DISCUSSION PAPER

LEARNING FROM INPUT-OUTPUT MIXES IN DEA:
A PROPORTIONAL MEASURE FOR SLACK-
BASED EFFICIENT PROJECTIONS

by

Laurens CHERCHYE
Tom VAN PUYENBROECK

Katholieke Universiteit Leuven

Departement Economie

Naamsestraat 69
B-3000 Leuven

Learning from Input-Output Mixes in DEA :
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Laurens CHERCHYE* and Tom VAN PUYENBROECK

Centrum voor Economische Studiën

KU Leuven

Naamsestraat 69

B-3000 Leuven, Belgium

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Abstract

Several Data Envelopment Analysis (DEA) models use a radial distance measure that is based on the Debreu-Farrell notion of (in)efficiency. While this measure has an attractive interpretation, its use may be problematic if slacks or zeros in the data are present. The additive DEA model can perfectly deal with these problems, but the meaning of its associated scores is less intuitive than the one attached to the radial measures. We introduce an alternative efficiency measure, based on the results of the additive model, that can be decomposed in a Debreu-Farrell component and a factor that captures differences in input-output mixes w.r.t. those of the best practice reference observation. On an aggregate level, this second component can be considered as an indicator of the dispersion between radial efficiency measurement and results based on the Pareto-Koopmans efficiency notion. On the individual level, the measure allows to regard relative inefficiency as resulting from (i) a divergence of implicit cost price vectors, and (ii) a cost level that is too high, even after adjustment for the implicit cost prices.

Keywords: Technical efficiency, additive DEA model, implicit prices

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*Corresponding author. E-mail: Laurens.Cherchye@econ.kuleuven.ac.be

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1 Introduction

Data Envelopment Analysis (DEA) as an efficiency measurement tool essentially starts from observed data, usually in the context of production, to (i) create a best practice envelope by means of mathematical programming techniques, and (ii) use this reference frontier and a distance concept to compute efficiency measures for all observations. While several DEA-models exist, many of them have in common the use of a relative radial distance measure to be employed in the second of these above steps. Indeed, the two benchmark DEA models of Charnes, Cooper and Rhodes (CCR; 1978,1979) and Banker, Charnes and Cooper (BCC; 1984) aimed to put into practice the notion of efficiency as expressed in the earlier work of Debreu (1951) and Farrell (1957). In an environment where multiple inputs are used to produce multiple outputs, the Debreu-Farrell input measure is computed as one minus the maximum *equiproportionate* reduction in all inputs which still allows for production of the given amount of output, while the Debreu-Farrell output measure is also defined with reference to the maximum feasible *equiproportionate* output expansion.

Next to this 'historical' explanation for their popularity, these measures have some attractive properties both for the practitioner who needs to communicate DEA results, and for the scholar interested in the relationship between DEA and the microeconomic theory of production. The former can summarize multiple dimensions in one scalar, and express the (in)efficiency 'score' of any observed productive unit as a degree. The latter may well appreciate the close link between the Debreu-Farrell measure and Shepard's distance function, and the related fact that —even if prices are not observed— the measures have a straightforward cost interpretation. Specifically, they can be written as the ratio of minimal to actual costs (input orientation) or actual to maximal revenues (output orientation), independently of the price vector which is used [see Russell (1985)].

Next to the Debreu-Farrell efficiency notion, there exists another widely used concept introduced by Koopmans (1951), which is commonly referred to as Pareto-Koopmans efficiency. In this case a decision making unit (DMU) is said to be technically efficient if an increase in any output or a decrease in any input requires a decrease in at least one other output or an increase in at least one other input. The radial (equiproportionate) nature of comparisons is clearly abandoned in the second of these notions. Evidently then, Debreu-Farrell efficiency does not necessarily imply Pareto-Koopmans efficiency. Indeed, if slacks are present, the Debreu-Farrell approach may lead to an observation being compared to a reference DMU which is not Pareto-Koopmans efficient. The problem is somewhat mitigated by appending any slacks, weighted by the "infinitesimal" constant ε, to the value of the radial measure. It is known that this often employed solution does have some drawbacks. First, as in any practical application ε must be given some value, it can give rise to computational problems [see Ali (1994)]. A second, more fundamental critique is that, though the use of ε excludes Pareto-Koopmans dominated observations from being identified as efficient, it falls short in eliminating the total impact of slacks on the eventual efficiency scores. Especially when slacks are large and occur frequently they could influence the efficiency scores and rankings considerably, which in turn could result in wrong management conclusions.

In some applications slacks indeed appear systematically while not being insignificantly small. In such cases it seems more appropriate to call for *non-radial measures* computed from comparison of each observation to a reference DMU which is not dominated in the Pareto-Koopmans sense. The Färe-Lovell (1978) and the Zieschang (1984) measures are two examples belonging to this class. A comparative study by Ferrier, Kerstens and Vanden Eeckaut (1994) shows that both measures have some favourable characteristics, making them a recommendable alternative for the radial measure when substantial amounts of slacks are present.

Ferrier, Kerstens and Vanden Eeckaut noted however another problem associated with the Färe-Lovell and Zieschang efficiency measures, viz. the fact that they, just as radial measures [see Thompson, Dharmapala and Thrall (1993)], cannot easily handle zero input and output data[1]. While at first sight a solution may be offered by affine displacements of the data, Ali and Seiford (1990) made clear that translation invariance pertains only to the classification of DMUs as efficient or inefficient. The actual efficiency scores and rankings can be altered. Moreover, this limited invariance only holds for models such as the BCC-model, containing a convexity constraint which reflects the assumption of variable returns to scale. These features carry over to the Färe-Lovell and the Zieschang classification.

A model that handles both the slack-problem and the occurrence of zeros in data is the *additive* model of Charnes et al. (1985), which we will discuss in somewhat more detail below. For the moment we point at some problems associated with the scores obtained from the additive DEA-model as it was originally formulated. First, unlike the Debreu-Farrell measures the scores lack an attractive "degree" interpretation. They equal zero for an efficient observation and are negative for an inefficient observation. No dimensionless meaning attaches to the magnitude of the negative value, the latter being constructed as a sum of slacks. However, this can be and has been overcome by a particular rescaling of the scores (see below). A second inconvenience is that efficiency analysis based on this model is not 'oriented'. Unlike e.g. the BCC model, the original additive DEA model does not split up global efficiency of an observation into an input and output component. Pastor (1994) showed that this second drawback can easily be solved by a simple modification of the original additive model, which will be introduced in the next section.

We will use the latter modification to introduce an alternative relative (input and output) efficiency measure computed from the additive results. The associated efficiency scores can be decomposed in an appealing way. Specifically, they are the product of a classical radial measure and a factor that captures differences in input-outputmixes of an evaluated observation *vis-à-vis* its best practice reference. Thus, even if the nature of a dataset prevents or complicates the use of a Debreu-Farrell measure, related information can be obtained from a slack-based measure. Moreover, if the same dataset justifies the use of non-radial projections on the efficient frontier, this implies that an inefficient observation can in principle also learn from the input-output mix of its best practice

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[1] In fact Färe, Lovell and Zieschang (1983) proposed a solution for the zero problem. The idea was to modify the measures so that any influence of zero dimensions on the efficiency measure was avoided. This implied, however, that the performance of different observations was measured relative to spaces of different dimensionality.

reference observation. In general, aggregate information about the importance of the mix-deviation factor can be considered as an indicator of the dispersion between Debreu-Farrell efficiency measurement and results obtained by referring to the Pareto-Koopmans efficiency notion. In the third section the output measure will be illustrated using the application of Nash and Sterna-Karwat (1996). The final section will reproduce our main findings and briefly comment on some possible generalizations.

2 Proportional slack based measures: construction and interpretation

As a preliminary step let us first consider the original additive model as it was formulated by Charnes, Cooper, Golany, Seiford and Stutz (1985). This model relates efficiency results to the notion of Pareto-Koopmans efficiency. Referring to the previous section, inefficiency in the Pareto-Koopmans sense implies that for the examined DMU there exists at least one other feasible input-output vector which weakly exceeds all of the DMU's output levels while simultaneously not using more of any of the DMU's input levels, with strict inequality of at least one component (input or output). The test for Pareto-Koopmans optimality for an observation o thus amounts to solving the following linear program[2]:

<div align="center">

Model 1 : **Additive Model**

$$\min_{\lambda, s^+, s^-} z_o = -e^T s^+ - e^T s^-$$

such that:

$$\mathbf{Y}\lambda - s^+ = Y_o$$

$$-\mathbf{X}\lambda - s^- = -X_o$$

$$e^T \lambda = 1$$

$$\lambda, s^+, s^- \geq 0$$

</div>

where $\mathbf{Y}$ is a non-negative output matrix and $\mathbf{X}$ a non-negative input matrix, with ith columns Y_i and X_i, respectively. The vectors s^+ and s^- contain the output and input slacks. The vector e^T is the sum vector.

Clearly the efficiency score, z_o, is exclusively composed of a sum of slacks. As such z_o is not units invariant, a problem that can be solved by prescaling the data. A commonly used scaling procedure is to divide each input (or output) in a DMU's vector by the

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[2]For the purpose of this paper it suffices to consider only the so-called envelopment formulation of the additive model. Note that one could equally well use the dual, "multiplier" formulation.

average observed input (or output) in its category. The analysis can then be performed on the scaled data. This procedure boils down to substituting the objective function of model 1 by the following:

$$\min_{\lambda, s^+, s^-} z_o = -e^T D_{y_A}^{-1} s^+ - e^T D_{x_A}^{-1} s^-$$

with D_{y_A} and D_{x_A} representing the diagonal matrices of output and input averages, respectively. Still, this formulation lacks the degree interpretation associated with the Debreu-Farrell measure. To obtain a relative score, the following measure, first proposed by Charnes, Cooper, Golany, Seiford and Stutz (C^2GS^2;1984), can be computed from the additive results:

$$C^2GS^2 = \frac{e^T D_{y_A}^{-1} \mathbf{Y} \lambda^* + e^T D_{x_A}^{-1} \mathbf{X} \lambda^*}{e^T D_{y_A}^{-1} \mathbf{Y} \lambda^* + e^T D_{x_A}^{-1} \mathbf{X} \lambda^* + e^T D_{y_A}^{-1} s^{+*} + e^T D_{x_A}^{-1} s^{-*}} \tag{1}$$

where the superscript (*) denotes an optimal value. Loosely speaking, the measure is computed as the 'sum of targets' divided by a sum of the 'sum of targets' and the 'sum of slacks' [see also Haag, Jaska and Semple (1992)]. The C^2GS^2 measure has the property that scores are 'dimensionless' in that they lie between zero and one.

Model 1 provides a measure for the overall efficiency of an observation. It is conceivable that in some applications one is interested in allowing for a separate reference point for input and output (in)efficiency. This information would be obtained by an input oriented and an output oriented additive model respectively [Pastor (1994)]. Essentially, this requires each observation to be compared with all Pareto-Koopmans efficient feasible input-output combinations that use at most the same amount of input (output orientation) and with all Pareto-Koopmans efficient members of the production set that produce at least the same amount of output (input orientation), respectively. Overall Pareto-Koopmans efficiency is achieved by an observation which is as well input as output efficient.

In a first step one should identify the Pareto-Koopmans efficient DMUs. To do so the original additive model should be applied. Suppose there are N observations of which the additive model labels i as efficient. Let us denote by $\mathbf{X}^*$ and $\mathbf{Y}^*$ the input and output matrices associated with these efficient observations. For each of the $N - i$ inefficient observations the following input and output oriented additive problems should then be solved:

$$\text{Model 2}:$$
$$\textbf{Input Oriented Additive Model}$$

$$\min_{\lambda, s^+, s^-} \ z_o = -e^T s^-$$

such that:

$$\mathbf{Y}^* \lambda - s^+ = Y_o$$

$$-\mathbf{X}^* \lambda - s^- = -X_o$$

$$e^T \lambda = 1$$

$$\lambda, s^+, s^- \geq 0$$

$$\text{Model 3}:$$
$$\textbf{Output Oriented Additive Model}$$

$$\min_{\lambda, s^+, s^-} \ z_o = -e^T s^+$$

such that:

$$\mathbf{Y}^* \lambda - s^+ = Y_o$$

$$-\mathbf{X}^* \lambda - s^- = -X_o$$

$$e^T \lambda = 1$$

$$\lambda, s^+, s^- \geq 0$$

The input and output efficiency measures we present below are easily computed using the results of models 2 and 3. We will first concentrate on the input measure, so as to sketch the basic intuition. Afterwards we will introduce and apply the output measure.

Solving model 2 results in an input efficiency rating which measures the maximum ℓ_1 or one-norm distance that a particular DMU lies from the reference vector on the (input) efficiency frontier. The same reference vector is also used for determining our efficiency measure, which explains the "slack-basedness" appearing in the title of this paper. Now, however, we switch from the one-norm orientation to the more popular two-norm, ℓ_2 or euclidean distance to arrive at a proportional measure. The input efficiency of an observation with input vector x and slack vector s^- is then computed using the following simple formula:

$$\frac{\|x_R\|}{\|x\|} = \frac{\sqrt{x_R^T x_R}}{\sqrt{x^T x}}, \text{ with } x_R = x - s^- \tag{2}$$

Like the Debreu-Farrell measure, (2) shows that the "degree" of inefficiency is computed by dividing two radial distances. However, the numerator is no longer found using equiproportionate input projections. Taken together it implies that the directions of the two radial vectors will in general not coincide. This deviation has a clear intuition, as we will now demonstrate.

In figure 1 an input vector $x_a \ (= (7, 4))$ is drawn. Suppose that the additive model has selected vector $x_b (= (2, 3))$ as its reference vector. The vectors x_a and x_b differ in two dimensions. First, there is the obvious fact that x_b Pareto-dominates x_a. Second, the input mixes of x_a and x_b differ. Indeed, the proportion $\frac{x_1}{x_2}$ for x_a is $\frac{7}{4}$ whereas it only amounts to $\frac{2}{3}$ for x_b. The latter results from different implicit cost price vectors, respectively $v_a \ (= (1, 7/4))$ and $v_b (= (1, 2/3))^3$.

Applying (2) yields an efficiency score of 44.72 % for x_a. This score can be decomposed in two factors. To see this we first construct the lines perpendicular to the radials through

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[3]The first input is taken as the numéraire.

Figure 1: An illustrative example
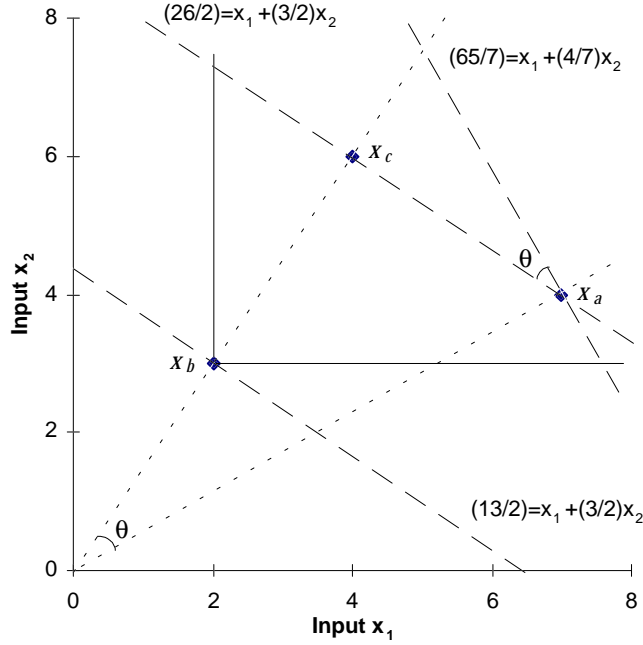
x_a and x_b. These lines take the form $\beta_i = w_i^T x_i$ $(i = a, b)$, with w_a and w_b equalling $(1, 4/7)$ and $(1, 3/2)$, respectively. Both perpendiculars are depicted in figure 1, together with a line through x_a which is parallel to the perpendicular through x_b. This allows us to identify x_c, which is the closest projection of x_a on the radial through x_b.

In fact x_c corrects for the deviation between v_a and v_b. In order to get from x_a to x_b one first needs to adjust the input-mix $(x_a \rightarrow x_c)$, so as to apply an equiproportionate reduction of all inputs $(x_c \rightarrow x_b)$ afterwards. The inefficiency of x_a thus results from (i) a deviation from the reference cost price vector and (ii) a cost level which is too high, even after adjustment for the implicit cost prices.

An appropriate measure for (i) is the cosine of the angle between the vectors x_b (or x_c) and x_a. The ratio of the cost level of x_b over the cost level of x_c constitutes a measure for (ii). In fact, the latter is a measure for the ratio of minimal to actual costs after correction for the implicit price vector. This clearly refers to the cost interpretation of the radial efficiency measures stressed by Russell (1985). We can now decompose the input efficiency measure (2) as follows:

Proportional, Slack Based Input Efficiency Score for x_a

$$= \frac{\text{cost level b}}{\text{cost level c}} \times \cos\theta, \text{ with } \theta \text{ the angle between } x_a \text{ and } x_b \text{ (or } x_c)$$

$$= \frac{w_b^T x_b}{w_b^T x_a} \times \frac{x_a^T x_b}{\|x_a\| \times \|x_b\|}$$

$$= 0.5 \times 0.8944$$

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$$= \quad 44.72 \ \%$$

The above example illustrates how the input efficiency measure (2) can be decomposed into two components. The first component captures the radial inefficiency due to a higher cost level after adjustment for the implicit vector (*Debreu-Farrell component (DF)*). The second component captures the inefficiency due to the deviation of the actual (implicit) cost price vector from the reference cost price vector (*Implicit Price Correspondence component (IPC)*). Its value will equal one if both cost price vectors coincide, and it will be lower as the price vectors deviate. Measure (2) thus expresses the idea that in a multiple input setting there are two sources of inefficiency. Indeed, efficiency does no longer only depend on a ratio between minimal to actual inputs but also on the proportion between the several inputs. Stated otherwise, as compared to the Debreu-Farrell measure the input-outputmixes of observations that are found to be inefficient (or equivalently, their implicit price vectors) are no longer sacrosanct when looking for a best practice reference DMU. Instead, the picture is reversed and inefficient DMUs can learn from the input-output mix of their identified reference DMUs.

In the general case the additive model will identify vector x_R as the reference input vector for x. Denoting the vectors which define the perpendicular on the radials through x and x_R as respectively w_R and w, we get:

$$\textit{Input Efficiency Measure} = \frac{\|x_R\|}{\|x\|} = \frac{w_R^T x_R}{w_R^T x} \times \frac{x^T x_R}{\|x\| \times \|x_R\|} \tag{3}$$

An analogous formulation holds for the slack based output efficiency measure[4]:

$$\textit{Output Efficiency Measure} = \frac{u_R^T y}{u_R^T y_R} \times \frac{y^T y_R}{\|y\| \times \|y_R\|} \tag{4}$$

where y and y_R denote the actual and reference output vector, respectively. The vectors u and u_R define the perpendiculars on the radials. Note that these perpendiculars coincide with the implicit isorevenue lines, so that u and u_R are in fact implicit revenue price vectors. The interpretation of the output measure follows immediate. It is notably the output measure (4) which will be used in the next section.

3 Application

To illustrate the measure and its decomposition we will apply it on a problem taken from Nash and Sterna-Karwat (1996). These authors analysed the efficiency of financial product cross-selling for a sample of 75 bank branches. Cross-selling efficiency refers to the amount of related products that is sold with the sale of a major product. In the study housing loans were identified as being the major product. Four associated products

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[4]Note that one cannot take $\frac{\|y\|}{\|y_R\|}$ as a measure for output efficiency as this ratio can be decomposed in the following way: $\frac{\|y\|}{\|y_R\|} = \frac{u_R^T y}{u_R^T y_R} \times \left[1 / \left(\frac{y^T y_R}{\|y\| \times \|y_R\|} \right) \right]$. Deviations from the reference output proportions would thus be rewarded, which is clearly undesirable.

were selected. The ratio of each of these products over the amount of housing loans sold provides a partial measure of cross-selling efficiency. The Nash and Sterna-Karwat data are (also) reported in the appendix.

The sample has two particular features. First there are zeros in the data set as some branches did not cross-sell certain products during the period considered. This is a first reason why the popular radial models cannot be employed. But there is also a second reason why they do not seem appropriate. A number of branches perform relatively well on some products while there is only a poor performance with respect to other products. Obviously, in these cases it does seem hard to defend that a measure capturing the maximum equiproportionate expansion in all outputs will adequately reflect the overall relative cross-selling efficiency.

In a first exercise the original additive model was employed. From the additive results the C^2GS^2 scores were computed. It appeared that the many zeros in the data distorted the analysis of the branch cross-selling power considerably. Some branches entirely specialized in selling large amounts of two or three products. This feature apparently disadvantaged branches that provided each of the four products, as they were largely outperformed by linear combinations of specialized DMU's. No branch selling all four of the products was identified as being relatively efficient. Therefore, Nash and Sterna-Karwat introduced a modified additive model, designed so as to mitigate the impact of the many zero entries. A new 'output' was introduced in the form of a count of how many different types of products were sold by a branch over the period considered. Its value could range from 1 to 4. If the corresponding 'counts'-vector is denoted by c (with c_j as its jth element), the eventual model, which we will refer to as the "trade-off"-DEA model in the following, is model 1 (or, equivalently, model 3) with all reference to input being omitted and the following constraint included:

$$c^T \lambda \geq c_o$$

As noted by Nash and Sterna-Karwat, the idea here is that there exists a trade-off between offering more products and achieving high loan-product ratios for each of the products provided[5]. The above restriction is important in so far that it restricts the set of potential reference points. The associated slack value is not captured by the objective function, however.

We also applied a third model to compute cross-selling efficiency scores, viz. the additive FDH model [see Bardhan et al. (1996)]. The applied model equals model 1 (or model 3) without references to input and adds to this the integrality constraint $\lambda_j \in \{0, 1\}$ ($\forall j$). Every inefficient observation is now compared to only one dominating observed output combination. This entails that an observation which provides, e.g., four products will be evaluated against an actually observed reference point providing all of the four products. The reference vector can no longer be a convex combination of two or more specialized observations. Clearly the FDH additive model is more generous than the two

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[5]To some extent, this view can be reconciled with the argument of Byrnes, Färe and Grosskopf [1984] that zeros in data represent fundamental characterisations of the observed decision making process. There it is argued that a manager who does not use a particular input or produce a particular output has made a conscious decision choice to do so and that it is this decision choice which should be analysed.

alternatives discussed above. All observations which are not strictly dominated in each of the four dimensions by an observed output vector will be labelled efficient.

C^2GS^2 output scores were computed for all three reference frontiers. Note that in this particular application formula (1) reduces to:

$$C^2GS^2 = \frac{e^T D_{y_A}^{-1} \mathbf{Y} \lambda^*}{e^T D_{y_A}^{-1} \mathbf{Y} \lambda^* + e^T D_{y_A}^{-1} s^{+*}},$$

so that the 'sum of slacks' can at most equal the 'sum of targets'. Consequently the lower bound for the C^2GS^2 measure equals one half, which seems odd in comparison with other efficiency measures. While rescaling the scores is of course always a possibility, there is still no clear intuition for the cardinal C^2GS^2 scores. All that can be said a priori is that they equal one for efficient DMUs and that they will be lower the further an observation lies from its reference point. The C^2GS^2 output scores were compared with the efficiency scores obtained by the output oriented measure (4), and the two parts of which it is composed.

Table 1 provides some summary statistics about the C^2GS^2 and the price-adjusted proportional, slack based efficiency results (and their DF and IPC components) for each of the three models discussed above (respectively referred to as DEA, trade-off DEA and FDH). We find that the Spearman rank correlations are much higher for the FDH application than for the DEA and trade-off DEA applications. This contrasts with the overall score correlations, which are more or less the same. This result mainly follows from the fact that the DEA and trade-off DEA scores are mostly very low and for a great deal packed together, so that small shifts in scores may give rise to substantial shifts in rankings. The FDH scores, on the contrary, are more evenly distributed over the whole [0,1]-range resulting in less sensitive rankings.

Still other observations can be made from table 1, such as the relatively high average and minimum values associated with the FDH model, which reflect the generosity of the methodology. There is also the relatively low correlation between the C^2GS^2 and IPC results. This finding has straightforward intuition as the C^2GS^2 measure only captures the (relative) distance from the target point and does not consider deviations from the target's implicit price vector.

Unlike the C^2GS^2 measure, the measure (4) also captures the deviation from the reference implicit price vector. Figure 2, which displays the histograms for the IPC components in the DEA, trade-off DEA and FDH applications, suggests that these price deviations can differ considerably across observations. It is seen that the most stringent model, the original DEA additive model, implies generally low IPC scores. Application of the most generous model, i.e. the additive FDH model, on the other hand, leads to higher IPC scores. But even then price deviations are distributed over a substantially large range. In general, such results may act as a signal that, as far as the examined dataset is concerned, there is a wide dispersion between Debreu-Farrell efficiency measurement and results obtained by referring to the Pareto-Koopmans efficiency notion.

Table 1: Summary statistics

	DEA	trade-off DEA	FDH
Scores correlation (Pearson):			
C^2GS^2-price-adjusted	0.9890	0.9859	0.9808
C^2GS^2-DF	0.9913	0.9861	0.9812
C^2GS^2-IPC	0.4965	0.6437	0.6179
Rankings correlation (Spearman):			
C^2GS^2-price-adjusted	0.2701	0.7590	0.9557
C^2GS^2-DF	0.5377	0.8336	0.9714
C^2GS^2-IPC	-0.0137	0.6050	0.8761
Standard deviation of scores:			
C^2GS^2	0.1092	0.1479	0.1971
price-adjusted	0.2258	0.3216	0.4064
DF	0.2263	0.3167	0.3957
IPC	0.2980	0.3180	0.2090
Average score value:			
C^2GS^2	0.5522	0.6051	0.7108
price-adjusted	0.0692	0.1779	0.4449
DF	0.0817	0.2019	0.4617
IPC	0.3799	0.5251	0.8380
Minimum score value:			
C^2GS^2	0.5063	0.5202	0.5216
price-adjusted	0.0002	0.0009	0.0151
DF	0.0026	0.0090	0.0378
IPC	0.0605	0.0985	0.2947

Analysing the price deviations somewhat further, we find for the DEA model and (to a somewhat lesser extent) for the trade-off version that especially the implicit price deviations for the fourth product are substantial (see figure 3, where output 2 is taken as the numéraire). This mainly follows from the fact that one observation (branch 3), which provided only two products, is heavily specialised in the fourth product. When searching for reference output combinations for inefficient DMUs, the weight attributed to this observation is generally high, which causes considerable differences between actual and reference implicit prices for the fourth product. On the other hand, the FDH model does not allow to compare with linear combinations of efficient DMUs. Only actually observed combinations can be used as reference points. The highly specialised DMU thus only constitutes a possible reference for similarly specialized branches. Apparently this applies for only two observations (44 and 61).

Let us, finally, take a closer look at some individual results. The appendix gives the FDH results for the whole set of DMUs. Consider branch 18 which is compared to observation 11. This observation has a moderate DF score of 69 percent. Its IPC score, on the contrary, amounts to 98 percent. Indeed, the output proportions of observation 18 are close to the reference output proportions. This is hardly the case for observation 57, which is dominated by observation 4. The low IPC score can be explained by the

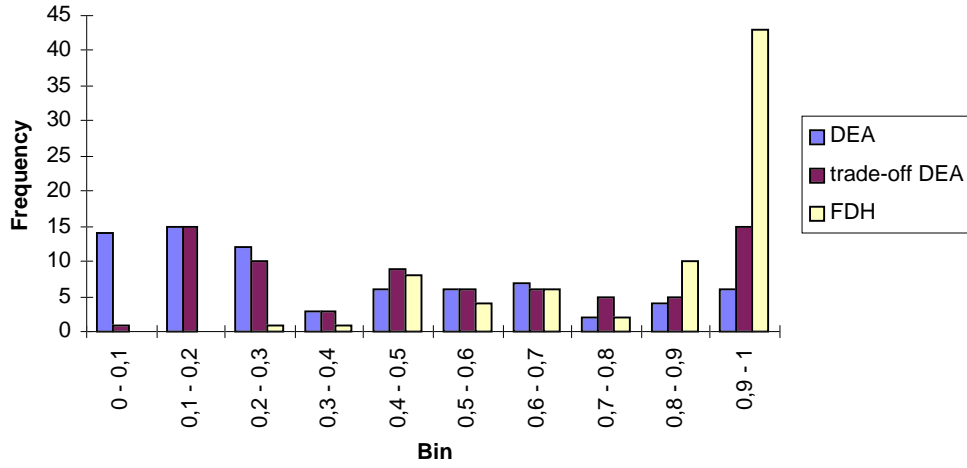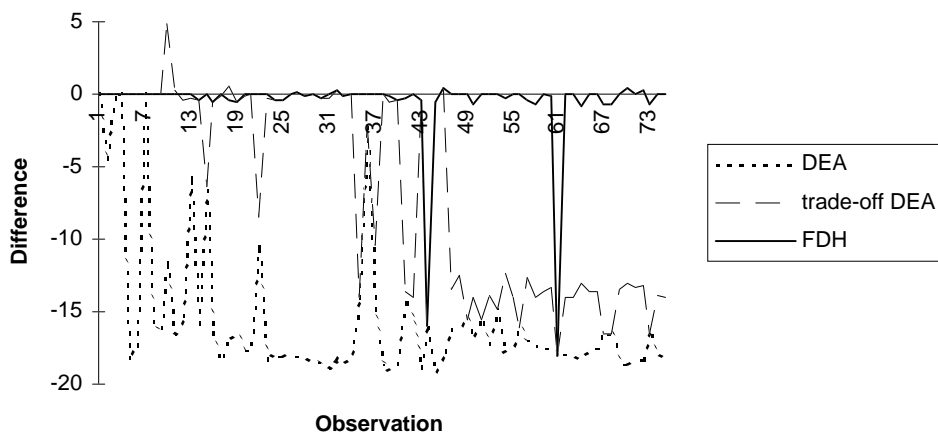Figure 2: Histograms implicit price correspondence (IPC) components

Figure 3: Differences between own and reference implicit prices (product 4)

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fact that branch 4, while attaining only moderately higher values for the product ratios 1, 2 and 4, achieves a very high cross-selling ratio for product 3, which is not provided by DMU 57. The 'mix efficiency' of the latter is therefore very low. Observation 48, which has 1 as reference branch, does not deviate much from its best practice peer w.r.t. the relative product prices that underly the observed mix. The Debreu-Farrell distance from the reference point (after correction for the implicit prices) is considerable, however. In this respect, branch 15, which shares the same reference observation, fares much better. Observation 48 could also be compared to branch 14. Both have bad scores on the "level-test", but branch 48 performs better w.r.t. mimicking the output-composition of its best practice reference. Such examples indicate that both components of the measure may provide the researcher with interesting information about (i) whether radial measurement would yield results that deviate to a considerable extent from Pareto-Koopmans efficiency measurement, and (ii) the inefficiency of a DMU in relation to its reference observation, both in terms of input/output composition as well as in level terms.

4   Comments and conclusion

While radial efficiency measures possess an attractive interpretation, they do not always conform to the Koopmans definition of technical efficiency. Moreover, in many cases, such as the classical CCR and BCC model formulations, strictly positive data entries are required throughout the complete data domain. Both of these problems restrict their field of applicability.

A model which does not face the above difficulties is the additive model. The associated score lacks an intuitive interpretation, however. Therefore we introduced another efficiency measure which is constructed from the additive results. In this paper we proposed to solve an (input- or output-) oriented additive model in a first step. Note that this is not strictly necessary -one could equally well compute input and output scores directly from the original additive results- but it allows to identify different reference sets according to the orientation.

The measure discussed in this paper, which has both an input and an output version, can be decomposed into a *Debreu-Farrell* and an *implicit price correspondence* component. As we tried to made clear using the Nash and Sterna-Karwat (1996) application, *both* components offer information that can be interesting when evaluating productive activities.

While we have opted to introduce this measure using an oriented additive model, we note here that in general such a decomposable measure need not necessarily be slack based. In fact it can be computed as soon as one disposes of a reference point for the DMU under study. One could, for instance, first identify the reference point used to calculate the Färe-Lovell (1978) or Zieschang (1984) efficiency score and compute a measure similar to (4) or (5) in a second step. Especially when the reference price vector differs from the actual implicit price vector (i.e. when the reference DMU does not coincide with an equiproportionate projection) the measure presented here captures useful information precisely on the deviation between these implicit relative prices (and their associated

input-outputmixes).

Of course, DEA is an efficiency measurement *tool* rather than a panacea. Options concerning the choice of measure(s) or assumptions on the reference technology are finally in the hands of the evaluator, who should therefore have an eye for the particular problem and data at hand. This is all the more true if one recognizes, as in our example, the substantial impact of such options on the eventual efficiency scores and, consequently, on the possible management conclusions.

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Appendix: Data and FDH results

	Product 1		Product 2		Product 3		Product 4		Score		
Branch	Actual	Reference	Actual	Reference	Actual	Reference	Actual	Reference	DF	PD	slack based
1	1176	1176	158	158	183	183	0	0	1,0000	1,0000	1,0000
2	255	255	297	297	0	0	125	125	1,0000	1,0000	1,0000
3	0	0	198	198	0	0	3900	3900	1,0000	1,0000	1,0000
4	215	215	85	85	767	767	62	62	1,0000	1,0000	1,0000
5	77	77	168	168	31	31	38	38	1,0000	1,0000	1,0000
6	158	158	126	126	12	12	19	19	1,0000	1,0000	1,0000
7	0	0	465	465	79	79	0	0	1,0000	1,0000	1,0000
8	120	120	82	82	49	49	156	156	1,0000	1,0000	1,0000
9	85	85	86	86	61	61	146	146	1,0000	1,0000	1,0000
10	80	80	22	22	67	67	146	146	1,0000	1,0000	1,0000
11	57	57	73	73	68	68	139	139	1,0000	1,0000	1,0000
12	94	94	87	87	259	259	20	20	1,0000	1,0000	1,0000
13	0	0	57	57	24	24	780	780	1,0000	1,0000	1,0000
14	215	215	85	85	87	767	27	62	0,1887	0,6103	0,1152
15	851	1176	112	158	0	183	0	0	0,7066	0,9883	0,6983
16	125	215	66	85	152	767	16	62	0,2324	0,8971	0,2085
17	41	41	119	119	72	72	28	28	1,0000	1,0000	1,0000
18	53	57	53	73	65	68	80	139	0,6897	0,9759	0,6730
19	14	215	57	85	232	767	6	62	0,2884	0,9679	0,2791
20	100	100	92	92	41	41	42	42	1,0000	1,0000	1,0000
21	39	215	77	85	105	767	57	62	0,1533	0,8360	0,1282
22	501	501	66	66	203	203	0	0	1,0000	1,0000	1,0000
23	59	59	109	109	40	40	25	25	1,0000	1,0000	1,0000
24	55	215	58	85	90	767	21	62	0,1349	0,8870	0,1197
25	90	215	79	85	37	767	23	62	0,0865	0,5457	0,0472
26	78	215	61	85	21	767	43	62	0,0631	0,4609	0,0291
27	43	215	63	85	29	767	59	62	0,0627	0,5006	0,0314
28	35	215	59	85	61	767	37	62	0,0955	0,7749	0,0740
29	32	215	69	85	30	767	52	62	0,0604	0,5007	0,0302
30	33	215	73	85	47	767	35	62	0,0798	0,6460	0,0516
31	29	77	97	168	25	31	25	38	0,5540	0,9878	0,5473
32	17	215	53	85	27	767	54	62	0,0499	0,4883	0,0244
33	25	215	71	85	16	767	46	62	0,0411	0,3683	0,0151
34	41	215	54	85	17	767	40	62	0,0448	0,4470	0,0200
35	92	92	122	122	374	374	0	0	1,0000	1,0000	1,0000
36	0	0	408	465	0	79	0	0	0,8528	0,9859	0,8408
37	327	1176	150	158	21	183	0	0	0,2859	0,9525	0,2723
38	10	215	61	85	23	767	48	62	0,0433	0,4265	0,0185
39	35	77	101	168	14	31	7	38	0,5570	0,9858	0,5491
40	42	215	55	85	34	767	17	62	0,0633	0,6437	0,0407
41	0	215	72	85	359	767	28	62	0,4387	0,9599	0,4211
42	75	92	87	122	311	374	0	0	0,8199	0,9991	0,8192
43	29	215	63	85	23	767	21	62	0,0473	0,4998	0,0236
44	0	0	102	198	0	0	378	3900	0,0980	0,9774	0,0958
45	11	215	84	85	29	767	18	62	0,0509	0,4478	0,0228
46	28	215	25	85	19	767	27	62	0,0378	0,6073	0,0229
47	0	0	88	88	236	236	42	42	1,0000	1,0000	1,0000
48	215	1176	28	158	87	183	0	0	0,1895	0,9740	0,1846
49	0	0	148	148	304	304	0	0	1,0000	1,0000	1,0000
50	84	215	77	85	203	767	0	62	0,2793	0,9640	0,2692
51	0	0	132	148	304	304	0	0	0,9793	0,9991	0,9784
52	116	1176	137	158	117	183	0	0	0,1245	0,6976	0,0869
53	341	1176	87	158	0	183	0	0	0,2877	0,9816	0,2825
54	0	120	76	82	37	49	120	156	0,5592	0,8335	0,4661
55	67	1176	68	158	172	183	0	0	0,0840	0,5124	0,0430
56	0	92	110	122	270	374	0	0	0,7009	0,9712	0,6807
57	177	215	70	85	0	767	22	62	0,0703	0,2947	0,0207
58	46	215	54	85	186	767	0	62	0,2434	0,9825	0,2391
59	137	1176	108	158	55	183	0	0	0,1306	0,8572	0,1119
60	137	255	108	297	0	0	28	125	0,4176	0,9712	0,4055
61	0	0	132	198	0	0	208	3900	0,0549	0,8704	0,0478
62	52	1176	85	158	136	183	0	0	0,0690	0,4914	0,0339
63	79	1176	104	158	96	183	0	0	0,0880	0,6521	0,0574
64	0	85	85	86	51	61	79	146	0,5536	0,8698	0,4815
65	130	1176	106	158	35	183	0	0	0,1221	0,8557	0,1045
66	137	1176	54	158	55	183	0	0	0,1247	0,9522	0,1187
67	0	215	79	85	243	767	0	62	0,2991	0,9405	0,2813
68	0	215	70	85	246	767	0	62	0,3015	0,9471	0,2855
69	0	0	118	118	64	64	56	56	1,0000	1,0000	1,0000

Branch	Product 1 Actual	Reference	Product 2 Actual	Reference	Product 3 Actual	Reference	Product 4 Actual	Reference	Score DF	PD	slack based
70	15	255	100	297	0	0	89	125	0,2644	0,8066	0,2133
71	56	1176	148	158	76	183	0	0	0,0716	0,4894	0,0350
72	47	255	89	297	0	0	67	125	0,2771	0,9418	0,2610
73	0	215	40	85	243	767	0	62	0,2940	0,9591	0,2819
74	116	1176	76	158	47	183	0	0	0,1089	0,8932	0,0973
75	108	1176	118	158	33	183	0	0	0,1052	0,7736	0,0814

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