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Public Economics

Center for Economic Studies
Discussions Paper Series (DPS) 99.24
http://www.econ.kuleuven.be/ces/discussionpapers/default.htm

November 1999

DISCUSSION
PAPER

A Shadow Price Approach to Technical Efficiency Measurement

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November 1999

Abstract

The axiomatic literature on technical efficiency measurement has drawn attention to the indication problem of the Debreu-Farrell (DF) measure. We follow a shadow price approach to preserve the DF benchmark while reconciling it with the Koopmans efficiency characterization. First, we define a set of Koopmans efficient references that can be rationalized in a similar way as the DF projection. The indication problem is then captured using a measure of implicit allocative or mix efficiency, also interpretable as a dominance measure in price space. We consequently present a mix-adjusted DF framework for efficiency measurement in which e.g. the Zieschang [24] procedure can be fitted.

1 Introduction

As a rule one is ultimately concerned with the concept of allocative efficiency, but there are instances where focusing on technical efficiency is not without its interest. This is e.g. the case when market prices for inputs and outputs are not readily available. Or, as could be the case for non-profit or government organizations, when it is conceivable that such prices differ from the shadow prices implicitly considered by those in charge. A technical efficiency gauge has been introduced by Debreu [8] and Farrell [14] and has since then been applied quite extensively. This Debreu-Farrell (DF) measure, which is the inverse of Shephard's distance function, is obtained by equiproportionate shrinkage or expansion of input respectively output vectors onto their best practice frontiers. It is however no longer the unique technical efficiency measure available. Some other proposals emerged following the axiomatic approach to technical efficiency measurement as initiated by Färe and Lovell [12]. The most important reason why such alternative measures were introduced pertains to the problem that the DF measure does not always satisfy the rather fundamental condition known as 'indication of efficient vectors'. That is, for some types of production technologies the radial projection yielded by the DF method can in fact still be a technically inefficient point in the commonly accepted sense of Koopmans [17]. This is immediately obvious if one considers the example of a Leontief technology. In an input-oriented context, radial shrinkage to a point on the L-shaped isoquant does not necessarily imply that all 'waste' has been eliminated. One can usually proceed by a further non-radial reduction to arrive at a vertex point where no input can be further decreased without decreasing at least one output. If this (potential) deficiency would only be associated with the Leontief technology then one is facing a fairly minor problem indeed. But it also extends to more general and frequently employed cases where the reference technology is constructed nonparametrically, for example when this technology is obtained as the convex monotonic hull of observed input and output vectors. Since Afriat [1] this last representation is of particular importance if one follows a 'revealed preference' approach to producer behavior. It has been advocated by Varian [23] for its use in testing regularity conditions and for non-parametric efficiency measurement itself by Banker et al. [3]. Below we will mainly work in this setting, but it should be recalled that the indication problem of the DF technical efficiency measure, sometimes labeled its 'slack problem', may appear in any technology which allows for the presence of zero marginal rates of technical substitution (or transformation).

The DF technical efficiency measure remains very popular, even when applied in problem-prone technologies. There are at least four good reasons for this. First, on the empirical level, reference technologies are normally approximative in nature and therefore the indication problem may be considered as an accidental and somewhat artificial consequence. Second, on the theoretical level, the axiomatic approach as further developed by various authors such as e.g. Bol [2], Russell [20, 21] and Christensen et al. [6] has forwarded the insight that no universally best measure exists. Third, these axioms mostly refer to desirable mathematical characteristics of a (efficiency) gauge function, whereas some authors (e.g. Kopp [18] or Russell [19]) clearly took a favorable stance towards the DF measure given its economic interpretation for the class of convex monotonic technologies. And fourth, it has been claimed e.g. by Ferrier and Lovell [15] that slacks may be viewed essentially as resulting from allocative rather than technical inefficiency. In this paper we depart from the first of these arguments and thus consider the indication problem as a genuine problem indeed. We also put less emphasis on the purely axiomatic approach. This leaves us to draw on the last two arguments when aiming to reconcile the economically attractive DF benchmark with the Koopmans characterization of technical efficiency.

To do so we first provide in section 2 a quick refresher to the potential conflict between the two, phrasing the indication problem in terms of the shadow prices implicitly present in the production technology. In these particular terms —which were notably the ones used by both Debreu [8] and Koopmans [17]— it is easily seen that the DF method may yield best practice reference vectors that implicitly allow for zero shadow prices. On the other hand, consistent with the two fundamental theorems of welfare economics in the case of a convex monotonic hull, one needs a vector of strictly positive shadow prices to eliminate any further (non-radial) waste and be technically efficient in the Koopmans sense.[1] Once this stage has been set we hold on to the shadow price approach in the remainder of the paper. In section 3 we use it to identify a set of reference vectors that are Koopmans efficient and at the same time retain the economic interpretation behind the DF measure. We also show that the method proposed by Zieschang [24] selects reference vectors that belong to this set. This method has the characteristic that one first radially projects on a reference isoquant, and then corrects for residual waste along the isoquant to obtain an adequate reference. Note that the compensated price response to which this second stage amounts falls in line with the above claim that slacks result from (implicit) allocative inefficiency rather than purely technical inefficiency.

This leads us in section 4 to explicitly address the question regarding the measurement of implicit allocative efficiency, or 'mix efficiency' for short. Several equivalent representations for gauging this kind of efficiency are forwarded: directly in terms of product mixes when drawing on a directional cosine representation of vectors, as a relative Euclidean distance measure in commodity space (i.e. in the same way the original DF measure is usually represented), or as a relative distance measure between isocost hyperplanes (which directly reveals the dominating character of 'mix efficient' references). A short discussion pertaining to the commensurability of the proposed mix efficiency measure forms the

[1] On the most general level, the second welfare theorem as defined with respect to convex technologies does allow for the presence of some zero prices. This special, somewhat pathological case occurs when the marginal rate of technical substitution (transformation) smoothly goes to zero at certain efficient points. While our discussion below focuses on technologies with non-smooth transitions for sake of clarity, it can easily be extended to cover such a case.

subject of section 5.

As regards the question of technical efficiency measurement, adhering to the shadow price perspective thus implies that the indication problem of the otherwise intuitive DF measure can be overcome by complementing it with the mix efficiency measure. Or stated otherwise, that at least for some non-radial efficiency gauges the earlier critique pertaining to their vague economic intuition seems less valid. In section 6 we show that the traditional DF measure can be complemented with the kind of mix efficiency component identified in section 4 to yield a mix-adjusted DF measure. Our main findings as well as some additional comments are summarized in section 7.

Throughout the paper we focus on input efficiency but an analogous treatment applies to the output oriented case. Also, while only the Afriat technology is explicitly considered, the insights straightforwardly carry over to other convex piecewise linear formulations surveyed in Färe, Grosskopf and Lovell [11]. Furthermore, the obtained results can be readily adapted to a parametric setting. The following vector inequality conventions will be used: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, (i) $\mathbf{x} > \mathbf{y}$ if and only if $x_i > y_i$, $i = 1, ..., n$, (ii) $\mathbf{x} \geqq \mathbf{y}$ if and only if $x_i \geqq y_i$, and (iii) $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geqq y_i$ and $\mathbf{x} \neq \mathbf{y}$. We use (x_i, x_j) to represent the (column) vector with (row) elements x_i and x_j.

2 Characterizing technical efficiency

The starting point is the production technology T transforming an input vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$ into an output vector $\mathbf{u} = (u_1, ..., u_m) \in \mathbb{R}^m_+$ with at least one element of each $\mathbf{x}$ and $\mathbf{u}$ strictly positive:

$$T \equiv \left\{ (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{n+m}_+ \mid \mathbf{x} \text{ can produce } \mathbf{u} \right\}.$$

Suppose a set of N input-output combinatiuons belonging to T is observed. To simplify exposition we will further concentrate on the Afriat [1] technology representation T^A, which is the smallest convex set that includes all observations and satisfies the monotonicity property:

$$T^A \equiv \left\{ (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{n+m}_+ \mid \mathbf{x} \geqq \sum_{j=1}^{N} \lambda_j \mathbf{x}_j, \mathbf{u} \leqq \sum_{j=1}^{N} \lambda_j \mathbf{u}_j, \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \geqq 0 \ \forall \lambda_j \right\}.$$

Because of its convex nature and piecewise (linear) construction T^A is particularly well suited for illustrating the difference between the DF and Koopmans efficiency characterizations. Moreover it is well-grounded both in the literature on testing for regularity conditions of production (see [23] and [4]) and the axiomatic non-parametric approach to production technology estimation (see [3]).

Suppose the observed input-output vector $(\mathbf{x}_o, \mathbf{u}_o) \in T$ is to be evaluated and factor prices are unknown. Technical input efficiency is estimated by comparing $\mathbf{x}_o$ to the input correspondence:

$$L^A(\mathbf{u}_o) \equiv \left\{ \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} \geqq \sum_{j=1}^N \lambda_j \mathbf{x}_j, \mathbf{u}_o \leqq \sum_{j=1}^N \lambda_j \mathbf{u}_j, \sum_{j=1}^N \lambda_j = 1, \lambda_j \geqq 0 \ \forall \lambda_j \right\}.$$

Two subsets of $L^A(\mathbf{u}_o)$ that are particularly relevant in the following discussion are its isoquant and efficient subset, respectively:

$$Isoq\ L^A(\mathbf{u}_o) \equiv \left\{ \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} \in L^A(\mathbf{u}_o) \text{ and } \lambda x \notin L^A(\mathbf{u}_o) \text{ for } \lambda \in [0,1) \right\}, \tag{2}$$

$$Eff\ L^A(\mathbf{u}_o) \equiv \left\{ \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} \in L^A(\mathbf{u}_o) \text{ and } \mathbf{x}' \leq \mathbf{x} \text{ then } \mathbf{x}' \notin L^A(\mathbf{u}_o) \right\}. \tag{3}$$

The DF measure of technical efficiency for an observation $(\mathbf{x}_o, \mathbf{u}_o)$, denoted as $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$, is defined as follows for $L^A(\mathbf{u}_o)$:

$$E_{DF}(\mathbf{x}_o, \mathbf{u}_o) \equiv \min \left\{ \lambda \in \mathbb{R}_+ \mid \lambda \mathbf{x}_o \in L^A(\mathbf{u}_o) \right\}.$$

In words, $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ gives the maximum equiproportionate reduction of all inputs in $\mathbf{x}_o$ that still allows to produce $\mathbf{u}_o$. From (2), $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ attains the maximum value of one if and only if $\mathbf{x}_o \in Isoq\ L^A(\mathbf{u}_o)$. The indication problem relates to the fact that $E_{DF}(\mathbf{x}_o, \mathbf{u}_o) = 1$ is not sufficient for $\mathbf{x}_o \in Eff\ L^A(\mathbf{u}_o)$, where (3) is consistent with the commonly accepted Koopmans [17] characterization. The latter declares an input combination efficient if and only if, for a given output level, it is impossible to reduce any input without simultaneously increasing another. Indeed, while $Eff\ L^A(\mathbf{u}_o) \subseteq Isoq\ L^A(\mathbf{u}_o)$ both subsets do generally not coincide and it follows that observations identified as DF efficient are not necessarily efficient in the Koopmans sense. Further, following the DF procedure inefficient input-output combinations may be evaluated with respect to a reference that is Koopmans dominated by another feasible point.

The potential conflict between the two concepts is illustrated for a two-input situation in figure 1 where the input efficiency of a vector $(\mathbf{x}_i, \mathbf{u}_i)$ is to be measured. In the diagram $Eff\ L^A(\mathbf{u}_i)$ corresponds to the facet $\overline{AB}$ whereas $Isoq\ L^A(\mathbf{u}_i)$ also contains the vertical and horizontal facets $\overline{AC}$ and $\overline{BD}$. Maximal radial contraction of the two inputs results in projecting $\mathbf{x}_i$ on $\mathbf{x}_e$ and correspondingly $E_{DF}(\mathbf{x}_i, \mathbf{u}_i) = \|\mathbf{x}_e\| / \|\mathbf{x}_i\|$. Clearly $\mathbf{x}_e$ is not Koopmans efficient as it exhibits a slack in the second input.

Both efficiency characterizations also have a specific shadow price representation. Let us first define the correspondence capturing all shadow price vectors under which $\mathbf{x} \in Isoq\ L^A(\mathbf{u}_o)$ is cost minimizing over $L^A(\mathbf{u}_o)$:

$$\forall \mathbf{x} \in Isoq\ L^A(\mathbf{u}_o) : P(\mathbf{x}, \mathbf{u}_o) \equiv \left\{ \mathbf{p} \in \mathbb{R}^n_+ \mid 0 < \mathbf{p} \cdot \mathbf{x} \leqq \mathbf{p} \cdot \mathbf{x}' \ \forall \mathbf{x}' \in L^A(\mathbf{u}_o) \right\}.$$

$P(\mathbf{x}, \mathbf{u}_o)$ defines the set of the supporting hyperplane(s) tangent to $L^A(\mathbf{u}_o)$ in $\mathbf{x}$.[2]

[2] There is an immediate link between our definition of the correspondence $P(\mathbf{x}, \mathbf{u}_o)$ and the popular, single-valued cost function concept. For our purposes the use of $P(\mathbf{x}, \mathbf{u}_o)$ is more adequate given that a similar approach was followed by Debreu [8].

Two further definitions are used for notational convenience:

$$\begin{aligned}\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) &\equiv E_{DF}(\mathbf{x}_o, \mathbf{u}_o)\,\mathbf{x}_o, \\ P_{DF}(\mathbf{x}_o, \mathbf{u}_o) &\equiv P(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), \mathbf{u}_o).\end{aligned}$$

Or, $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ is the reference yielded by the radial DF projection and $P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ its associated set of implicitly cost minimizing price vectors.

We can now state the equivalent formulation of $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ which Debreu [8] derived in order to justify the collinear projection:[3]

Proposition 1 *(Debreu [8]) For all* $\mathbf{x}_o \in L^A(\mathbf{u}_o), \mathbf{x}' \in Isoq\ L^A(u_o), \mathbf{p}_{DF} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ *and* $\mathbf{p}' \in P(\mathbf{x}', \mathbf{u}_o)$:

$$E_{DF}(\mathbf{x}_o, \mathbf{u}_o) = \frac{\mathbf{p}_{DF} \cdot \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)}{\mathbf{p}_{DF} \cdot \mathbf{x}_o} \geqq \frac{\mathbf{p}' \cdot \mathbf{x}'}{\mathbf{p}' \cdot \mathbf{x}_o}.$$

The DF procedure thus applies an implicit 'benefit-of-the-doubt' weighting when looking for a best practice reference. That is, it yields a maximal ratio of reference to actual costs with prices chosen so as to make $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ cost minimizing. As such, the DF measure possesses an evocative interpretation as an upper bound to economic efficiency (see also Russell [19]). This benefit-of-the-doubt characterization is also clear from figure 1. Indeed, $\|\mathbf{x}_e\| / \|\mathbf{x}_i\|$ for example exceeds $\left\|\mathbf{x}_e'\right\| / \|\mathbf{x}_i\|$ and –to an even greater extent– $\left\|\mathbf{x}_e''\right\| / \|\mathbf{x}_i\|$, where both $\mathbf{x}_e'$ and $\mathbf{x}_e''$ respectively lie on the supporting ('isocost') hyperplanes of the isoquant facets $\overline{AB}$ and $\overline{BD}$.

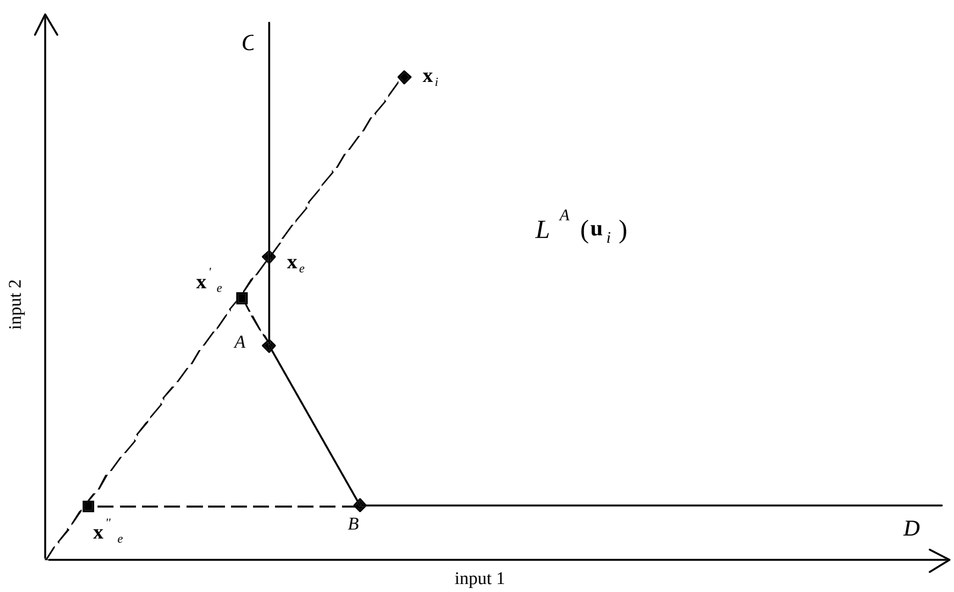

FIGURE 1

[3] Shephard [22] obtained the same result from the duality between his input distance function and the cost function. For a recent discussion of the shadow price interpretation of the Shephard distance functions (and consequently the DF measures) we refer to [10].

It should be noted that $\mathbf{p}_{DF}$ in proposition 1 may contain some zero entries, whereas Koopmans [17] showed:

Proposition 2 *(Koopmans [17]) For* $\mathbf{x} \in L^A(\mathbf{u}_o)$:

$$\mathbf{x} \in Eff\ L^A(\mathbf{u}_o) \Leftrightarrow \exists \mathbf{p} \in P(\mathbf{x}, \mathbf{u}_o) : \mathbf{p} \in \mathbb{R}^n_{++}.$$

So, for each element of the efficient subset one can construct a strictly positive price vector under which it becomes cost minimizing. In the illustration of figure 1 all elements on the segment $\overline{AB}$ clearly meet this Koopmans condition. However, this is no longer so for $\mathbf{x}_e$, for which a zero shadow price is accorded to the second input in the corresponding cost ratio of proposition 1.

This implicit price characterization provides an economic intuition for the indication problem associated with DF measures. In the next section we tackle the question which references combine the attractive interpretation of the DF projection with the intuitive notion of Koopmans efficiency.

3 Benefit-of-the-doubt pricing and Koopmans efficiency

Proposition 1 provides a shadow cost efficiency characterization for $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ which makes it interesting from an economic point of view. However, it may still be that $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) \notin Eff\ L^A(\mathbf{u}_o)$. We now show that references which preserve both a benefit-of-the-doubt interpretation and pass the test for Koopmans efficiency belong to:

$$X_D(\mathbf{x}_o, \mathbf{u}_o) \equiv \left\{\mathbf{x} \mid \mathbf{x} \in Eff\ L^A(\mathbf{u}_o) \text{ and } \mathbf{x} \leqq \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)\right\}.$$

This set contains all $\mathbf{x} \in Eff\ L^A(\mathbf{u}_o)$ that weakly dominate $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. In the illustration of figure 1 we have $X_D(\mathbf{x}_i, \mathbf{u}_i) = \{A\}$.

With respect to $X_D(\mathbf{x}_o, \mathbf{u}_o)$ we can state a result very similar to that formulated in proposition 1:[4]

Proposition 3 *For all* $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o), \mathbf{x}' \in Eff\ L^A(\mathbf{u}_o) \backslash X_D(\mathbf{x}_o, \mathbf{u}_o), \mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u}_o)$ *and* $\mathbf{p}' \in P(\mathbf{x}', \mathbf{u}_o)$:

$$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}_o} \geqq \frac{\mathbf{p}' \cdot \mathbf{x}'}{\mathbf{p}' \cdot \mathbf{x}_o}.$$

Thus, for each $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ the associated shadow cost ratio will never be lower than for any $\mathbf{x}' \in Eff\ L^A(\mathbf{u}_o) \setminus X_D$. The interpretation is analogous to the one of proposition 1. The important difference is that proposition 3 is defined with reference to $Eff\ L^A(\mathbf{u}_o)$, whereas proposition 1 concerns $Isoq\ L^A(\mathbf{u}_o)$. Hence, recalling proposition 2, the result just stated is appealing since it allows to use strictly positive shadow price vectors in comparisons.

[4] Proofs of our results are given in the appendix.

To make the result somewhat more intuitive, take again figure 1. Shadow cost ratios of $\mathbf{x}_i$ obtained using any $\mathbf{p}_A \in P(A,\mathbf{u}_i)$ are not below $\left\|\mathbf{x}_e^{'}\right\| / \left\|\mathbf{x}_i\right\|$. In geometric terms, one can always find a supporting hyperplane through A that crosses the radial somewhere between $\mathbf{x}_e^{'}$ and $\mathbf{x}_e$. Obviously, no other element belonging to the facet $\overline{AB}$ would yield a strictly higher ratio value.

While proposition 3 indicates that an appropriate reference (from a shadow price perspective) can be found in $X_D(\mathbf{x}_o, \mathbf{u}_o)$, it does not immediately identify a unique reference. Of course, if $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) \in Eff\ L^A(\mathbf{u}_o)$ then $X_D(\mathbf{x}_o, \mathbf{u}_o) = \{\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)\}$. Also, even when $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) \notin Eff\ L^A(\mathbf{u}_o)$ the set $X_D(\mathbf{x}_o, \mathbf{u}_o)$ will always be a singleton if $n = 2$ (see our illustration). In general it could be that X_D contains more elements. The following proposition demonstrates that each input vector $x_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ is justifiable from a shadow cost efficiency perspective:

Proposition 4 *For all $\mathbf{x}_D, \mathbf{x}_D^{'} \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ with $\mathbf{x}_D \neq \mathbf{x}_D^{'}$, given a strictly positive $\mathbf{p}_D^{'} \in P(\mathbf{x}_D^{'}, \mathbf{u}_o)$, there always exists a strictly positive $\mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u}_o)$ such that*:

$$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}_o} \geqq \frac{\mathbf{p}_D^{'} \cdot \mathbf{x}_D^{'}}{\mathbf{p}_D^{'} \cdot \mathbf{x}_o}.$$

As proposition 4 applies to all $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ there is no a priori reason to further discriminate in terms of implicit cost efficiency.

In the literature there have been some proposals that in fact yield references belonging to $X_D(\mathbf{x}_o, \mathbf{u}_o)$. Conceptually, they can be interpreted as multi-stage procedures where the DF method is applied in the first stage. If there remains an indication problem with the obtained $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ it is subsequently overcome by moving downward along the isoquant until $Eff\ L^A(\mathbf{u}_o)$ is reached. An example is the procedure of Zieschang [24], where the second stage applies the non-radial projection proposed by Färe and Lovell [12] to $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. That is, $\mathbf{x}_Z(\mathbf{x}_o, \mathbf{u}_o) \equiv \boldsymbol{\lambda}^Z(\mathbf{x}_o, \mathbf{u}_o) \cdot \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ with $\boldsymbol{\lambda}_Z(\mathbf{x}_o, \mathbf{u}_o) = \left(\lambda_1^Z, ..., \lambda_n^Z\right) \in R_+^n$ solving:

$$\min_{\lambda_1,...,\lambda_n} \left\{ \sum_{l=1}^{n} \lambda_l \mid (\lambda_1 x_{DF1}, ..., \lambda_n x_{DFn}) \in L^A(\mathbf{u}_o), \lambda_i \in [0,1] \text{ for } i = 1, ..., n \right\},$$

where $x_{DF1}, ..., x_{DFn}$ are the row elements of $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. The sum of unidimensional shrinkage factors (contained in $\boldsymbol{\lambda}_Z$) is minimized. Of course, when $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) \in Eff\ L^A(\mathbf{u}_o)$ each of these factors equals unity (i.e. $\boldsymbol{\lambda}_Z = (1, ..., 1)$ and the second stage is redundant). It is easily checked that $\mathbf{x}_Z \in Eff\ L(\mathbf{u}_o)$ and $\mathbf{x}_Z(\mathbf{x}_o, \mathbf{u}_o) \leqq \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ by construction, so that $\mathbf{x}_Z(\mathbf{x}_o, \mathbf{u}_o) \in X_D(\mathbf{x}_o, \mathbf{u}_o)$.[5,6]

[5] Strictly speaking the possibility of zero input values is excluded. However, Färe, Lovell and Zieschang [13] proposed a way to circumvent this problem (see also Zieschang [24]).

[6] Two other examples of multi-stage procedures that select references in $X_D(\mathbf{x}_o, \mathbf{u}_o)$ are presented in [3] and [7].

So far we have shown that vectors $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ have a sound economic grounding as input references for $(\mathbf{x}_o, \mathbf{u}_o)$, and we also presented a procedure to obtain such an appropriate reference. However, we have not yet related this to an efficiency measure. The attractiveness of the DF gauge (see proposition 1) does not straightforwardly extend when using $X_D(\mathbf{x}_o, \mathbf{u}_o)$. In particular, for $\mathbf{x}_D \neq \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ typically $P(\mathbf{x}_D, \mathbf{u}_o)$ will be multi-valued resulting in different shadow cost ratios. This prevents direct construction of an efficiency measure with an associated traditional shadow price representation.

We therefore opt for another avenue in this paper, and explicitly split up inefficiency in the potential radial improvement as captured by the popular DF measure and the residual Koopmans inefficiency resulting from the possibility that $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) \notin Eff\ L^A(\mathbf{u}_o)$. In fact, building on the results hitherto obtained, we hold on to the DF procedure to project on $Isoq\ L^A(\mathbf{u}_o)$ and proceed by evaluating any presence of slacks in a second stage. The question is then how this residual inefficiency is to be treated. We take the perspective that zero shadow prices (or zero marginal rates of technical substitution) reflect implicit allocative inefficiency. This in turn refers to suboptimal proportions of the different amounts of inputs consumed. Putting it differently, the remaining Koopmans inefficiency essentially pertains to input mixes and could thus also be termed 'mix inefficiency'. In the next section we propose a way to evaluate this additional source of suboptimal behavior.

4 The measurement of mix efficiency

To enable the measurement of mix efficiency we need a characterization of input mixes. A possibility is to construct the vector of directional cosines, which is defined as follows for a vector $\mathbf{x} \in L^A(\mathbf{u}_o)$:

$$\mathbf{cos}_{\mathbf{x}} \equiv \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{4}$$

In $\mathbf{cos}_{\mathbf{x}} = (\cos_{x_1}, ..., \cos_{x_n}) \in \mathbb{R}^n_+$ each $\cos_{xl} \in [0,1]$ $(l = 1, ..., n)$ gives the cosine of the angle between the vector $\mathbf{x}$ and the lth input axis. To clarify the concept we return to our illustration as recaptured in figure 2. Using $A = (a_1, a_2)$ and $\mathbf{x}_e = (x_{e1}, x_{e2})$ the directional cosines corresponding to A and $\mathbf{x}_e$ are respectively $\cos_{a_l} = \cos\alpha_l = a_l / \|A\|$ and $\cos_{x_{el}} = \cos\beta_l = x_{el} / \|\mathbf{x}_e\|$ $(l = 1, 2)$.

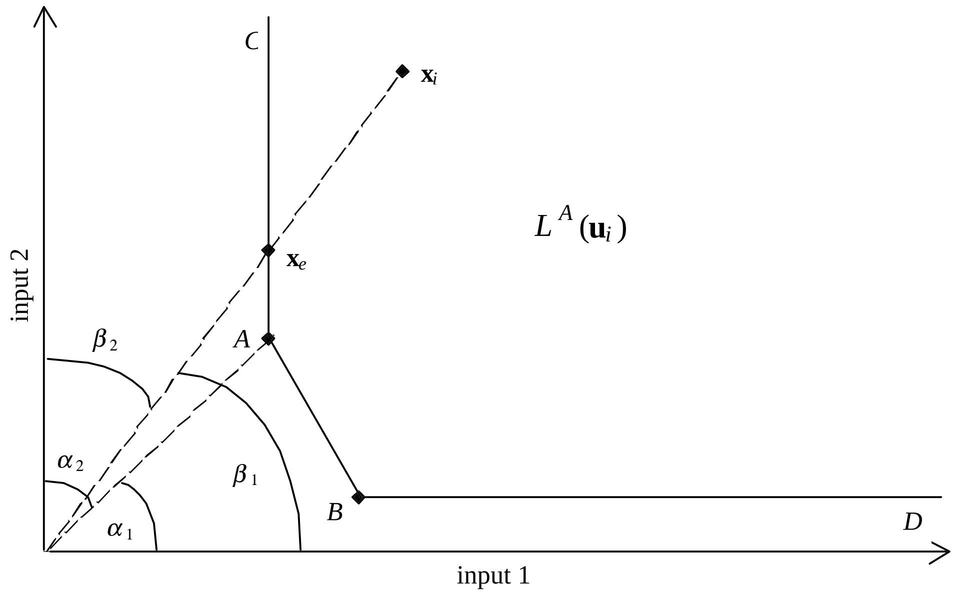

FIGURE 2

The vector $\mathbf{cos_x}$ contains dimensional characterizations of the mix of $\mathbf{x}$. The proportions between each pair of row entries of $\mathbf{x}$ can equivalently be represented in terms of directional cosines as $x_i/x_j = \cos_{x_i}/\cos_{x_j}$ for all $i, j \in \{1, ..., n\}$. Two input vectors are collinear if and only if they share the same vector of directional cosines. Formally,

$$\text{for } \mathbf{x}, \mathbf{x}^{'} \in L^A(\mathbf{u}_o) : \mathbf{x} = \lambda \mathbf{x}^{'} \text{ with } \lambda \in \mathbb{R}_{++} \Longleftrightarrow \mathbf{cos_x} = \mathbf{cos_{x'}}. \tag{5}$$

Hence, given (5), when evaluating the input mix efficiency of the vector $(\mathbf{x}_o, \mathbf{u}_o)$ we can confine attention to its collinear input projection $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. Suppose an input reference $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o) \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ has been selected for $(\mathbf{x}_o, \mathbf{u}_o)$ (e.g. $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o) = \mathbf{x}_Z(\mathbf{x}_o, \mathbf{u}_o)$). Slightly abusing notation, we denote the directional cosine vectors corresponding to $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ and $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)$ by respectively $\mathbf{cos}_{\mathbf{x}_{DF}}$ and $\mathbf{cos}_{\mathbf{x}^*_D}$. Mix efficiency evaluation amounts to comparing these two vectors.

Each ratio $\cos_{\mathbf{x}_{DF,l}}/\cos_{\mathbf{x}^*_{D,l}}$ $(l = 1, ..., n)$ gives an angular representation of mix deviation (e.g. in figure 2 these are $\cos_{x_{e1}}/\cos_{a_1} < 1$ and $\cos_{x_{e2}}/\cos_{a_2} > 1$). To obtain an overall mix efficiency measure these dimension-specific values should be combined. In analogy to the DF technical efficiency measure we propose to take the ratio of price-weighted sums of the different row elements of $\mathbf{cos}_{\mathbf{x}_{DF}}$ and $\mathbf{cos}_{\mathbf{x}^*_D}$, using $\mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ as the weighting vector. The overall mix efficiency measure is hence defined as:[7]

$$E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}_{DF}}}{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}^*_D}} \; \forall \mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o). \tag{6}$$

Of course, in view of (5) we can rewrite (6) as:

$$E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}_o}}{\mathbf{p} \cdot \mathbf{cos}_{\mathbf{x}^*_D}} \; \forall \mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o), \tag{7}$$

[7] It will become clear in what follows why (6) holds for all $\mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$.

which links $E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$ directly to the input mix properties of $(\mathbf{x}_o, \mathbf{u}_o)$.

The fact that we use $P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ rather than e.g. $P(\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o), \mathbf{u}_o)$ as the set of admissible shadow price weighting vectors in (6) allows for a convenient reformulation of $E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$. As it implies $\mathbf{p} \cdot \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) = \mathbf{p} \cdot \mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)$ and recalling (4) we get a third definition of $E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$:

$$E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{\|\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)\|}{\|\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)\|}, \tag{8}$$

from which it is immediate that $E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \leqq 1$ and that $E_{ME}(\mathbf{x}_o, \mathbf{u}_o) = 1$ if and only if $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) \in Eff\ L(\mathbf{u}_o)$. This norm ratio formulation clearly reveals the analogy with the way the DF technical efficiency measure is commonly presented.

The 'indifference' result as identified in Proposition 4 carries over to the measurement of mix efficiency. That is, definitions (6), (7) and (8) apply to any selection of $\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o) \in X_D(\mathbf{x}_o, \mathbf{u}_o)$. Conversely, equivalence of the three characterizations crucially depends on the fact that $\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)$ belongs to $X_D(\mathbf{x}_o, \mathbf{u}_o)$.

The use of $P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ implies that zero weights are assigned to those input dimensions in which $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ exhibits slack. The intuition can be sharpened by means of figure 2. Only the directional cosines with respect to the first axis are weighted positively as $\mathbf{x}_{e1}$ implicitly attributes a zero price to the second input. The mix efficiency estimate is $\cos\alpha_1 / \cos\beta_1 = \|A\| / \|\mathbf{x}_e\|$. In general, when $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ exhibits slack in a particular dimension the corresponding directional cosine value will be lower for $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ than for a vector belonging to $X_D(\mathbf{x}_o, \mathbf{u}_o)$ (e.g. in figure 2 $\cos_{x_{e2}} > \cos_{a_2}$). As this cannot be viewed as mix dominance it is natural to accord a zero weight to these cosines.

$E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$ can also be viewed as a measure of dominance in price space. We know that the cost level associated with $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ and $\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)$ is the same for this price vector. Let us then reverse the picture and consider all price vectors that imply the same cost level (arbitrarily fixed at unity) for both input vectors. That is, we consider the following two hyperplanes in price space:

$$\begin{aligned} H(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), 1) &\equiv \left\{\boldsymbol{\pi}_{DF} \in \mathbb{R}_+^n \mid \boldsymbol{\pi}_{DF} \cdot \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) = 1\right\}, \\ H(\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o), 1) &\equiv \left\{\boldsymbol{\pi}_D^* \in \mathbb{R}_+^n \mid \boldsymbol{\pi}_D^* \cdot \mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o) = 1\right\}. \end{aligned}$$

We present these hyperplanes for input vectors $\mathbf{x}_e$ and A of our illustration in figure 3. At the shadow price vector corresponding to $\mathbf{x}_e$ (with zero weight for the second input) both hyperplanes intersect. The implicit allocative (or mix) inefficiency of $\mathbf{x}_e$ is here revealed by the 'dominance' of the hyperplane associated with A over the one associated with $\mathbf{x}_e$. The reason is that, because it uses strictly less of the second input, observation A allows higher (positive) second input prices to imply the same cost level for each possible price assigned to the first input. This last point can be generalized: throughout the domain $\mathbb{R}_+^n$ isocost hyperplanes associated with $\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o) \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ are located at least as far away from the origin as those corresponding to $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. Mix efficiency decreases the more these distances differ.

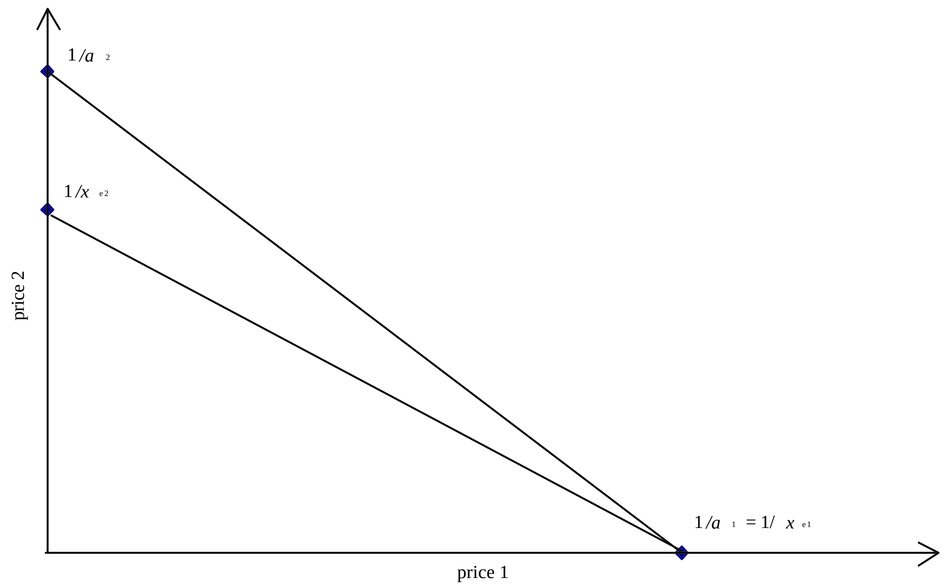

FIGURE 3

A natural way to evaluate this dominance in price space is to compare the Euclidean distances from these hyperplanes to the origin. These distances are given by:

$$\begin{aligned} d_{\mathbf{x}_{DF}}(\mathbf{x}_o, \mathbf{u}_o) &\equiv \min_{\boldsymbol{\pi}_{DF} \in \mathbb{R}^n_+} \{\|\boldsymbol{\pi}_{DF}\| \mid \boldsymbol{\pi}_{DF} \cdot \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) = 1\}, \\ d_{\mathbf{x}^*_D}(\mathbf{x}_o, \mathbf{u}_o) &\equiv \min_{\boldsymbol{\pi}^*_D \in \mathbb{R}^n_+} \{\|\boldsymbol{\pi}^*_D\| \mid \boldsymbol{\pi}^*_D \cdot \mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o) = 1\}, \end{aligned}$$

and we have:

$$E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{d_{\mathbf{x}_{DF}}(\mathbf{x}_o, \mathbf{u}_o)}{d_{\mathbf{x}^*_D}(\mathbf{x}_o, \mathbf{u}_o)}. \tag{9}$$

The equivalence of this ‘price based’ dominance characterization with the ‘quantity based’ characterization in (8) follows directly from the fact that the distance from a hyperplane to the origin can be retrieved using its normal. In this case, substituting

$$d_{\mathbf{x}_{DF}}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{1}{\|\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)\|} \text{ and } d_{\mathbf{x}^*_D}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{1}{\|\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)\|}$$

in (9) immediately yields (8).

A convenient by-product of characterization (9) is that it allows to come full circle with our particular choice of $\mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ as the weighting vector in (6). For this purpose we switch to the Hesse normal form representations of $H(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), 1)$ and $H(\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o), 1)$:

$$\begin{aligned} H^n(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), 1) &\equiv \left\{\boldsymbol{\pi}_{DF} \in \mathbb{R}^n_+ \mid \boldsymbol{\pi}_{DF} \cdot \mathbf{cos}_{\mathbf{x}_{DF}} = d_{\mathbf{x}_{DF}}(\mathbf{x}_o, \mathbf{u}_o)\right\}, \\ H^n(\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o), 1) &\equiv \left\{\boldsymbol{\pi}^*_D \in \mathbb{R}^n_+ \mid \boldsymbol{\pi}^*_D \cdot \mathbf{cos}_{\mathbf{x}^*_D} = d_{\mathbf{x}^*_D}(\mathbf{x}_o, \mathbf{u}_o)\right\}. \end{aligned}$$

Using this representation it can be seen that $E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv [\boldsymbol{\pi}_{DF} \cdot \mathbf{cos}_{\mathbf{x}_{DF}}] / \left[\boldsymbol{\pi}_D^* \cdot \mathbf{cos}_{\mathbf{x}_D^*}\right]$ for $\boldsymbol{\pi}_{DF} \in H^n(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), 1)$ and $\boldsymbol{\pi}_D^* \in H^n(\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o), 1)$. The only legitimate way to employ a common price-weighting vector in the numerator and the denominator as is done in (6), is to take vectors that lie in the intersection of $H^n(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), 1)$ and $H^n(\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o), 1)$. These vectors are indeed all $\mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ for which $\mathbf{p} \cdot \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) = \mathbf{p} \cdot \mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o) = 1$.[8]

5 A digression on commensurability

The mix efficiency measure as it has been introduced above is not invariant to the units in which the different input quantities are measured. The commensurability property has been especially advocated by Russell [20], and is indeed of interest in cases where it is not immediately clear which particular measurement unit to choose. Before addressing the issue more thoroughly, remember that so far we have assumed the projection $\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)$ to be given. A first point of interest then concerns the projection procedure itself. The Zieschang procedure presented in section 3, for example, does satisfy the commensurability property (see [20]). As such, commensurability poses a problem only to the extent that it affects the eventual mix efficiency estimate.

To obtain units invariance of $E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$ data rescaling can be applied. The three examples presented below yield particularly intuitive reformulations of the mix efficiency measure (computed with respect to the rescaled data) in terms of the original input values.

The norm representation as given in definition (8) provides a convenient point of departure. We re-express it as:

$$E_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \left[\frac{\sum_{l=1}^{n} (x_{Dl}^*)^2}{\sum_{l=1}^{n} (x_{DFl})^2}\right]^{1/2}. \tag{10}$$

We further use hats to denote the rescaled counterparts of an input vector $\mathbf{x}$ and its entries x_l $(l = 1, ..., n)$. We similarly use $\widehat{E}_{ME}(\mathbf{x}_o, \mathbf{u}_o)$ to indicate the mix efficiency estimate computed with respect to these rescaled vectors.

Example 1 *For* $\widehat{x}_l \equiv \left(x_l / \sqrt{x_l x_{DFl}}\right)$ *(i.e. we divide each input value by the geometric mean of itself and the corresponding value in* $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$*) we get from (10)*:

$$\widehat{E}_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \left[\frac{\sum_{l=1}^{n} (\widehat{x}_{Dl}^*)^2}{\sum_{l=1}^{n} (\widehat{x}_{DFl})^2}\right]^{1/2} \equiv \left[\sum_{l=1}^{n} \left(\frac{x_{Dl}^*}{x_{DFl}}\right) / n\right]^{1/2}. \tag{11}$$

[8] It is immediate that the validity of (9) does not depend on the fact that we considered a cost level of unity. This also implies that this legitimation of (6) holds for all $\mathbf{p} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$.

This is a monotonic transformation of the arithmetic mean of the original input proportions. It is precisely this value which is minimized in the second (Färe-Lovell) stage of the Zieschang projection procedure (cfr. supra). For this particular rescaling, one can thus re-interpret the Zieschang reference selection procedure as one that minimizes mix efficiency.

Example 2 *For* $\widehat{x}_l \equiv \left(x_l/\sqrt{x_l x^*_{Dl}}\right)$ *(i.e. we divide each input value by the geometric mean of itself and the corresponding value in* $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)$*) we get from (10):*

$$\widehat{E}_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \left[n/\sum_{l=1}^{n}\left(\frac{x_{DFl}}{x^*_{Dl}}\right)\right]^{1/2}, \tag{12}$$

This is a monotonic transformation of the harmonic mean of the original input proportions. The fact that the harmonic mean is obtained instead of the arithmetic mean directly builds on the orientation change implicit in choosing $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)$ *as a basis of comparison. Whereas in (11) one averages over input reductions when going from* $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ *to* $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)$*, we now look at input expansions to get from* $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)$ *to* $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ *and a mix efficiency estimate is consequently obtained as the inverse of an arithmetic mean. The orientation is thus reversed, resulting in an 'inverse Zieschang procedure' and a correspondingly different mix efficiency estimate (see also [5]).*

Example 3 *For* $\widehat{x}_l \equiv \left(x_l/\sqrt{x_{DFl} x^*_{Dl}}\right)$ *(i.e. we divide each input value by the geometric mean of the corresponding values in* $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ *and* $\mathbf{x}^*_D(\mathbf{x}_o, \mathbf{u}_o)$*) we get from (10):*

$$\widehat{E}_{ME}(\mathbf{x}_o, \mathbf{u}_o) \equiv \left[\left(\sum_{l=1}^{n}\left(\frac{x^*_{Dl}}{x_{DFl}}\right)/n\right)\cdot\left(n/\sum_{l=1}^{n}\left(\frac{x_{DFl}}{x^*_{Dl}}\right)\right)\right]^{1/2}, \tag{13}$$

which is the geometric mean of the estimates in (11) and (12) (ignoring the square roots for simplicity). Somewhat similar to a Fisher ideal index as a means to answer the question about the most appropriate price or quantity base, this alternative may have particular appeal to some as it avoids having to choose between the two previous orientations.

6 A mix-adjusted Debreu-Farrell measure

As we have indicated in the introduction, our aim was to preserve the commonly employed DF measure of technical efficiency in view of its economic intuition while at the same time dealing with its potential 'axiomatic' indication problem. This can now be achieved by combining the measure $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ and the $E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$ measure of implicit allocative efficiency into a mix-adjusted Debreu-Farrell index:[9]

$$E_{MA}(\mathbf{x}_o, \mathbf{u}_o) \equiv E_{DF}(\mathbf{x}_o, \mathbf{u}_o)\cdot E_{ME}(\mathbf{x}_o, \mathbf{u}_o). \tag{14}$$

[9] For ease of exposition we assume that the commensurability problem of the mix efficiency component has adequately been dealt with. Also note that the DF component $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ satisfies the commensurability property (see [20]).

In view of definition (8) and given that $E_{DF}(\mathbf{x}_o, \mathbf{u}_o) \equiv \|\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)\| / \|\mathbf{x}_o\|$ we can rewrite (14) as:

$$E_{MA}(\mathbf{x}_o, \mathbf{u}_o) \equiv \frac{\|\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)\|}{\|\mathbf{x}_o\|},$$

which indicates that $E_{MA}(\mathbf{x}_o, \mathbf{u}_o)$ can also be computed directly. The composite index $E_{MA}(\mathbf{x}_o, \mathbf{u}_o)$ then acts as a non-radial measure for the relative distance from $\mathbf{x}_o$ to $Eff\ L(\mathbf{u}_o)$. Its DF component $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ gives the traditional shadow cost efficiency estimate for $\mathbf{x}_o$ and measures the distance from $\mathbf{x}_o$ to $Isoq\ L^A(\mathbf{u}_o)$. It does not only estimate the (maximum) cost efficiency of $\mathbf{x}_o$ with respect to $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ but also with respect to $\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o)$ (as $P_{DF}(\mathbf{x}_o, \mathbf{u}_o) \equiv P(\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o), \mathbf{u}_o) \subseteq P(\mathbf{x}_D^*(\mathbf{x}_o, \mathbf{u}_o), \mathbf{u}_o)$ by definition). On the other hand, $E_{ME}(\mathbf{x}_o, \mathbf{u}_o)$ deals with the presence of any slack in $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ and pertains to the mix properties of $\mathbf{x}_o$.

In fact, for the particular rescaling considered in example 2 the mix-adjusted DF measure defined above is quasi-identical to the Zieschang [24] measure (except for the square root in (11)). In this respect example 2 is interesting insofar as it links the shadow price approach followed here to the search for desirable efficiency gauges in the axiomatic literature.[10]

7 Summary and concluding remarks

In this paper we have addressed the well-known indication problem of the DF technical efficiency measure focusing on the often employed class of convex monotonic technologies, which allowed us to follow a shadow price approach. Our main findings can be summarized as follows.

First, we have characterized the set X_D of Koopmans efficient references that maintain the Debreu shadow cost legitimation. We have subsequently indicated that the procedure introduced by Zieschang [24] always selects such well-grounded references.

Second, we have presented a measure E_{ME} of implicit allocative or mix efficiency, which can also be interpreted as a measure of dominance in the price space. Commensurability of this measure can be ascertained by appropriately rescaling the original data. One such rescaling yielded a new characterization of the Zieschang projection and measure. Another one seems appealing given its 'orientation-independence'.

Third, we have introduced a mix-adjusted DF measure E_{MA} that is decomposable in the DF and mix efficiency estimate. In line with Russell's defense of the original DF measure in [19] we suggest to work notably with the decomposed representation in order to reveal the two-stage nature of the non-radial projection (first towards the isoquant, and next along the isoquant towards the efficient subset) more clearly.

[10] Of course, the axiomatic literature considers a yet broader class of technologies than we do in this paper. For a convex monotonic hull, the attractiveness of the Zieschang measure on axiomatic grounds is discussed by Ferrier et al. [16].

The precise value of E_{MA} of course depends on the choice of a reference $\mathbf{x}_D \in X_D$. Proposition 4 learns that, when X_D is not a singleton, each of its elements is equally well justifiable from a shadow cost efficiency perspective. Throughout we have deliberately left open the question which (non-radial) reference belonging to X_D is the most preferred one. Different perspectives could be adopted. One could take into account feasibility restrictions on mix adjustments. Or one could return to the axiomatic literature (see e.g. our discussion of the Zieschang measure). The above discussion makes clear that there is probably no unique a priori answer. Indeed, it may well be that from one perspective the Zieschang procedure is most recommendable while another starting point leads to another $\mathbf{x}_D$. Nevertheless, we hope that the insights forwarded in this paper may serve as a guide when addressing this issue more thoroughly.

Acknowledgements

We have benefited from constructive remarks of participants at the European Workshop on Efficiency and Productivity and Efficiency Analysis (Copenhagen, 28-31 October 1999). We are particularly indebted to T. Kuosmanen, C.A.K. Lovell, T. Post and H. Scheel for valuable comments.

Appendix

Proof of Proposition 3 For $\mathbf{x}_{DF} \in Eff\ L^A(\mathbf{u}_o)$ the result follows directly from proposition 1. Let us then consider $\mathbf{x}_{DF} \notin Eff\ L^A(\mathbf{u}_o)$. We have to compare input vectors $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ and $\mathbf{x}' \in Eff\ L^A(\mathbf{u}_o) \setminus X_D$. To facilitate exposition we consider scaled shadow price vectors $\mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u}_o)$ and $\mathbf{p}' \in P\left(\mathbf{x}', \mathbf{u}_o\right)$ such that $\mathbf{p}_D \cdot \mathbf{x}_D = \mathbf{p}' \cdot \mathbf{x}' = 1$ in the following. We are led to proof that for all pairs $\left(\mathbf{p}_D, \mathbf{p}'\right)$:

$$\frac{1}{\mathbf{p}_D \cdot \mathbf{x}_o} \geqq \frac{1}{\mathbf{p}' \cdot \mathbf{x}_o} \text{ or } \mathbf{p}' \cdot \mathbf{x}_o \geqq \mathbf{p}_D \cdot \mathbf{x}_o. \tag{15}$$

Multiplying both sides of the second expression in (15) by $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ we obtain:

$$\mathbf{p}' \cdot \mathbf{x}_{DF} \geqq \mathbf{p}_D \cdot \mathbf{x}_{DF}, \tag{16}$$

where $\mathbf{x}_{DF} = \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ to save on notation. Of course, from proposition 1 (16) holds for $\mathbf{p}_D \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. Let us therefore focus on $\mathbf{p}_D \notin P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$. As a preliminary step remember that $P_{DF}(\mathbf{x}_o, \mathbf{u}_o) \subseteq P(\mathbf{x}, \mathbf{u}_o)$ while $P_{DF}(\mathbf{x}_o, \mathbf{u}_o) \nsubseteq P\left(\mathbf{x}', \mathbf{u}_o\right)$ by construction. That is, for $\mathbf{p}_{DF} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ (with again for simplicity $\mathbf{p}_{DF} \cdot \mathbf{x}_{DF} = 1$) the following condition is satisfied:

$$\mathbf{p}' \cdot \mathbf{x}_{DF} > 1 = \mathbf{p}_{DF} \cdot \mathbf{x}_{DF},. \tag{17}$$

Define $\mathbf{p}_\rho \equiv \rho\mathbf{p}_{DF} + (1-\rho)\mathbf{p}_D$. For all $\rho \in [0,1]$ we have $\mathbf{p}_\rho \in P(\mathbf{x}_D, \mathbf{u}_o)$ Indeed, $\mathbf{p}_{DF} \cdot \mathbf{x}_D \leqq \mathbf{p}_{DF} \cdot \mathbf{x}$ and $\mathbf{p}_D \cdot \mathbf{x}_D \leqq \mathbf{p}_D \cdot \mathbf{x}$ for all $\mathbf{x} \in L(\mathbf{u}_o)$, so also $\mathbf{p}_\rho \cdot \mathbf{x}_D \leqq \mathbf{p}_\rho \cdot \mathbf{x}$ for all $\rho \in [0,1]$.

Now suppose (16) is not met so that:

$$\mathbf{p}^{'} \cdot \mathbf{x}_{DF} < \mathbf{p}_D \cdot \mathbf{x}_{DF}. \tag{18}$$

From (17) and (18) there should exist a $\mathbf{p}_{\overline{\rho}}$ for $\overline{\rho} \in (0,1)$ such that $\mathbf{p}^{'} \cdot \mathbf{x}_{DF} = \mathbf{p}_{\overline{\rho}} \cdot \mathbf{x}_{DF}$. Solving for $\overline{\rho}$ yields:

$$\overline{\rho} = \frac{\mathbf{p}^{'} \cdot \mathbf{x}_{DF} - \mathbf{p}_D \cdot \mathbf{x}_{DF}}{1 - \mathbf{p}_D \cdot \mathbf{x}_{DF}}. \tag{19}$$

As $\mathbf{p}_\rho \in P(\mathbf{x}_D, \mathbf{u}_o)$ we should always have $1 = \mathbf{p}_{\overline{\rho}} \cdot \mathbf{x}_D \leqq \mathbf{p}_{\overline{\rho}} \cdot \mathbf{x}_{DF}$. However, under (19) this condition is equivalent to $1 \geqq \mathbf{p}^{'} \cdot \mathbf{x}_{DF}$, which contradicts (17). We therefore conclude that (18) does not hold and (16) is indeed satisfied. ■

Proof of Proposition 4 We consider two input vector $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ and $\mathbf{x}_D^{'} \in X_D(\mathbf{x}_o, \mathbf{u}_o) \setminus \{\mathbf{x}_D\}$. We assume $\mathbf{p}_D^{'} \in P(\mathbf{x}, \mathbf{u}_o)$ with $\mathbf{p}_D^{'} \in \mathbb{R}_{++}^n$ given. We have to proof that a $\mathbf{p}_D \in P(\mathbf{x}_D, \mathbf{u}_o)$ with $\mathbf{p}_D \in \mathbb{R}_{++}^n$ can always be constructed such that:

$$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}_o} \geqq \frac{\mathbf{p}_D^{'} \cdot \mathbf{x}_D^{'}}{\mathbf{p}_D^{'} \cdot \mathbf{x}_o}. \tag{20}$$

Dividing both sides of (20) by $E_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ gives the equivalent condition:

$$\frac{\mathbf{p}_D \cdot \mathbf{x}_D}{\mathbf{p}_D \cdot \mathbf{x}_{DF}} \geqq \frac{\mathbf{p}_D^{'} \cdot \mathbf{x}_D^{'}}{\mathbf{p}_D^{'} \cdot \mathbf{x}_{DF}}, \tag{21}$$

where $\mathbf{x}_{DF} = \mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ to save on notation. We can decompose $\mathbf{x}_{DF}$, $\mathbf{x}_D$ and $\mathbf{x}_D^{'}$ in a slack and non-slack subvector. Let us denote $\mathbf{x}_D = (\overline{\mathbf{x}}_D, \widetilde{\mathbf{x}}_D)$, $\mathbf{x}_D^{'} = \left(\overline{\mathbf{x}}_D^{'}, \widetilde{\mathbf{x}}_D^{'}\right)$ and $\mathbf{x}_{DF}(\mathbf{x}_o, \mathbf{u}_o) = (\overline{\mathbf{x}}_{DF}, \widetilde{\mathbf{x}}_{DF})$ such that (for the non-slack subvectors) $\overline{\mathbf{x}}_D^{'} = \overline{\mathbf{x}}_D = \overline{\mathbf{x}}_{DF}$ while (for the slack subvectors) $\widetilde{\mathbf{x}}_D > \widetilde{\mathbf{x}}_{DF}$ and $\widetilde{\mathbf{x}}_D^{'} > \widetilde{\mathbf{x}}_{DF}$. We restate (21):

$$\frac{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_D}{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_{DF}} \geqq \frac{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_D^{'}}{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_{DF}}, \tag{22}$$

with corresponding decompositions $\mathbf{p}_D = (\overline{\mathbf{p}}_D, \widetilde{\mathbf{p}}_D)$ and $\mathbf{p}_D^{'} = \left(\overline{\mathbf{p}}_D^{'}, \widetilde{\mathbf{p}}_D^{'}\right)$. Obviously, for $\mathbf{p}_D, \mathbf{p}_D^{'} \in \mathbb{R}_{++}^n$:

$$\frac{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF}}{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF}} = \frac{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF}}{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF}} = 1 \text{ and } \frac{\widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_D}{\widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_{DF}} < 1,\ \frac{\widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_D^{'}}{\widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_{DF}} < 1. \tag{23}$$

Both the left and right hand side of (22) are strictly smaller than one. Let:

$$\frac{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_D^{'}}{\overline{\mathbf{p}}_D^{'} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^{'} \cdot \widetilde{\mathbf{x}}_{DF}} = 1 - \varepsilon, \text{ with } \varepsilon > 0. \tag{24}$$

We now proceed by constructing a $\mathbf{p}_D$ such that (20) is satisfied. First note that each $\mathbf{p}_{DF} \in P_{DF}(\mathbf{x}_o, \mathbf{u}_o)$ can be decomposed similarly as $\mathbf{p}_D$ and $\mathbf{p}_D^{'}$ above so that

$\mathbf{p}_{DF} = (\overline{\mathbf{p}}_{DF}, 0)$. As $\mathbf{x}_D \in X_D(\mathbf{x}_o, \mathbf{u}_o)$ we can define $\mathbf{p}_D^* = (\overline{\mathbf{p}}_{DF}, \widetilde{\mathbf{p}}_D^*)$ with $\widetilde{\mathbf{p}}_D^* \in \mathrm{IR}_{++}^n$ appropriately chosen such that $\mathbf{p}_D^* \in P(\mathbf{x}_D, \mathbf{u}_o)$. We have for all $\mathbf{x} \in L(\mathbf{u}_o)$:

$$\overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}_{DF} \leqq \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}, \tag{25}$$

$$\overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D^* \cdot \widetilde{\mathbf{x}}_D \leqq \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}} + \widetilde{\mathbf{p}}_D^* \cdot \widetilde{\mathbf{x}}, \tag{26}$$

where again we use the decomposition $\mathbf{x} = (\overline{\mathbf{x}}, \widetilde{\mathbf{x}})$. Convex combinations of (25) and (26) yield for all $\kappa \in [0, 1]$:

$$\overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}}_{DF} + \kappa (\widetilde{\mathbf{p}}_D^* \cdot \widetilde{\mathbf{x}}_D) \leqq \overline{\mathbf{p}}_{DF} \cdot \overline{\mathbf{x}} + \kappa (\widetilde{\mathbf{p}}_D^* \cdot \widetilde{\mathbf{x}}). \tag{27}$$

Denote $\mathbf{p}_D^\kappa = (\overline{\mathbf{p}}_{DF}, \kappa \widetilde{\mathbf{p}}_D^*)$. From (27) we know that $\mathbf{p}_D^\kappa \in P(\mathbf{x}_D, \mathbf{u}_o)$ for all $\kappa \in [0, 1]$. Moreover, $\mathbf{p}_D^\kappa \in \mathrm{IR}_{++}^n$ for $\kappa \in (0, 1]$. So, we can shrink the shadow price vector associated with $\widetilde{\mathbf{x}}_D$ independently of that corresponding to $\overline{\mathbf{x}}_{DF}$. In fact, it can be made infinitesimally small while still remaining positive. Using (23) and for a given value of $\varepsilon > 0$, we can always set $\mathbf{p}_D = \mathbf{p}_D^{\overline{\kappa}}$ with $\overline{\kappa} \in (0, 1]$ sufficiently small such that:

$$\frac{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_D}{\overline{\mathbf{p}}_D \cdot \overline{\mathbf{x}}_{DF} + \widetilde{\mathbf{p}}_D \cdot \widetilde{\mathbf{x}}_{DF}} \geqq 1 - \varepsilon. \tag{28}$$

Combining (24) and (28) yields (20). ■

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