ldots < F_n(0)$ and that s is such the $F_j(0) < F_{S_n^c}(s) \leq F_{j+1}(0)$, this implies that participants $j+1, \ldots, n$ do not have to contribute to S_n. This may lead to problematic situations in P2P insurance schemes since it is reasonable to expect that all participants putting the pool at risk must contribute to S_n ex post. Assume for instance that the realizations of losses $X_1, \ldots, X_{n-1}$ are all 0 but that $X_n = x_n > 0$. If x_n is such that $F_1(0) < F_{S_n^c}(x_n) < F_2(0)$ then participant 1 will have to pay x_n (alone) while participant n is the only one having caused some loss within the pool. This may cause moral hazard issues.

If all no-claim probabilities are equal then the problem disappears. This is the case when all $F_i(0)$ are zero, or can be considered as being negligible (for instance when $X_1, \ldots, X_n$ correspond to business lines or insurance portfolios). The problem also disappears when the number of participants to the pool becomes sufficiently large, as shown next. The probability that every participant contributes to the total loss is given by

$$\mathrm{P}[H_i^{\text{quant}}(S_n) > 0 \text{ for all } i] = \mathrm{P}[F_{S_n^c}(S_n) > \max\{F_1(0), \ldots, F_n(0)\}].$$

If participants are numbered so that $F_1(0) < F_2(0) < \ldots < F_n(0)$ then this probability is also equal to

$$\mathrm{P}\big[S_n > F_{S_n^c}^{-1}\big(F_n(0)\big)\big] = \mathrm{P}\left[S_n > \sum_{j=1}^{n} F_j^{-1}\big(F_n(0)\big)\right].$$

Let us now consider large pools, i.e., the number n of participants becomes larger and larger, and assume that, for all $i = 1, \ldots, n$, the inequality $F_i(0) \leq \eta$ holds for some $0 < \eta < 1$. If for all $i = 1, \ldots, n$ and for some $0 < \varepsilon < 1$,

$$F_i^{-1}(\eta) \leq (1 - \varepsilon) \operatorname{E}[X_i],$$

then, provided a law of large numbers applies to individual losses X_i, the probability that every participant contributes to the total loss under the quantile risk-sharing rule tends to 1 as n tends to infinity.

Example 5.5 (scale proportional risk-sharing rule)**.** *If $F_i(x) = F(x/\sigma_i)$ for some distribution function F and some particular scale factors $\sigma_i > 0$, then*

$$H_i^{\text{quant}}(\boldsymbol{X}_n) = \sigma_i F^{-1(\alpha_{S_n})}\left(F_{S_n^c}(S_n)\right) \text{ for } i = 1, \ldots, n.$$

The full allocation condition then gives

$$F^{-1(\alpha_{S_n})}\left(F_{S_n^c}(S_n)\right) = \frac{S_n}{\sum_{j=1}^n \sigma_j}$$

which allows us to conclude that

$$H_i^{\text{quant}}(\boldsymbol{X}_n) = \frac{\sigma_i}{\sum_{j=1}^n \sigma_j} S_n,$$

whatever the dependence structure of the individual losses X_i. The quantile risk-sharing rule then coincides with the scale proportional risk-sharing rule $\boldsymbol{H}^{\text{scale}}$ where participants agree to share the total loss S_n according to ratios based on their respective scale parameter, divided by the sum of all scale parameters.

5.2 Properties

Let us now establish the validity of the properties presented in Section 3 for the quantile risk-sharing rule.

Proposition 5.6. *The quantile risk-sharing rule $\boldsymbol{H}^{\text{quant}}$ satisfies the reshuffling property, the normalization property, the translativity property, the positive homogeneity property, the constancy property, the no-ripoff property, the comonotonic property, the stand-alone property for comonotonic losses, and the uniformity property for exchangeable losses. The quantile risk-sharing rule does not necessarily satisfy the actuarial fairness property, the willingness-to-join property, the fair-bilateral-redistributing property, the fair-merging property, the fair-splitting property, and the fair-redistributing property.*

The proof of Proposition 5.6 is given in Appendix B.

Remark 5.7. Notice that the quantile risk-sharing rule $\boldsymbol{H}^{\text{quant}}$ satisfies the fair-merging property when X_k and X_l are comonotonic since the identity $F_{X_k+X_l}^{-1(\alpha)}(p) = F_{X_l}^{-1(\alpha)}(p) + F_{X_l}^{-1(\alpha)}(p)$ holds true for all probability levels p and all α in this case, and because of full allocation. Similarly, the quantile risk-sharing rule satisfies the fair-splitting property in the particular case where X'_k and X_{n+1} are comonotonic.

Remark 5.8. As the quantile risk-sharing rule only considers the marginal distributions and not the dependency structure, a stronger property than the uniformity property for exchangeable losses holds true. Indeed, for the quantile risk-sharing rule we only need equal marginals to ensure that the risk-sharing rule reduces to the uniform risk-sharing rule.

5.3 Links between conditional mean and quantile risk-sharing rules

5.3.1 Comonotonic pools

Comonotonic pools play an important role in the study of risk-sharing rules. Within comonotonic pools, every individual loss is in fact an increasing function of the aggregate loss of the entire pool, as shown next.

Proposition 5.9. *The pool $\boldsymbol{X}_n$ is comonotonic if, and only if,*

$$\boldsymbol{X}_n = \big(h_1(S_n), h_2(S_n), \ldots, h_n(S_n)\big) \tag{5.14}$$

with the non-decreasing and continuous functions $h_i, i = 1, 2, \ldots, n$, given by

$$h_i(s) = F_i^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right), \quad \textit{for any } s \in \mathbb{R}, \tag{5.15}$$

where α_s is defined by (5.8)*.*

The proof of the new characterization of comonotonicity in Proposition 5.9 is given in Appendix C.

Consider a comonotonic pool $\boldsymbol{X}_n$. From equations (5.14) and (5.15) in Proposition 5.9, we find that

$$\big(\mathrm{E}\left[X_1 \mid S_n\right], \mathrm{E}\left[X_2 \mid S_n\right], \ldots, \mathrm{E}\left[X_n \mid S_n\right]\big) = (h_1(S_n), h_2(S_n), \ldots, h_n(S_n)) = \boldsymbol{X}_n.$$

Conversely, if

$$\boldsymbol{X}_n = \big(\mathrm{E}\left[X_1 \mid S_n\right], \mathrm{E}\left[X_2 \mid S_n\right], \ldots, \mathrm{E}\left[X_n \mid S_n\right]\big) \tag{5.16}$$

holds with $\mathrm{E}\left[X_i \mid S_n\right]$ non-decreasing in S_n for all i then it follows that $\boldsymbol{X}_n$ is comonotonic. This leads to the following new characterization of comonotonic pools in terms of those left unchanged by the conditional mean risk-sharing rule.

Proposition 5.10. *The pool $\boldsymbol{X}_n$ is comonotonic if, and only if, it admits the representation* (5.16) *where the conditional expectations $\mathrm{E}\left[X_i \mid S_n\right]$, $i = 1, 2, \ldots, n$, are non-decreasing functions of S_n.*

Proposition 5.10 gives a useful characterization of comonotonicity in terms of the conditional expectations defining the conditional mean risk-sharing rule. Propositions 5.9 and 5.10 clearly show that a pool is comonotonic if, and only if, the individual losses it comprises can be considered as increasing functions of the aggregate loss, meaning in fact that the randomness of any individual loss is only caused by the randomness of the aggregate loss.

Propositions 5.9 and 5.10 can be summarized as follows.

Proposition 5.11. *The following statements are equivalent:*

(1) The pool $\boldsymbol{X}_n$ is comonotonic.

(2) $\boldsymbol{X}_n = \boldsymbol{H}^{\text{quant}}(\boldsymbol{X}_n)$.

(3) $\boldsymbol{X}_n = \boldsymbol{H}^{\text{cm}}(\boldsymbol{X}_n)$ with all $H_i^{\text{cm}}(\boldsymbol{X}_n)$ non-decreasing functions of S_n.

5.3.2 Marginal VaR

The risk-sharing rules $\boldsymbol{H}^{\text{cm}}$ and $\boldsymbol{H}^{\text{quant}}$ can both be obtained from marginal VaR, as shown next. For $\boldsymbol{\lambda} \in \mathbb{R}_{+}^{n}$, let $S_n(\boldsymbol{\lambda}) = \sum_{i=1}^{n} \lambda_i X_i$ and $S_n^c(\boldsymbol{\lambda}) = \sum_{i=1}^{n} \lambda_i X_i^c$. Assume that $\boldsymbol{X}_n$ has a strictly positive density function. It is well-known that $F_{S_n(\boldsymbol{\lambda})}^{-1}(\epsilon)$ and $F_{S_n^c(\boldsymbol{\lambda})}^{-1}(\epsilon)$ are positive-homogeneous function of order 1 with respect to $\boldsymbol{\lambda}$. By Euler's rule, we deduce

$$F_{S_n(\boldsymbol{\lambda})}^{-1}(\epsilon) = \sum_{i=1}^{n} \lambda_i \frac{\partial F_{S_n(\boldsymbol{\lambda})}^{-1}(\epsilon)}{\partial \lambda_i} \quad \text{and} \quad F_{S_n^c(\boldsymbol{\lambda})}^{-1}(\epsilon) = \sum_{i=1}^{n} \lambda_i \frac{\partial F_{S_n^c(\boldsymbol{\lambda})}^{-1}(\epsilon)}{\partial \lambda_i}.$$

We then have

$$\frac{\partial F_{S_n(\boldsymbol{\lambda})}^{-1}(\epsilon)}{\partial \lambda_i} = \mathrm{E}[X_i | S_n(\boldsymbol{\lambda}) = F_{S_n(\boldsymbol{\lambda})}^{-1}(\epsilon)] \quad \text{and} \quad \frac{\partial F_{S_n^c(\boldsymbol{\lambda})}^{-1}(\epsilon)}{\partial \lambda_i} = F_i^{-1}(\epsilon).$$

Hence, we see that

$$\begin{aligned} \left.\frac{\partial F_{S_n(\boldsymbol{\lambda})}^{-1}(\epsilon)}{\partial \lambda_i}\right|_{\boldsymbol{\lambda}=\mathbf{1}, \epsilon=F_{S_n}(S_n)} &= H_i^{\text{cm}}(\boldsymbol{X}_n) \\ \left.\frac{\partial F_{S_n^c(\boldsymbol{\lambda})}^{-1}(\epsilon)}{\partial \lambda_i}\right|_{\epsilon=F_{S_n^c}(S_n)} &= H_i^{\text{quant}}(\boldsymbol{X}_n). \end{aligned}$$

6 Generalized conditional mean risk-sharing rules

6.1 Change of distribution

It is possible to offer an additional degree of freedom to the group of participants by allowing them to base the risk-sharing rule on another distribution for $\boldsymbol{X}_n$ they agree about, leaving the total loss S_n to be shared unchanged. Thus, all participants agree to replace the actual (distribution of) loss random vector $\boldsymbol{X}_n$ with (the distribution of) another random vector for the sake of risk sharing. The latter is called the modification of $\boldsymbol{X}_n$ and henceforth denoted as $\boldsymbol{M}(\boldsymbol{X}_n) = (M_1(\boldsymbol{X}_n), \ldots, M_n(\boldsymbol{X}_n))$. Possible reasons why participants choose a modification of the pool to share could be that the joint distribution function of the original pool is only partly known, or that they do not want to take into account the dependency structure (maybe because it is unknown). The comonotonic version $\boldsymbol{X}_n^c$ is an example of such a modified loss random vector that only depends on the marginal distribution functions of the original pool. This suggests to adopt the following definition.

Definition 6.1 (generalized conditional mean risk-sharing rule)**.** *A risk-sharing rule $\boldsymbol{H}$ is a generalized conditional mean risk-sharing rule in case for any pool $\boldsymbol{X}_n$, there is a modified loss random vector $\boldsymbol{M}(\boldsymbol{X}_n)$ with sum $T_n = \sum_{i=1}^{n} M_i(\boldsymbol{X}_n)$, such that the ex-post contribution of individual i is given by*

$$H_i(s) = \mathrm{E}\left[M_i(\boldsymbol{X}_n) \mid T_n = s\right], \qquad i = 1, 2, \ldots, n, \tag{6.1}$$

where s is the realization of S_n. The generalized conditional mean risk-sharing rule for a pool $\boldsymbol{X}_n$ based on modified losses $\boldsymbol{M}(\boldsymbol{X}_n)$ is henceforth denoted as $\boldsymbol{H}^{\text{gcm}}(\boldsymbol{X}_n; \boldsymbol{M}(\boldsymbol{X}_n))$.

Some compatibility conditions are needed so that the conditional expectation in (6.1) is well defined and (6.1) defines a risk-sharing rule. Henceforth, we always assume that the modified loss random vector $\boldsymbol{M}(\boldsymbol{X}_n)$ is such that any outcome of S_n is also a possible outcome for T_n. Stated more formally, the support for S_n (defined as the set of all the possible values for S_n, loosely speaking) has to be included in the support of T_n so that the conditioning makes sense. This is generally satisfied in applications to insurance where the supports of S_n and T_n are often $[0,\infty)$. Henceforth, we tacitly assume that the compatibility condition is fulfilled.

By a suitable choice of $\boldsymbol{M}(\boldsymbol{X}_n)$, we can then relate many risk-sharing rules to the conditional mean risk-sharing one. For instance, we know from implication (A.1) that the uniform risk-sharing rule is equivalent to the conditional mean risk-sharing rule with $\boldsymbol{M}(\boldsymbol{X}_n)$ exchangeable. This is the approach proposed by Skogh and Wu (2005) who argue that agents presuming that they are faced with exchangeable risks can share them mutually beneficially.

6.2 Quantile risk-sharing rule as a generalized conditional mean risk-sharing rule

In this section, we consider the generalized conditional mean risk-sharing rule, where for any pool $\boldsymbol{X}_n$, we set $\boldsymbol{M}(\boldsymbol{X}_n)$ equal to its comonotonic version $\boldsymbol{X}_n^c = \left(F_1^{-1}(U), \ldots, F_n^{-1}(U)\right)$, with U uniformly distributed over the interval $[0,1]$. This leads to the following definition.

Definition 6.2 (comonotonic conditional mean risk-sharing rule)**.** *A risk-sharing rule $\boldsymbol{H}$ is the comonotonic conditional mean risk-sharing rule in case for any pool $\boldsymbol{X}_n$, the contribution for individual i is well-defined and given by*

$$H_i(s) = \mathrm{E}\left[X_i^c \mid S_n^c = s\right], \qquad i = 1, 2, \ldots, n, \tag{6.2}$$

where $S_n^c = \sum_{i=1}^n X_i^c$ and s is the realization of the pool's aggregate loss S_n. The comonotonic conditional mean risk-sharing rule is henceforth denoted as $\boldsymbol{H}^{\mathrm{ccm}}$.

The risk-sharing rule $\boldsymbol{H}^{\mathrm{ccm}}$ is thus obtained from $\boldsymbol{H}^{\mathrm{ccm}}(\boldsymbol{X}_n) = \boldsymbol{H}^{\mathrm{gcm}}(\boldsymbol{X}_n; \boldsymbol{X}_n^c)$, where S_n^c corresponds to T_n in (6.1).

Notice that even with the comonotonic version of $\boldsymbol{X}_n$, some compatibility conditions are needed to properly define the generalized conditional mean risk-sharing rule. Consider for instance $n = 2$ with X_1 and X_2 independent and uniformly distributed over the union of intervals $[0,1] \cup [2,3]$. The support of S_2 is $[0,6]$ whereas the support of S_2^c is $[0,2] \cup [4,6]$. If the outcome of S_2 is 3 then the conditional expectation (6.2) is not defined. This kind of problem does not happen with losses having strictly increasing distribution functions over the interval $[F_i^{-1+}(0), F_i^{-1}(1)]$. Another simple sufficient condition is that the support of all losses X_i is the set of non-negative real values. The risk-sharing rule $\boldsymbol{H}^{\mathrm{ccm}}$ is thus applicable in insurance studies where losses generally obey zero-augmented distributions (as defined in Remark 5.4).

Let us now investigate the relation between the risk-sharing rules $\boldsymbol{H}^{\mathrm{quant}}$ and $\boldsymbol{H}^{\mathrm{ccm}}$. From the definitions of $\boldsymbol{H}^{\mathrm{quant}}$ and $\boldsymbol{H}^{\mathrm{ccm}}$, together with Proposition 5.11, we see that for any pool $\boldsymbol{X}_n$ with aggregate loss S_n,

$$H_i^{\mathrm{ccm}}(s) = \mathrm{E}\left[X_i^c \mid S_n^c = s\right] = F_i^{-1(\alpha_s)}\left(F_{S_n^c}(s)\right) = H_i^{\mathrm{quant}}(s),$$

for any $i = 1, 2, \ldots, n$. This leads to the following result.

Proposition 6.3. *For any pool $\boldsymbol{X}_n$ for which the comonotonic conditional mean risk-sharing rule is well-defined, one has that $\boldsymbol{H}^{\text{quant}}(\boldsymbol{X}_n) = \boldsymbol{H}^{\text{ccm}}(\boldsymbol{X}_n)$.*

In the following example, we compare the conditional mean risk-sharing rule, the comonotonic conditional mean risk-sharing rule, and the quantile risk-sharing rule in a simple case.

Example 6.4 (Continuation of Example 5.3)**.** *Consider again the pair of individual losses $\boldsymbol{X}_2 = (X_1, X_2)$ with $X_1 = 2U$ and $X_2 = 1 - U$. The comonotonic conditional mean risk-sharing rule $\boldsymbol{H}^{\text{ccm}}$ defined in (6.2) applied to $\boldsymbol{X}_2$ gives rise to*

$$H_1^{\text{ccm}}(s) = \mathrm{E}\left[X_1^c \mid S_2^c = s\right] = \frac{2}{3}s$$

and

$$H_2^{\text{ccm}}(s) = \mathrm{E}\left[X_2^c \mid S_2^c = s\right] = \frac{1}{3}s,$$

which means that

$$H_1^{\text{ccm}}(S_2) = \frac{2}{3}S_2 = \frac{2}{3}(1+U) \neq X_1$$

and

$$H_2^{\text{ccm}}(S_2) = \frac{1}{3}S_2 = \frac{1}{3}(1+U) \neq X_2.$$

In this example, the comonotonic conditional mean risk-sharing rule leads to comonotonic contributions. Summarizing what has been obtained in this example and Example 5.3 for the pool $\boldsymbol{X}_2 = (X_1, X_2) = (2U, 1-U)$, we have that

$$\boldsymbol{H}^{\text{quant}}(\boldsymbol{X}_2) = \boldsymbol{H}^{\text{ccm}}(\boldsymbol{X}_2) = \left(\frac{2}{3}(1+U), \frac{1}{3}(1+U)\right)$$

while

$$\boldsymbol{H}^{\text{cm}}(\boldsymbol{X}_2) = \boldsymbol{X}_2 = (2U, 1-U),$$

which is in line with Proposition 6.3.

6.3 Generalized proportional rules as generalized conditional mean risk-sharing rules

The change of distribution can also operate on the random vector of proportions. To this end, notice that the conditional mean risk-sharing rule can be rewritten as

$$H_i^{\text{cm}}(S_n) = \mathrm{E}\left[X_i \mid S_n\right] = S_n \mathrm{E}\left[\Theta_i(S_n)\right]$$

where the vector of proportions $\boldsymbol{\Theta}_n(s)$ is defined as

$$\boldsymbol{\Theta}_n(s) = \left(\frac{X_1}{S_n}, \ldots, \frac{X_n}{S_n}\right)\Bigg| S_n = s.$$

An extension of the conditional mean risk-sharing rule can then be obtained by replacing $(\boldsymbol{\Theta}_n(s))_{s>0}$ with a modified vector of proportions $\big(\boldsymbol{M}\big(\boldsymbol{\Theta}_n(s)\big)\big)_{s>0}$, but leaving the distribution of S_n unchanged. This defines the generalized conditional mean risk-sharing rule $\boldsymbol{H}^{\text{gcm}}(\cdot;\boldsymbol{M}(\boldsymbol{\Theta}_n))$ as

$$H_i^{\text{gcm}}\big(S_n;\boldsymbol{M}(\boldsymbol{\Theta}_n)\big) = S_n \mathrm{E}\left[M_i(\boldsymbol{\Theta}_n(S_n)\,|S_n\right]$$

where $\boldsymbol{M}(\boldsymbol{\Theta}_n)$ is the modified vector of proportions that may be correlated with S_n. An alternative to replacing $\boldsymbol{X}_n$ with $\boldsymbol{M}(\boldsymbol{X}_n)$ thus consists in adopting another distribution for the random vector of proportions $\boldsymbol{\Theta}_n$.

Example 6.5. *Consider absolutely continuous individual losses X_i with marginal distribution functions F_i, $i = 1, \ldots, n$. Assume that $\boldsymbol{M}(\boldsymbol{\Theta}_n(s))$ obeys the Dirichlet distribution with parameter $\boldsymbol{\alpha}(s) = (\alpha_1(s;F_1), \alpha_2(s;F_2), \ldots, \alpha_n(s;F_n))$ with $\alpha_i(s;F_i) > 0$ for $i = 1, \ldots, n$. Then*

$$\mathrm{E}\left[M_i\big(\boldsymbol{\Theta}_n(s)\big)\right] = \frac{\alpha_i(s;F_i)}{\sum_{j=1}^n \alpha_j(s;F_j)}.$$

Total losses S_n are thus distributed among participants according to the proportions $\alpha_i(S_n;F_i)/\sum_{j=1}^n \alpha_j(S_n;F_j)$. In particular, if $\alpha_i(s,F_i) = \mathrm{E}[X_i]$ then

$$H_i^{\text{gcm}}\left(S_n;\boldsymbol{M}\big(\boldsymbol{\Theta}_n\big)\right) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n = H_i^{\text{prop}}(S_n)$$

and we recover the mean proportional risk-sharing rule.

7 Some non-aggregate risk-sharing rules

In the previous sections, we mainly presented and discussed aggregate risk-sharing rules. Let us now introduce several risk-sharing rules that do not depend only on the sum S_n.

7.1 Convex combinations

Participants could combine several risk-sharing rules since any convex combination of risk-sharing rules is itself a risk-sharing rule. This is precisely stated next.

Property 7.1. *If $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are risk-sharing rules, then, for any $\delta \in [0,1]$, $\delta \boldsymbol{H}_1 + (1-\delta)\boldsymbol{H}_2$ is a risk-sharing rule.*

Combining the stand-alone rule with an aggregate risk-sharing rule then produces a rule depending on $\boldsymbol{X}_n$, not only on S_n.

Example 7.2. *A convex combination of the stand-alone and the conditional mean risk-sharing rule produces*

$$H_i(\boldsymbol{X}_n) = \delta X_i + (1-\delta)\,\mathrm{E}[X_i|S_n].$$

Under exchangeable losses (or replacing the conditional mean risk-sharing rule with the uniform one), we obtain

$$H_i(\boldsymbol{X}_n) = \delta X_i + (1-\delta)\frac{S_n}{n}$$

which resembles a credibility premium. The latter risk-sharing rule has been proposed by Charpentier et al. (2021).

7.2 Network-based conditional mean risk-sharing rule

7.2.1 Definition

Another way to design new risk-sharing rules consists in locating participants on a network to describe the links existing between them. Network structures are particularly effective in the context of risk sharing within P2P insurance communities, as demonstrated by Charpentier et al. (2021 and Feng et al. (2020).

For participant i, we define the subset $\mathcal{C}(i)$ of participants that are connected with him or her, with associated weights $\{w_{ii}, w_{ij} : j \in \mathcal{C}(i)\}$ such that $w_{ii} > 0$, $w_{ij} > 0$ for $j \in \mathcal{C}(i)$, $w_{ii} + \sum_{j\in\mathcal{C}(i)} w_{ij} = 1$. The loss X_i is split into $\#\mathcal{C}(i) + 1$ parts. For $j \in \mathcal{C}(i)$, the parts $w_{ij}X_i$ will be considered for the sub-pool attached to participant j. The part $w_{ii}X_i$ will be considered for his or her own sub-pool.

Let

$$S_{\boldsymbol{w},i} = w_{ii}X_i + \sum_{j\in\mathcal{C}(i)} w_{ji}X_j$$

be the aggregated losses considered for participant i's conditional mean risk-sharing rule. Let us define respectively the contribution $H_{ii}(S_{\boldsymbol{w},i}) = \mathrm{E}\left[w_{ii}X_i \left| S_{\boldsymbol{w},i}\right.\right]$ of participant i to his or her sub-pool and the contribution $H_{ij}(S_{\boldsymbol{w},i}) = \mathrm{E}\left[w_{ji}X_j \left| S_{\boldsymbol{w},i}\right.\right]$ of participant j to the sub-pool of participant i. The contribution of participant i to the global pool is then given by

$$H_i(\boldsymbol{X}_n) = H_{ii}(S_{\boldsymbol{w},i}) + \sum_{j\in\mathcal{C}(i)} H_{ji}(S_{\boldsymbol{w},j}).$$

These contributions satisfy the full loss allocation condition; see Appendix D for a proof.

7.2.2 Properties

Let us now investigate the properties of the network-based conditional mean risk-sharing rule. Here, we assume that the network structure and associated weights w_{ij} are given and cannot be modified. As it was the case for the multiple layer rule, fair splitting and fair merging do not apply to the network-based conditional mean risk-sharing rule. This is because these properties would imply a modification of the network and weights. The same comments apply to fair redistribution. Also, the normalization property needs to be properly interpreted. Indeed, eliminating participant j from the network could be viewed as assuming that $w_{ij} = 0$ for $i \in \mathcal{C}(j)$ and therefore as modifying the other weights of the network. For the normalization property, we implicitly assume that the network and its associated weights are not modified. Setting the jth component of $\boldsymbol{X}_n$ equal to 0 is not equivalent to eliminating participant j.

Proposition 7.3. *For a given network and set of associated weights, the network-based conditional mean risk-sharing rule satisfies the reshuffling property provided the weights are reshuffled according to the components of $\boldsymbol{X}_n$, the normalization property, the translativity property, the positive homogeneity property, the constancy property, the no-ripoff property, the actuarial fairness property, the willingness-to-join property, and the stand-alone property for comonotonic losses. The network-based conditional mean risk-sharing rule does not necessarily satisfy the uniformity property for exchangeable losses.*

The proof of Proposition 7.3 is given in Appendix E.

7.3 Claim-only conditional mean risk-sharing rule

7.3.1 Definition

We could operate the sharing only among participants who suffer a positive loss, letting the claim-free ones out of the exchange. This is particularly useful to distribute a given amount of resources among the individuals in need as it is the case in contingent risk funds, as explained next.

Define

$$I_i = \begin{cases} 1 \text{ if } X_i > 0, \\ \\ 0 \text{ otherwise.} \end{cases}$$

Also, denote as $\boldsymbol{I}_n$ the random vector $(I_1, \ldots, I_n)$. To restrict the sharing to claimants, we can use the modified conditional mean risk-sharing rule defined as $\mathrm{E}[X_i|S_n, \boldsymbol{I}_n]$. Since $I_i = 0$ implies $\mathrm{E}[X_i|S_n, \boldsymbol{I}_n] = 0$ claim-free participants do not bear any losses. The claim-only conditional mean risk-sharing rule $\boldsymbol{H}^{\mathrm{cm},+}$ is defined as

$$H_i^{\mathrm{cm},+}(\boldsymbol{X}_n) = \mathrm{E}[X_i|S_n, \boldsymbol{I}_n], \;\; i = 1, 2, \ldots, n. \tag{7.1}$$

Obviously, $\boldsymbol{H}^{\mathrm{cm},+}$ satisfies the full allocation condition.

The rule $\boldsymbol{H}^{\mathrm{cm},+}$ appears to be useful within contingent risk funds where a fixed budget b is distributed among participants who suffer from losses. Participant i then receives the share $\mathrm{E}[X_i|S_n, \boldsymbol{I}_n]/S_n$ of b.

7.3.2 Properties

Let us now investigate the properties of the claim-only conditional mean risk-sharing rule. It is implicitly assumed here that, for some $i = 1, 2, \ldots, n$, $0 < \mathrm{P}[I_i = 1] < 1$. Indeed if $\mathrm{P}[I_i = 1] = 1$ for all $i = 1, 2, \ldots, n$, then the claim-only conditional mean risk-sharing rule reduces to the conditional mean risk-sharing rule.

Proposition 7.4. *The claim-only conditional mean risk-sharing rule satisfies the reshuffling property, the normalization property, the positive homogeneity property, the constancy property, the no-ripoff property, the actuarial fairness property, the willingness-to-join property, and the stand-alone property for comonotonic losses. The claim-only conditional mean risk-sharing rule does not necessarily satisfy the translativity property, the fair-bilateral-redistributing property, the fair-merging property, the fair-splitting property, the fair-redistributing property, the comonotonicity property, and the uniformity property for exchangeable losses.*

The proof of Proposition 7.4 is given in Appendix F. Notice that even if the claim-only conditional mean risk-sharing rule does not satisfy the uniformity property for exchangeable losses, it does satisfy a slightly adapted version of this property since (F.1) shows that total losses are uniformly distributed over all claimants in the exchangeable case.

8 Discussion

8.1 Summary

Based on a list of desirable properties for risk sharing, this paper offers a systematic treatment of different risk-sharing rules for insurance losses, including the conditional mean risk-sharing rule and the newly proposed quantile risk-sharing rule. Several modifications of the conditional mean risk-sharing rule have also been suggested.

Some rules considered in this paper are nonparametric in the sense that they do not require the knowledge of the joint distribution of the losses comprised in the pool. This is the case for the order statistics, the multiple layer and the uniform risk-sharing rules. Other ones are very easy to implement since they only involve elementary quantities, such as the mean proportional risk-sharing rule that only uses expected losses or the scale-proportional risk-sharing rule. The quantile risk-sharing rule requires the knowledge of the marginal distribution of the losses comprised in the pool, but it does not use their dependence structure. In that respect, the quantile rule is copula-free. Finally, the conditional mean risk-sharing rule requires full knowledge of the joint distribution of the losses in the pool. Its implementation obviously necessitates more computational efforts but all properties discussed in Section 3, except comonotonicity, are satisfied. Change of distributions may ease computations and the conditional mean risk-sharing rule reduces to simpler rules under some modified distribution.

The properties established in this paper for the risk-sharing rules under consideration are summarized in Tables 8.1-8.2, where check marks indicate that the corresponding risk-sharing rule fulfills the property under consideration. Table 8.1 summarizes the conservation properties according to the terminology introduced in the introduction to this paper. These are properties fulfilled by the stand-alone risk-sharing rule (as shown by all check marks appearing in the first column) that may also be desirable for other rules. Table 8.2 considers the improvement, local redistribution and specific-pool properties for non-trivial risk-sharing rules (that is, all rules except the trivial stand-alone one). Notice that the absence of mark means that the property is not relevant (redistribution for multiple layer or network-based conditional mean risk-sharing rules, for instance) or does not necessarily hold true. In the latter case, it can nevertheless be fulfilled in some particular cases. For instance, the conditional mean risk-sharing rule satisfies the comonotonicity property when losses are independent with log-concave densities or when their number tends to infinity, under mild regularity conditions.

Of course, one can imagine other properties than those proposed in this paper, such as the monotonicity property proposed by Chen et al. (2017) which requires introducing risk measures whereas comparisons are based on the convex order in this paper. The monotonicity property could be expressed in the present setting as follows: a new entry in the pool will not lead to higher risks allocated to existing participants in terms of convex order. Precisely, considering a pool $\boldsymbol{X}_n$ including participant i and a larger pool $\boldsymbol{X}_{n+k}$, $k \geq 1$, supplementing the pool $\boldsymbol{X}_n$ with additional losses $X_{n+1}, \ldots, X_{n+k}$, then $H_i(\boldsymbol{X}_{n+k}) \preceq_{\text{CX}} H_i(\boldsymbol{X}_n)$. In some

	Stand-alone	Unif.	Mean-prop.	Order stat.	Mult. layer	Cond. mean	Quantile	Network	Claim-only
Reshuffling	✓	✓	✓			✓	✓	✓	✓
Normalization	✓		✓			✓	✓	✓	✓
Translativity	✓					✓	✓	✓	
Pos. hom.	✓	✓	✓	✓		✓	✓	✓	✓
Constancy	✓					✓	✓	✓	✓
No-ripoff	✓					✓	✓	✓	✓
Act. fair.	✓		✓			✓		✓	✓

Table 8.1: Summary of the conservation properties fulfilled by the risk-sharing rules considered in this paper.

sense, this is a generalization of the willingness-to-join property. The conditional mean risk-sharing rule satisfies this property for independent losses and the uniform risk-sharing rule fulfills it for exchangeable losses, for instance. It is also worth to recall from the introduction that we do not consider optimality criteria in the present paper. Several comparison criteria have been proposed in the literature. The interested reader is referred e.g. to Abdikerimova and Feng (2022) for a discussion of altruistic transfer plans and to Charpentier et al. (2021) for a study of maximum coverage for non-linear contracts, for examples in the most recent literature.

8.2 From risk sharing to insurance

The risk-sharing rules studied in this paper are very helpful to better understand commercial insurance. Consider individual losses X_i such that $s \mapsto \mathrm{E}[X_i|S_n = s]$ are continuously increasing for all i. If not then individual i should not join the pool because he or she could benefit from an increase in S_n and thus has interests conflicting with other participants. Consider the case of a commercial insurance company having to pay a rate of return $\mathrm{r}_{\mathrm{roc}}$ on the solvency capital put at its disposal. Assume that the solvency capital is obtained from the Value-at-Risk at probability level 99.5% as under Solvency 2 in the European Union.

	Unif.	Mean-prop.	Order stat.	Mult. layer	Cond. mean	Quantile	Network	Claim-only
Will. to join					✓		✓	✓
Comonot.	✓	✓		✓		✓		
Bil. redist.	✓	✓		✓	✓			
Merging		✓			✓			
Splitting		✓			✓			
Redist.		✓			✓			
Stand alone for comonot.					✓	✓	✓	✓
Uniform. for exchang.	✓	✓			✓	✓		

Table 8.2: Summary of the improvement, local redistribution and specific-pool properties fulfilled by the risk-sharing rules considered in this paper.

The total solvency capital $\text{VaR}[S_n; 99.5\%] - \text{E}[S_n]$ can be decomposed into

$$\begin{aligned}\text{VaR}[S_n; 99.5\%] - \text{E}[S_n] &= \text{VaR}\left[\sum_{i=1}^{n} \text{E}[X_i|S_n]; 99.5\%\right] - \sum_{i=1}^{n} \text{E}[X_i] \\ &= \sum_{i=1}^{n} \Big(\text{VaR}\big[\text{E}[X_i|S_n]; 99.5\%\big] - \text{E}[X_i]\Big) \\ &= \sum_{i=1}^{n} \Big(\text{E}\big[X_i|S_n = \text{VaR}[S_n; 99.5\%]\big] - \text{E}[X_i]\Big).\end{aligned}$$

Policyholders then receive the loss amount X_i ex post in exchange of the ex-ante premium

$$\pi[X_i; S_n] = \text{E}[X_i] + \text{r}_{\text{roc}}\Big(\text{E}\big[X_i|S_n = \text{VaR}[S_n; 99.5\%]\big] - \text{E}[X_i]\Big)$$

where we recognize the conditional mean risk-sharing rule applied to $s = \text{VaR}[S_n; 99.5\%]$. This appears to be an appropriate economic premium calculation principle. Similar formulas hold true replacing the conditional mean risk-sharing rule with any comonotonic risk-sharing rule.

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APPENDIX

A Proof of Proposition 4.2

Reshuffling property The reshuffling property is valid since individual contributions $H_i^{\text{cm}}(S_n)$ only depend on (X_i, S_n) which is not modified by reshuffling.

Normalization property The normalization property is valid since $S_n = S_{n-1}$ when $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_{n-1}, 0)$ and $\boldsymbol{X}_{n-1} = (X_1, X_2, \ldots, X_{n-1})$.

Translativity property Translativity holds true since, for all $i \neq n$,

$$H_i^{\text{cm}}\left(\boldsymbol{X}_n + c\ \mathbf{1}_{n,n}\right) = \text{E}[X_i|S_n + c] = \text{E}[X_i|S_n] = H_i^{\text{cm}}\left(\boldsymbol{X}_n\right).$$

Positive homogeneity property Positive homogeneity holds true since

$$H_i^{\text{cm}}(c\boldsymbol{X}_n) = \text{E}[cX_i|cS_n] = c\text{E}[X_i|S_n] = cH_i^{\text{cm}}(S_n).$$

Constancy property Let $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_{n-1}, c)$, for some constant c, and let $\boldsymbol{X}_{n-1} = (X_1, X_2, \ldots, X_{n-1})$. Then we have that, for any $i \neq n$,

$$H_i^{\text{cm}}\left(\boldsymbol{X}_n\right) = \text{E}[X_i|S_n] = \text{E}[X_i|S_{n-1} + c] = \text{E}[X_i|S_{n-1}] = H_i^{\text{cm}}\left(\boldsymbol{X}_{n-1}\right).$$

No-ripoff property The no-ripoff property is valid since

$$H_i^{\text{cm}}\left(\boldsymbol{X}_n\right) = \text{E}[X_i|S_n] \leq \text{E}[F_i^{-1}\left(1\right)|S_n] = F_i^{-1}\left(1\right),\ \text{for any } i = 1, 2, \ldots, n.$$

Actuarial fairness property Actuarial fairness holds true because

$$\text{E}\left[H_i^{\text{cm}}\left(\boldsymbol{X}_n\right)\right] = \text{E}\left[\text{E}[X_i|S_n]\right] = \text{E}\left[X_i\right],\ \text{for any } i = 1, 2, \ldots, n.$$

Willingness-to-join property The willingness-to-join property is valid since

$$\text{E}[X_i|S_n] \preceq_{\text{CX}} X_i \text{ holds for every } i = 1, 2, \ldots, n,$$

Fair-bilateral-redistributing property This property is valid because $H_i^{\text{cm}}(\boldsymbol{X}_n)$ only depends on (X_i, S_n) and S_n is not modified by a bilateral redistribution.

Fair-merging property The fair-merging property holds true since, for any $k \neq l$,

$$H_i^{\text{cm}}\big(\boldsymbol{X}_n + X_l \times \left(\mathbf{1}_{k,n} - \mathbf{1}_{l,n}\right)\big) = \text{E}[X_i|S_n] = H_i^{\text{cm}}\left(\boldsymbol{X}_n\right) \text{ if } i \text{ is different from } k \text{ and } l,$$

Fair-splitting property The fair-splitting property holds true since, for any $k \in 1, 2, \ldots, n$, and any non-negative random variables X_k' and X_{n+1} satisfying $X_k = X_k' + X_{n+1}$, one has that, for any i different from k and $n+1$

$$H_i^{\mathrm{cm}}\Big((\boldsymbol{X}_n + (X_k' - X_k) \times \mathbf{1}_{k,n}, X_{n+1}) \Big) = \mathrm{E}[X_i|S_n] = H_i^{\mathrm{cm}}\left(\boldsymbol{X}_n\right).$$

Fair-redistributing property This property is valid because $H_i^{\mathrm{cm}}(\boldsymbol{X}_n)$ only depends on (X_i, S_n) and S_n is not modified by redistribution.

Stand-alone property for comonotonic losses The stand-alone property for comonotonic losses follows from Proposition 5.9. Indeed, if $\boldsymbol{X}_n$ is comonotonic then $X_i = h_i(S_n)$ for the non-decreasing functions h_i defined in (5.15). Therefore,

$$H_i^{\mathrm{cm}}\left(S_n\right) = \mathrm{E}\left[h_i\left(S_n\right) \mid S_n\right] = h_i\left(S_n\right) = X_i.$$

Uniformity property for exchangeable losses If $\boldsymbol{X}_n$ is exchangeable then we have for any $i \neq j$ in $\{1, 2, \ldots, n\}$ that

$$\mathrm{E}[X_i|S_n] = \mathrm{E}[X_j|S_n] = \frac{1}{n}\sum_{k=1}^{n} \mathrm{E}[X_k|S_n] = \frac{S_n}{n}.$$

This shows that

$$\boldsymbol{X}_n \text{ exchangeable} \Rightarrow \boldsymbol{H}^{\mathrm{cm}}(\boldsymbol{X}_n) = \boldsymbol{H}^{\mathrm{uni}}(\boldsymbol{X}_n). \tag{A.1}$$

Comonotonicity property The conditional mean risk-sharing rule satisfies the comonotonicity property only if, $s \mapsto \mathrm{E}[X_i|S_n = s]$ is non-decreasing for any $i \in 1, 2, \ldots, n$. This is not true in general. We refer the reader to Denuit and Robert (2021a) for a counter-example involving independent zero-augmented Gamma-distributed risks.

B Proof of Proposition 5.6

Reshuffling property The quantile risk-sharing rule satisfies the reshuffling property since the distributions of S_n and S_n^{c} are not modified by reshuffling.

Normalization property The quantile risk-sharing rule satisfies the normalization property since $S_n = S_{n-1}$ and $S_n^{\mathrm{c}} = S_{n-1}^{\mathrm{c}}$ when $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_{n-1}, 0)$ and $\boldsymbol{X}_{n-1} = (X_1, X_2, \ldots, X_{n-1})$.

Translativity property The quantile risk-sharing rule satisfies the translativity property since, for all $i \neq n$,

$$H_i^{\mathrm{quant}}\left(\boldsymbol{X}_n + c\ \mathbf{1}_{n,n}\right) = F_i^{-1(\alpha_{S_n+c})}\left(F_{S_n^{\mathrm{c}}+c}\left(S_n + c\right)\right) = F_i^{-1(\alpha_{S_n+c})}\left(F_{S_n^{\mathrm{c}}}\left(S_n\right)\right)$$

with

$$\begin{aligned}\alpha_{S_n+c} &= \frac{F_{S_n+c}^{-1+}\left(F_{S_n+c}(S_n+c)\right)-S_n-c}{F_{S_n+c}^{-1+}\left(F_{S_n+c}(S_n+c)\right)-F_{S_n+c}^{-1}\left(F_{S_n+c}(S_n+c)\right)}\\ &= \frac{F_{S_n}^{-1+}\left(F_{S_n}(S_n)\right)-S_n}{F_{S_n}^{-1+}\left(F_{S_n}(S_n)\right)-F_{S_n}^{-1}\left(F_{S_n}(S_n)\right)}=\alpha_{S_n}.\end{aligned}$$

Positive homogeneity property The proof of the positive homogeneity property follows the same lines as the one of the translativity property.

Constancy property Considering the constancy property, let $\boldsymbol{X}_n=(X_1,X_2,\ldots,X_{n-1},c)$, for some constant c, and let $\boldsymbol{X}_{n-1}=(X_1,X_2,\ldots,X_{n-1})$. Then we have that, for any $i\neq n$,

$$H_i^{\text{quant}}\left(\boldsymbol{X}_n\right)=F_i^{-1(\alpha_{S_{n-1}+c})}\left(F_{S_{n-1}^c+c}\left(S_{n-1}+c\right)\right)=F_i^{-1(\alpha_{S_{n-1}+c})}\left(F_{S_{n-1}^c}\left(S_{n-1}\right)\right)$$

with

$$\begin{aligned}\alpha_{S_{n-1}+c} &= \frac{F_{S_{n-1}+c}^{-1+}\left(F_{S_{n-1}+c}(S_{n-1}+c)\right)-S_{n-1}-c}{F_{S_{n-1}+c}^{-1+}\left(F_{S_{n-1}+c}(S_{n-1}+c)\right)-F_{S_{n-1}+c}^{-1}\left(F_{S_{n-1}+c}(S_{n-1}+c)\right)}\\ &= \frac{F_{S_{n-1}}^{-1+}\left(F_{S_{n-1}}(S_{n-1})\right)-S_{n-1}}{F_{S_{n-1}}^{-1+}\left(F_{S_{n-1}}(S_{n-1})\right)-F_{S_{n-1}}^{-1}\left(F_{S_{n-1}}(S_{n-1})\right)}=\alpha_{S_{n-1}}.\end{aligned}$$

Hence the quantile risk-sharing rule satisfies the constancy property.

No-ripoff property The no-ripoff property is valid since

$$H_i^{\text{quant}}\left(\boldsymbol{X}_n\right)=F_i^{-1(\alpha_{S_n})}\left(F_{S_n^c}\left(S_n\right)\right)\leq F_i^{-1}\left(1\right)\text{ for any }i=1,2,\ldots,n.$$

Comonotonicity property The quantile risk-sharing rule satisfies the comonotonicity property by construction as $F_i^{-1(\alpha_s)}(F_{S_n^c}(s))$ is non-decreasing in s.

Stand-alone property for comonotonic losses Every comonotonic pool is left unchanged by the quantile risk-sharing pool so that the stand-alone property for comonotonic losses is valid. See Proposition 5.11.

Uniformity property for exchangeable losses If $X_1,X_2,\ldots,X_n$ are identically distributed with common distribution function F then the full allocation condition gives $S_n=nF^{-1(\alpha_{S_n})}\left(F_{S_n^c}\left(S_n\right)\right)$. This shows that

$$H_i^{\text{quant}}\left(\boldsymbol{X}_n\right)=H_i^{\text{uni}}\left(S_n\right)=\frac{S_n}{n},$$

whatever the dependence structure of individual losses X_i, so that this applies in particular to exchangeable losses.

Actuarial fairness property To show that the quantile risk-sharing rule does not necessarily satisfy the actuarial fairness property, let us consider the case where $n = 2$, X_1 has distribution function F and X_2 has distribution function $F \circ g^{-1}$ where g is a positive, continuous and increasing function. We have

$$\begin{aligned} \mathrm{P}\left[H_1^{\text{quant}}(\boldsymbol{X}_2) > v\right] &= \mathrm{P}\left[S_2 > \left(F^{-1} + g \circ F^{-1}\right) \circ F(v)\right] \\ &= \mathrm{P}\left[S_2 > v + g(v)\right] \\ &= \mathrm{P}\left[h^{-1}(S_2) > v\right] \end{aligned}$$

with $h(v) = v + g(v)$. It follows that

$$\mathrm{E}\left[H_1^{\text{quant}}(\boldsymbol{X}_2)\right] = \mathrm{E}\left[h^{-1}(X_1 + g(Z_1))\right] = \mathrm{E}\left[h^{-1}(X_1 - Z_1 + h(Z_1))\right]$$

where Z_1 has the same distribution as X_1. If X_1 and Z_1 are comonotonic or g is a linear function, then $\mathrm{E}\left[H_1^{\text{quant}}(\boldsymbol{X}_2)\right] = \mathrm{E}[X_1]$. But this is not true in general.

Willingness-to-join property The quantile risk-sharing rule cannot always be beneficial for all risk-averse individuals since it does not always satisfy the fairness property. Hence the willingness-to-join property is not necessarily valid.

Fair-bilateral-redistributing property This property cannot hold since the quantile risk-sharing rule satisfies the normalization property but not the fair-merging property (see below), in application of Remark 3.54.

Fair-merging property Moving to the fair-merging property, the quantile risk-sharing rule does not necessarily satisfy it since, for any $k \neq l$, $F^{-1}_{X_k+X_l}(p)$ is not always equal to $F^{-1}_{X_l}(p) + F^{-1}_{X_l}(p)$ if X_k and X_l are not comonotonic.

Fair-splitting property Following the same reasoning as for the fair-merging property, the quantile risk-sharing rule does not necessarily satisfy the fair-splitting property.

Fair-redistributing property This property cannot hold because the weaker fair-bilateral-redistributing, fair-merging and fair-splitting properties are not fulfilled. See Remark 3.70.

C Proof of Proposition 5.9

First, suppose that $\boldsymbol{X}_n$ is comonotonic. We define the connected support of $\boldsymbol{X}_n$ as follows:

$$\left\{\left(F_1^{-1(\alpha)}(p), F_2^{-1(\alpha)}(p), \ldots, F_n^{-1(\alpha)}(p)\right) \middle| p \in [0,1] \text{ and } \alpha \in [0,1]\right\}, \tag{C.1}$$

similar to the one introduced in formula (29) of Dhaene et al. (2002a). By convention, we set $F_i^{-1(\alpha)}(0) = F_i^{-1+}(0)$ and $F_i^{-1(\alpha)}(1) = F_i^{-1}(1)$. Let $\boldsymbol{x}_n = (x_1, x_2, \ldots, x_n)$ be an element of this connected support and let $s = \sum_{i=1}^n x_i$. Following a reasoning similar to the one of the proof of Theorem 7 in Dhaene et al. (2002a), we find that $\boldsymbol{x}_n$ is the unique point of the intersection of the connected support and the hyperplane $\{(y_1, y_2, \ldots, y_n) \mid \sum_{i=1}^n y_i = s\}$. As the point $(h_1(s), h_2(s), \ldots, h_n(s))$ with the h_i defined in (5.15) is an element of the connected support of $\boldsymbol{X}_n$, and moreover, $\sum_{i=1}^n h_i(s) = s$ so that $(h_1(s), h_2(s), \ldots, h_n(s))$ is a point of the hyperplane considered above, we find that

$$\boldsymbol{x}_n = (h_1(s), h_2(s), \ldots, h_n(s)) \,. \tag{C.2}$$

As this expression hold for any point $\boldsymbol{x}_n$ of the connected support of $\boldsymbol{X}_n$, we can conclude that (5.14) holds.

Conversely, let us assume that $\boldsymbol{X}_n$ is given by (5.14), with the functions h_i defined in (5.15). As the functions h_i are all non-decreasing, it follows immediately that $\boldsymbol{X}_n$ is comonotonic, see e.g. Theorem 3 in Dhaene et al. (2002a).

Remark C.1. The characterization of comonotonicity in Proposition 5.9 is similar to the one established in the bivariate case by Denneberg (1994, Prop. 4.5, item v). While Denneberg (1994) only proves the existence of non-decreasing and continuous functions h_i, explicit expressions of h_i are given here.

D Proof that the network-based conditional mean risk-sharing rule satisfies the full loss allocation condition

It suffices to write

$$\begin{aligned}
\sum_{i=1}^n H_i(\boldsymbol{X}_n) &= \sum_{i=1}^n H_{ii}(S_{\boldsymbol{w},i}) + \sum_{i=1}^n \sum_{j\in\mathcal{C}(i)} H_{ji}(S_{\boldsymbol{w},j}) \\
&= \sum_{i=1}^n \mathrm{E}\left[w_{ii}X_i \,|S_{\boldsymbol{w},i}\right] + \sum_{i=1}^n \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[w_{ij}X_i \,|S_{\boldsymbol{w},j}\right] \\
&= \sum_{i=1}^n \mathrm{E}\left[w_{ii}X_i \,|S_{\boldsymbol{w},i}\right] + \sum_{i=1}^n \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[w_{ji}X_j \,|S_{\boldsymbol{w},i}\right] \\
&= \sum_{i=1}^n \mathrm{E}\left[w_{ii}X_i + \sum_{j\in\mathcal{C}(i)} w_{ji}X_j \Big| S_{\boldsymbol{w},i}\right] \\
&= \sum_{i=1}^n S_i = \sum_{i=1}^n \left(w_{ii}X_i + \sum_{j\in\mathcal{C}(i)} w_{ji}X_j \right) \\
&= \sum_{i=1}^n w_{ii}X_i + \sum_{i=1}^n \sum_{j\in\mathcal{C}(i)} w_{ji}X_j = \sum_{i=1}^n w_{ii}X_i + \sum_{i=1}^n \sum_{j\in\mathcal{C}(i)} w_{ij}X_i \\
&= \sum_{i=1}^n X_i \left(w_{ii} + \sum_{j\in\mathcal{C}(i)} w_{ij} \right) = \sum_{j=1}^n X_j.
\end{aligned}$$

The full loss allocation condition (2.2) is therefore fulfilled.

E Proof of Proposition 7.3

The contribution for participant i to the global pool is given by

$$H_i(\boldsymbol{X}_n) = H_{ii}(S_{\boldsymbol{w},i}) + \sum_{j\in\mathcal{C}(i)} H_{ji}(S_{\boldsymbol{w},j}) = \mathrm{E}\left[w_{ii}X_i \left| S_{\boldsymbol{w},i}\right.\right] + \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[w_{ij}X_i \left| S_{\boldsymbol{w},j}\right.\right]$$

so that $H_i(\boldsymbol{X}_n)$ may be viewed as a sum of contributions obeying the conditional mean risk sharing rule within $\#\mathcal{C}\,(i)+1$ sub-pools.

Reshuffling property The reshuffling property is valid provided we assume that the family of weights are also reshuffled with the components of the loss random vector $\boldsymbol{X}_n$.

Normalization property The normalization property is valid since each sub-pool satisfies the normalization property.

Translativity property Translativity holds true because each sub-pool satisfies the translativity property and $w_{ii} + \sum_{j\in\mathcal{C}(i)} w_{ij} = 1$ for $i = 1, 2, \ldots, n$.

Positive homogeneity property Also, positive homogeneity holds true because each sub-pool satisfies the positive homogeneity property.

Constancy property Constancy is valid because each sub-pool satisfies the constancy property.

No-ripoff property The no-ripoff property is valid since

$$\begin{aligned} H_i(\boldsymbol{X}_n) &= \mathrm{E}\left[w_{ii}X_i \left| S_{\boldsymbol{w},i}\right.\right] + \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[w_{ij}X_i \left| S_{\boldsymbol{w},j}\right.\right] \\ &\le \mathrm{E}\left[w_{ii}F_i^{-1}(1) \left| S_{\boldsymbol{w},i}\right.\right] + \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[w_{ij}F_i^{-1}(1) \left| S_{\boldsymbol{w},j}\right.\right] = F_i^{-1}(1)\,. \end{aligned}$$

Actuarial fairness property Actuarial fairness holds true because each sub-pool satisfies the actuarial fairness property and $w_{ii} + \sum_{j\in\mathcal{C}(i)} w_{ij} = 1$ for $i = 1, 2, \ldots, n$.

Willingness-to-join property To establish the validity of the willingness-to-join property, let us consider a convex function ϕ. Jensen's inequality allows us to write

$$\begin{aligned} \mathrm{E}[\phi\,(X_i)] &= w_{ii}\mathrm{E}[\phi\,(X_i)] + \sum_{j\in\mathcal{C}(i)} w_{ij}\mathrm{E}[\phi\,(X_i)] \\ &\ge w_{ii}\mathrm{E}[\phi\,(\mathrm{E}\left[X_i \left| S_{\boldsymbol{w},i}\right.\right])] + \sum_{j\in\mathcal{C}(i)} w_{ij}\mathrm{E}[\phi\,(\mathrm{E}\left[X_i \left| S_{\boldsymbol{w},j}\right.\right])] \\ &= \mathrm{E}[w_{ii}\phi\,(\mathrm{E}\left[X_i \left| S_{\boldsymbol{w},i}\right.\right]) + \sum_{j\in\mathcal{C}(i)} w_{ij}\phi\,(\mathrm{E}\left[X_i \left| S_{\boldsymbol{w},j}\right.\right])]. \end{aligned}$$

Since ϕ is convex, we then get

$$w_{ii}\phi\left(\mathrm{E}\left[X_i\left|S_{\boldsymbol{w},i}\right.\right]\right)+\sum_{j\in\mathcal{C}(i)}w_{ij}\phi\left(\mathrm{E}\left[X_i\left|S_{\boldsymbol{w},j}\right.\right]\right)\geq\phi\left(\mathrm{E}\left[w_{ii}X_i\left|S_{\boldsymbol{w},i}\right.\right]+\sum_{j\in\mathcal{C}(i)}\mathrm{E}\left[w_{ij}X_i\left|S_{\boldsymbol{w},j}\right.\right]\right)$$

and finally

$$\mathrm{E}[\phi\left(X_i\right)]\geq\mathrm{E}\left[\phi\left(\mathrm{E}\left[w_{ii}X_i\left|S_{\boldsymbol{w},i}\right.\right]+\sum_{j\in\mathcal{C}(i)}\mathrm{E}\left[w_{ij}X_i\left|S_{\boldsymbol{w},j}\right.\right]\right)\right].$$

Since the latter inequality holds true for any convex function ϕ, we have $H_i(\boldsymbol{X}_n)\preceq_{\mathrm{CX}}X_i$.

Stand-alone property for comonotonic losses The stand-alone for comonotonic losses property holds true because each sub-pool satisfies the comonotonic losses property and $w_{ii}+\sum_{j\in\mathcal{C}(i)}w_{ij}=1$ for $i=1,2,\ldots,n$.

Uniformity property for exchangeable losses Uniformity property for exchangeable losses does not hold since weights are not necessarily uniform.

F Proof of Proposition 7.4

Reshuffling property The reshuffling property is valid since the rule only depends on $(X_i,S_n,\boldsymbol{I}_n)$ which is not modified by reshuffling.

Normalization property The normalization property is valid because assuming $X_j=0$ is equivalent to have $\mathrm{P}[I_j=0]=1$. Participant j has no claim a.s. and never bear losses.

Positive homogeneity property Positive homogeneity holds true because

$$H_i^{\mathrm{cm},+}(c\boldsymbol{X}_n)=\mathrm{E}[cX_i|cS_n,\boldsymbol{I}_n]=c\mathrm{E}[X_i|S_n,\boldsymbol{I}_n]=H_i^{\mathrm{cm},+}(c\boldsymbol{X}_n).$$

Constancy property Constancy is valid since, if $X_j=c>0$, then, for $i\neq j$,

$$\mathrm{E}[X_i|S_n,\boldsymbol{I}_n]=\mathrm{E}[X_i|S_{n-1}^{(\backslash j)},\boldsymbol{I}_{n-1}^{(\backslash j)}].$$

No-ripoff property The no-ripoff property is valid since

$$H_i^{\mathrm{cm},+}\left(\boldsymbol{X}_n\right)=\mathrm{E}[X_i|S_n,\boldsymbol{I}_n]\leq\mathrm{E}[F_i^{-1}\left(1\right)|S_n,\boldsymbol{I}_n]=F_i^{-1}\left(1\right),\ \text{for any } i=1,2,\ldots,n.$$

Actuarial fairness property Actuarial fairness holds true because

$$\mathrm{E}\left[H_i^{\mathrm{cm},+}\left(\boldsymbol{X}_n\right)\right]=\mathrm{E}\big[\mathrm{E}[X_i|S_n,\boldsymbol{I}_n]\big]=\mathrm{E}\left[X_i\right],\ \text{for any } i=1,2,\ldots,n.$$

Willingness-to-join property The willingness-to-join property is valid since

$$\mathrm{E}[X_i|S_n, \boldsymbol{I}_n] \preceq_{\mathrm{CX}} X_i \text{ holds for every } i = 1, 2, \ldots, n.$$

Stand-alone property for comonotonic losses The stand-alone for comonotonic losses property follows from Proposition 5.9.

Translativity property Assume that $\mathrm{P}[I_j = 1] < 1$ and that $c > 0$. If the vector of risks is now $\boldsymbol{X}_n + c\ \mathbf{1}_{j,n}$, then participant j always contributes to the payment of S_n and the contributions of the other participants change. It is clear that, for $i \neq j$, $H_i^{\mathrm{cm},+}(\boldsymbol{X}_n + c\ \mathbf{1}_{j,n}) \neq H_i^{\mathrm{cm},+}(\boldsymbol{X}_n)$.

Fair-bilateral-redistributing property This property cannot hold since the claim-only conditional mean risk-sharing rule satisfies the normalization property but not the fair-merging property (as shown next), see Remark 3.54.

Fair-merging property The fair-merging property does not hold since from the event $\{X_k + X_l > 0\}$ it is not possible to know if $\{X_k > 0\}$ and $\{X_l > 0\}$.

Fair-splitting property The fair-splitting property does not hold since the split of X_k into X_k' and X_{n+1} does not necessarily imply that $\{X_k > 0\}$, $\{X_k' > 0\}$ and $\{X_{n+1} > 0\}$ are equal.

Fair-redistributing property This property cannot hold because the weaker fair-bilateral-redistributing, fair-merging and fair-splitting properties are not fulfilled. See Remark 3.70.

Uniformity property for exchangeable losses In case $\boldsymbol{X}_n$ is exchangeable, the claim-only conditional mean risk sharing rule reduces to a uniform allocation among claimants, that is,

$$\boldsymbol{X}_n \text{ exchangeable} \Rightarrow H_i^{\mathrm{cm},+}(\boldsymbol{X}_n) = I_i \frac{S_n}{\sum_{j=1}^n I_j} \tag{F.1}$$

with the understanding that $H_i^{\mathrm{cm},+}(\boldsymbol{X}_n) = 0$ when $\sum_{j=1}^n I_j = 0$. Thus, we do not recover $\boldsymbol{H}^{\mathrm{uni}}$ in the exchangeable case.