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DISCUSSION
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An Affine Model for International Bond Markets

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Abstract

We present and estimate a parsimonious multi-factor affine term structure model for joint bond markets. We extend the standard affine models by focusing on joint markets and by incorporating the exchange rate dynamics in the estimation procedure. Estimation is done by means of a Kalman filter algorithm. We find that our particular three factor model is quite successful in fitting bond correlations, both within and between national bond markets. Moreover, the model sheds light on some of the most persistent puzzles in empirical finance, e.g. the UIP puzzle and the home bias puzzle. Finally, we apply the model to test for international diversification gains in unhedged bond portfolios, conditional on the information that is present in the term structures of both countries. We find that exchange rate risk is sufficiently priced such that the inclusion of foreign bonds allows for an improved risk-return trade-off from the perspective of a domestic investor.

Keywords: multi-factor affine term structure model, Kalman filter, common factor, uncovered interest rate parity, international diversification, exchange rate risk.
J.E.L.: C33, E43, G15

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1 Introduction

In this paper we propose a model to account for the interdependencies both within and between bond markets. The main motivation behind this paper lies in the observed high correlations between bond yields across the maturity spectrum and across countries. This observation at least suggests the presence of a limited number of latent factors that drive the yield curves across countries. The presented model belongs to the class of affine term structure models (Duffie and Kan, 1996) extended to an international framework. The international extension is accomplished through a theoretical specification of 'local' (domestic or foreign) and 'common' (international) factors driving the respective instantaneous interest rates in both countries. The basic idea of using common and local factors was originally presented by Ahn (1999). While the common factor will mainly model the international correlations, both the local and common factor will generate the within market correlations for each bond market respectively. Moreover, we assume the instantaneous drift of the pricing kernel of each country to be a function of the domestic and common factor only, whilst its diffusion term is assumed to be a function of all three factors (including the foreign local factor, that is). The justification for this framework follows from the idea that domestic investors not only care about domestic and international shocks in evaluating their bond portfolio. They also take foreign local shocks into account to the extent that their portfolio consists of foreign bonds. An implication of this foreign risk spill-over effect is that local factors play a possibly important role in explaining both the dynamics and the term structure of risk premia in international markets. This paper contributes to the literature by elaborating and estimating the model within a continuous-time conditional affine framework, along with an application.[1]

Our particular joint bond market model is in theory flexible enough to incorporate various stylized facts of the international bond markets, such as the forward premium puzzle and time-varying correlations between local and foreign yields. Traditional single market term structure models (such as Duan and Simonato, 1999, and de Jong, 2000) cannot do this and, hence, this work can be viewed as a theoretical extension to these type of models. We also extend the standard affine models by incorporating the exchange rate dynamics in the estimation procedure. Using arbitrage free pricing conditions we derive the price dynamics and interrelations as well as the implications for exchange rate predictions. By exploiting both the cross section information as well as the time series information, we are capable of efficiently identifying the key parameters that are left unidentified in an unconditional approach. More specifically, the model allows for identification of the domestic and foreign prices of risk on all factors, including the domestic price of risk on the foreign local factor and including the bond factor prices of risk for exchange rate uncertainty. The latter identification is accomplished by including the exchange rate dynamics endogenously in the estimation procedure, while the

[1] To the best of our knowledge, there is no similar paper that tries to fit the international bond markets within the proposed framework. Ahn restricts himself to an unconditional model estimation and advocates the conditional approach in his conclusions.

former is accomplished by assuming the latent factors to be dependent.

Full identification of the model parameters allows us to address important issues in international finance, such as the existence of international portfolio diversification gains and the uncovered interest rate parity (UIP) puzzle. We apply this model to the UK-US joint bond market. We choose this particular market as a first test case because very strong interrelations have been reported in the literature. Taking the viewpoint of a UK-investor who initially only invests in the UK, we study the question whether this investor can enhance the mean-variance characteristics of his portfolio by also investing in the US market on an instantaneous investment horizon, conditional on the information that is present in both term structures. Obviously, there is a trade-off between the additional spanning possibilities and the incurred unpriced exchange rate risk implied by the inclusion of foreign bonds in an unhedged portfolio. In essence, the outcome will depend on whether or not the exchange rate risk is sufficiently priced in the market. The part of the exchange rate risk that is orthogonal to the bond market factors remains unpriced in our model and will serve as a deterrent for international bond diversification. Finally, we also present the Fama beta statistic implied by our model. Our model predicts rational deviations from UIP, due to the presence of time-varying risk premia. Hence, our Fama beta statistic will be more in line with the empirical literature that documents this deviation from UIP (Fama, 1984).

The remainder of the paper is organized as follows. In section 2 we present the theoretical pricing equations implied by the factor and pricing kernel specification. In section 3 unconditional and conditional estimation results are presented, along with a discussion. In section 4 a mean variance efficiency analysis is performed on an international unhedged portfolio. This enables us to evaluate whether the inclusion of foreign bonds in your portfolio is beneficial in terms of the risk-return trade-off. Finally, section 5 presents conclusions.

2 A model for joint bond markets

Endogenous term structure models typically study a particular national bond market and derive zero coupon (domestic) bond prices as a function of the dynamics of the (domestic) instantaneous interest rate, $r(t)$. The modeling of joint bond markets naturally extends this framework and merely requires an assumption about the foreign instantaneous interest rate, $r^*(t)$, and about how these two rates are linked to each other. Assuming a complete probability space $(\Omega, \mathcal{F}, P)$ and in the absence of arbitrage opportunities, only technical conditions are required for the existence of an equivalent martingale measure Q (Duffie, 1996, pp. 110-111). Such a probability measure Q has the property that any domestic or foreign security with certain payoff 1 at time $t+\tau$ has a price at time t of:

$$p(\tau,t) = E_t^Q\left[\exp\left(-\int_t^{t+\tau} r(u)du\right)\right], \qquad p^*(\tau,t) = E_t^Q\left[\exp\left(-\int_t^{t+\tau} r^*(u)du\right)\right], \tag{1}$$

where E_t^Q denotes $\mathcal{F}_t$-conditional expectation under Q.

In this section, we use arbitrage free pricing theory to construct a particular three factor joint bond market model. More specifically, we derive the dynamics of the joint bond prices and the corresponding exchange rate within the Cox-Ingersoll-Ross (1985, CIR, henceforth) class of models. Moreover, inspired by Ahn (1999), we characterize factors as either 'domestic', 'foreign' or 'common'. Next to deriving the implied yield dynamics we analyze the implications for correlations among yields both within and between markets and for the exchange rate dynamics.

2.1 Common and local factors

Formal term structure models are based on assumptions about the number and specific dynamics of the underlying sources of uncertainty that drive the instantaneous interest rates. We start off by defining the domestic and foreign instantaneous interest rates ($r(t)$ and $r^*(t)$, respectively) as affine functions of their 'local' and 'common' factors. In particular, we assume one driving factor, $F_2(t)$, that is common to both the instantaneous interest rate in the home and foreign country and two strictly local factors, $F_1(t)$ and $F_3(t)$.[2] Formally:

$$\begin{aligned} r(t) &\equiv F_1(t) + F_2(t), \\ r^*(t) &\equiv F_2(t) + F_3(t). \end{aligned} \tag{2}$$

The dynamics of the latent factors are modeled by means of a system of square-root diffusion equations. Define $F(t)$ as the vector of factors $F(t) \equiv (F_1(t), F_2(t), F_3(t))'$ and let $W(t) \equiv (W_1(t), W_2(t), W_3(t))'$ be a vector of Wiener processes under the historical probability measure P. The dynamics of the factors are then governed by:

$$dF(t) = K(F(t) - \theta)\,dt + \Sigma_1 \Sigma_F dW(t), \tag{3}$$

where K, $K \equiv diag(-\kappa_1, -\kappa_2, -\kappa_3)$, determines the convergence speed of each factor towards his unconditional mean θ, $\theta \in \mathbb{R}^3$. The matrices Σ_1 and Σ_F model the variance component of the factors. Σ_F allows for time variability in the conditional variance function by modeling the level dependence of factor volatility (as in CIR, 1985). More specifically, we restrict Σ_F to be diagonal:

$$\Sigma_F = diag(\,\sigma_1\sqrt{F_1(t)}, \sigma_2\sqrt{F_2(t)}, \sigma_3\sqrt{F_3(t)}\,). \tag{4}$$

Allowing full identification of all parameters while still remaining parsimonious, we impose the following rather simple covariance structure among the factors:

[2] Obviously, the model can be easily extended to a higher number of driving factors.

$$\Sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ x_{21} & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

The local factors F_1 and F_3 are uncorrelated but each local factor is allowed to move together with the common factor F_2.[3] To keep the model as parsimonious as possible, we restrict the other off-diagonal elements to be zero. As long as Σ_1 is diagonal, all factors are independent. Positive (negative) entries for x_{21} and x_{23} generate positive (negative) unconditional covariance between the local and the common factor, while conditional covariances will also depend on the level of the factors themselves. As such the model is sufficiently flexible to generate unconditional and conditional covariation of either sign in the factors. Dai and Singleton (2000) conclude that it is important for a multifactor affine model to permit negative correlation among the factors. The dynamics of the factors also determine the conditional covariation among the factors. This covariation will be passed entirely on to the covariation of the asset prices that are linear in the factors (e.g. yields and expected exchange rate returns, see below). The first two conditional moments of the factors can be calculated as (de Jong, 2000):

$$\begin{aligned} E_t\left[F\left(t+h\right)\right] &= \theta + \exp\left\{Kh\right\}\left(F\left(t\right)-\theta\right), \\ var_t[F\left(t+h\right)] &= \int_0^h \exp\left\{K(h-s)\right\}\Sigma_1 E_t\left[\Sigma_{F(t+s)}\Sigma'_{F(t+s)}\right]\Sigma'_1 \exp\left\{K(h-s)\right\}ds, \end{aligned} \tag{6}$$

with E_t the expectation operator (under the historical probability measure P) conditional upon time t information, cov_t the conditional factor covariance matrix, and $\exp\{Kx\} \equiv diag(e^{-\kappa_1 x}, e^{-\kappa_2 x}, e^{-\kappa_3 x})$, $\forall x$.[4]

We change measure from P to Q by applying Girsanov's theorem[5], $dW_i(t) = d\tilde{W}_i(t) -$

[3] It is straightforward to show that non-diagonality of Σ_1 (an extension to the model as proposed by Ahn) is a necessary requirement to identify the foreign market prices of risk, λ_i^*, $i = \{1,2,3\}$, in a conditional setting.

[4] Defining a_{ij} ($\forall\ i,j$) as the $\mathbb{R}^3$-vector that solves $\forall\ x \in \mathbb{R}^3 : a'_{ij}x = [\Sigma_1 diag\left(x\right)\Sigma'_1]_{ij}$, we can reformulate this conditional variance-covariance matrix as:

$$\begin{aligned} var_t[F\left(t+h\right)]_{i,j} = {} & \frac{1-\exp\left\{-\left(\kappa_i+\kappa_j\right)h\right\}}{\kappa_i+\kappa_j}a'_{ij}\theta + \\ & \sum_{k=1}^{3}\frac{\exp\left\{-\left(\kappa_k\right)h\right\}-\exp\left\{-\left(\kappa_i+\kappa_j\right)h\right\}}{\kappa_i+\kappa_j-\kappa_k}a_{ij,k}\left(F_k\left(t\right)-\theta_k\right), \end{aligned} \tag{7}$$

where $a_{ij,k}$ is the k-th row element in the column vector a_{ij}. $[\Sigma_1 diag\left(x\right)\Sigma'_1]_{ij}$ denotes the ij-th element of the matrix $\Sigma_1 E_t[\Sigma_F \Sigma'_F]\Sigma'_1$ evaluated in any point $E_t[F\left(t+h\right)] = x$.

[5] Here and in the following, the *tilde* refers to the value under Q. Note that the foreign currency version of Girsanov's theorem is written as:

$$dW_i = d\tilde{W}_i - \Psi_i^*\sqrt{F_i(t)}dt, \qquad i = \{1,2,3,\}$$

where Ψ^* is defined analogous to Ψ, $\Psi^* = \left(\frac{\lambda_1^*}{\sigma_1}, \frac{\lambda_2^*}{\sigma_2}, \frac{\lambda_3^*}{\sigma_3}\right)$.

$\frac{\lambda_i}{\sigma_i}\sqrt{F_i(t)}dt$, where $\frac{\lambda_i}{\sigma_i}$ is to be understood as the price of risk for factor i (scaled by its standard deviation for convenience), $i = \{1, 2, 3\}$. Hence, the risk neutral factor dynamics can be written as:

$$dF(t) = \tilde{K}\left(F(t) - \tilde{\theta}\right)dt + \Sigma_1 \Sigma_F d\tilde{W}(t), \tag{8}$$

where $\tilde{K}$ and $\tilde{\theta}$ denote the risk adjusted analogues of K and θ:

$$\begin{aligned} \tilde{K} &= K - \Sigma_1 \Lambda, \\ \tilde{\theta} &= \tilde{K}^{-1} \Lambda \theta, \end{aligned} \tag{9}$$

and where $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$.

2.2 Zero coupon bond prices

In this section, we derive the implications of the above common-local structure for the international bond markets in the absence of arbitrage opportunities. More specifically, we analyze the implied covariation between bonds within a national and across national bond markets. Applying this methodology to bond holding (period) returns, we arrive at a parsimonious description of covariation of these individuals investment opportunities that can be used to construct various financial statistics on international (hedged) bonds (e.g. duration, Sharpe ratio's, ...).

Intuitively put, arbitrage free pricing implies that expected returns are equal to the instantaneous interest rate over all possible investment strategies, under Q. Letting $p(\tau, t) = p(\tau, F_1(t), F_2(t), F_3(t))$ denote the price at time t of a domestic bond maturing at time $t + \tau$ (note the dependence on the local foreign factor), equality of expected returns implies that:

$$\mathcal{D}^Q p(\tau, F(t)) = r(t)\, p(\tau, F(t)), \tag{10}$$

where $\mathcal{D}$ denotes the Dynkin operator. Using Itô's lemma to solve for the dynamics of the bond prices and taking the expectation under the risk neutral probability measure yields exponential bond price solutions:

$$\begin{aligned} p(\tau, F(t)) &= \exp\left\{-A(\tau) - B(\tau)' F(t)\right\}, \\ p^*(\tau, F(t)) &= \exp\left\{-A^*(\tau) - B^*(\tau)' F(t)\right\}, \end{aligned} \tag{11}$$

where $A(\tau)$ and $A^*(\tau)$ denote scalars and $B(\tau)$, $B^*(\tau)$ are both $\mathbb{R}^3$ vectors (e.g. $B(\tau) = (B_1(\tau), B_2(\tau), B_3(\tau))'$). The loadings are the solution of the following system of ordinary differential equations (ODEs):

$$
\begin{aligned}
\frac{dB(\tau)}{d\tau} &= B_0 - \tilde{K}'B(\tau) - 1/2\sum_{i=1}^{3}\sum_{j=1}^{3}B_i(\tau)\,B_j(\tau)\,a_{ij}, \\
\frac{dA(\tau)}{d\tau} &= A_0 + (\tilde{K}\tilde{\theta})'B(\tau), \\
\frac{dB^*(\tau)}{d\tau} &= B_0^* - \tilde{K}^{*\prime}B^*(\tau) - 1/2\sum_{i=1}^{3}\sum_{j=1}^{3}B_i^*(\tau)\,B_j^*(\tau)\,a_{ij}, \\
\frac{dA^*(\tau)}{d\tau} &= A_0^* + (\tilde{K}^*\tilde{\theta})'B^*(\tau),
\end{aligned}
\tag{12}
$$

where the boundary conditions are defined by equation (2): $B_0 = (1,1,0)'$, $B_0^* = (0,1,1)'$, $A_0 = A_0^* = 0$. These conditions ensure that equation (2) will hold continuously. Initial conditions for this system of ODEs are $B(0) = B^*(0) = (0,0,0)'$, and $A(0) = A^*(0) = 0$. Note that we allow the loading of each factor to be country specific, since prices of risk on each factor risk may differ. We will let the data determine whether financial markets in the two countries price the sources of risk differently. The system of ODEs cannot be solved in closed form. We use a standard fourth order Runge-Kutta procedure to solve for the factor loadings numerically. The procedure is found to be reliable and accurate for the typical problems encountered in the affine term structure literature.

If we have a solution for the bond price loadings, implications for yields (levels, covariances and correlations between yields), term premia and holding period returns follow naturally. Indeed, given that the yield of a maturity τ zero coupon bond is defined as $y(\tau, F(t)) \equiv -\frac{1}{\tau}\ln p(\tau, F(t))$, we are able to express yields as affine functions of the latent state variables:

$$
\begin{aligned}
y(\tau, F(t)) &= \frac{1}{\tau}A(\tau) + \frac{1}{\tau}B(\tau)'\,F(t), \\
y^*(\tau, F(t)) &= \frac{1}{\tau}A^*(\tau) + \frac{1}{\tau}B^*(\tau)'\,F(t).
\end{aligned}
\tag{13}
$$

The covariance between two yields with respective maturities τ_1 and τ_2 can be shown to be:

$$
\begin{aligned}
cov_t\left[y(\tau_1, F(t+h)), y(\tau_2, F(t+h))\right] &= B(\tau_1)'\,var_t\left[F(t+h)\right]B(\tau_2), \\
cov_t\left[y(\tau_1, F(t+h)), y^*(\tau_2, F(t+h))\right] &= B(\tau_1)'\,var_t\left[F(t+h)\right]B^*(\tau_2).
\end{aligned}
\tag{14}
$$

The foreign market case is similar.

While expected holding period returns under the risk neutral measure Q will be equal to the instantaneous riskless interest rate, this will not be the case under the historical probability measure. It is straightforward to show that:

$$
\mathcal{D}^{Q}p(\tau, F(t)) = \mathcal{D}^{P}p(\tau, F(t)) - B(\tau)'\,\Sigma_1\Lambda F(t)\;p(\tau, F(t)), \tag{15}
$$

implying the following affine endogenous risk premia on domestic bonds: $B(\tau)' \Sigma_1 \Lambda F(t)$. Note the impact of the foreign local factor level on domestic term premia (spill-over effect). In this kind of models, the foreign local factor may play a potentially important role in explaining both the dynamics and the term structure of risk premia in international markets. An analogous representation can be derived for the foreign risk premia.

Within the class of affine models the covariance between holding returns is linear in the factors. If we denote the return over a holding period h for a bond with time to maturity τ_i by $hr(h, \tau_i)$, $\tau_i > h$, then the covariance between the holding returns on two bonds with respective maturities τ_1 and τ_2 can be shown to be:

$$
\begin{aligned}
cov_t\left[hr(h,\tau_1), hr(h,\tau_2)\right] &= \int_0^h E_t\left[B(\tau_1 - u)' \Sigma_1 \Sigma_{F(t+u)} \Sigma'_{F(t+u)} \Sigma'_1 B(\tau_2 - u)\right] du, \\
cov_t\left[hr(h,\tau_1), hr^*(h,\tau_2)\right] &= \int_0^h E_t\left[B(\tau_1 - u)' \Sigma_1 \Sigma_{F(t+u)} \Sigma'_{F(t+u)} \Sigma'_1 B^*(\tau_2 - u)\right] du.
\end{aligned}
\tag{16}
$$

The foreign within market case is analogous.

Note that since $E_t\left[B(\tau_1 - u)' \Sigma_1 \Sigma_{F(t+u)} \Sigma'_{F(t+u)} \Sigma_1 B(\tau_2 - u)\right]$ is linear in the factors, the holding return covariance will be linear in the factors and thus by definition time-varying. The covariance matrix of holding returns thus depends on the factor dynamics as well as on the factor loadings (depending on both the factor dynamics and the prices of risk). No closed form expression is available in this setting, since our factors are dependent according to (5). However, the covariance matrix is easily calculated by means of numerical analysis. Obviously, instantaneous correlations can be computed analogously.

2.3 Incorporating the exchange rate dynamics

While the analysis in the previous section suffices to discuss most issues of hedged portfolios, a full description of the characteristics of international portfolios requires the modeling of the exchange rate dynamics. We extend the current version of the model by incorporating the relation between exchange rates and bond prices in the model. This analysis is pursued under the maintained assumption of arbitrage free pricing in conjunction with that of complete markets. However, we explicitly take into account that bond market factors explain little of the exchange rate variability. We do this by assuming driving factors for the exchange rate that are not in the set of the bond market factors.

If markets are complete, exchange rates play a fundamental role in 'equalizing' the domestic and foreign pricing kernels, $M(t)$ and $M^*(t)$, respectively. Formally, as shown in Backus, Foresi and Telmer (1998) and Ahn (1999), the exchange rate, $S(t)$, defined as the home currency price of a unit of foreign currency, obeys the no-arbitrage condition and is endogenously derived as:

$$\frac{S(t+\tau)}{S(t)} = \frac{M^*(t+\tau)}{M(t+\tau)}, \tag{17}$$

or

$$s(t+\tau) - s(t) = m^*(t+\tau) - m(t+\tau), \tag{18}$$

where the small scripts denote logs, $s(t) \equiv \ln S(t)$ and $m(t) \equiv \ln M(t)$.

We assume the following specification for the stochastic part in the pricing kernel dynamics under the historical probability measure:

$$\begin{aligned} \frac{dM(t)}{M(t)} &= -r(t)\,dt - \sum_{i=1}^{3} v_i \sqrt{F_i(t)} dZ_i(t) - \sigma_M dV_1(t), \\ \frac{dM^*(t)}{M^*(t)} &= -r^*(t)\,dt - \sum_{i=1}^{3} v_i^* \sqrt{F_i(t)} dZ_i(t) - \sigma_M^* dV_1^*(t). \end{aligned} \tag{19}$$

Several issues should be emphasized here. First, all three factors enter in the stochastic part of the pricing kernel dynamics. The fact that the foreign local factor appears in the diffusion of the domestic pricing kernel in both countries is essential to the model implications. As argued by Ahn (1999) and given integrated bond markets, domestic investors could possibly care about foreign local shocks to the extent that they are investing in foreign securities. Foreign local shocks do not affect the instantaneous interest rate and hence do not enter the drift term. Since term premia are equal to the covariance between bond returns and the pricing kernel changes, the foreign local factor could still play an important role in explaining both the dynamics and the term structure of risk premia in international markets, to the extent that agents incorporate these (foreign) shocks in their risk evaluation.

Second, in order to analyze the implied exchange rate dynamics implicit in the information of the yield term structure, it will be essential to model the correlation between the risks inherent in the term structure, $dW(t)$, and the risks that drive the pricing kernel. There is clearly no *a priori* link between these pricing kernel risks and the ones that describe the term structure. We model the relation between the pricing kernel shocks and the shocks in the factors by making distinction between pricing kernel shocks that are transmitted into the bond markets, i.e. $dZ(t)$, and those that are not. The latter group of shocks is, by definition, not identifiable within our proposed bond market framework and hence we aggregate them into some measure of aggregate idiosyncratic risk, $dV_1(t)$ and $dV_1^*(t)$. The relation among the factor shocks, $dW(t)$, and the pricing kernel shocks transmitted in the bond market, $dZ(t)$, is allowed to be partially, as proposed by Ahn (1999). This is rather intuitive. The pricing kernel, by definition, prices all assets in the economy (see the beginning of section 2), including, say, stocks. Hence, given the different volatility dynamics between, say, bonds and stocks, the underlying factors are deemed to be substantially different.

More formally, we impose the following instantaneous covariance structure on the shocks:

$$dZ_i dW_i = \rho_i dt, \qquad dZ_i dW_j = 0, \qquad dV_1 dW_i = 0, \qquad dV_1 dV_1^* = 0$$
$$dV_1^* dW_i = 0, \qquad dZ_i dV_1 = 0, \qquad dZ_i dV_1^* = 0, \tag{20}$$

for $i \neq j$, $i = \{1,2,3\}$ and $j = \{1,2,3\}$.

This correlation pattern will be important for the identification of the factors and the specification of the term premia. It will turn out that the magnitude of ρ_i, $i = \{1,2,3\}$, is important from an empirical point of view to end up with a Fama beta coefficient that is substantially different from the one implied by UIP (see below).

Given the definition of the pricing kernel dynamics, we can derive the exchange rate dynamics by applying Itô's lemma to $ds(t)$, $ds(t) = ds\left(m(t), m^*(t)\right)$:[6]

$$\begin{aligned} ds\left(t\right) &= \left(r\left(t\right) - r^*\left(t\right)\right) dt + \frac{1}{2}\sum_{i=1}^{3}\left(v_i^2 - v_i^{*2}\right) F_i\left(t\right) dt + \frac{1}{2}\left(\sigma_M^2 - \sigma_M^{*2}\right) dt \\ &+ \sum_{i=1}^{3}\left(v_i - v_i^*\right)\sqrt{F_i\left(t\right)} dZ_i\left(t\right) + \sigma_M dV_1\left(t\right) - \sigma_M^* dV_1^*\left(t\right). \end{aligned} \tag{21}$$

In **appendix A**, we show that the expected exchange rate return over a discrete time interval is an affine function of the current bond factors, $F\left(t\right)$. A similar result is shown to hold for the conditional variance of the exchange rate changes. Formally for all τ:

$$\begin{aligned} E_t\left[s\left(t+\tau\right) - s\left(t\right)\right] &= A_S\left(\tau\right) + B_S\left(\tau\right)' F\left(t\right), \\ var_t\left[s\left(t+\tau\right) - s\left(t\right)\right] &= A_V\left(\tau\right) + B_V\left(\tau\right)' F\left(t\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} A_S\left(\tau\right) &= \tau\sum_{i=1}^{3}\theta_i\alpha_i\left(1 - \frac{1}{\tau\kappa_i}\left(1 - \exp\left\{-\kappa_i\tau\right\}\right)\right) + \frac{1}{2}\left(\sigma_M^2 - \sigma_M^{*2}\right)\tau, \\ B_{S,i}\left(\tau\right) &= \tau\frac{\alpha_i}{\tau\kappa_i}\left(1 - \exp\left\{-\kappa_i\tau\right\}\right), \qquad i = \{1,2,3\}, \\ \alpha_1 &= 1 + \frac{1}{2}\left(v_1^2 - v_1^{*2}\right), \\ \alpha_2 &= \frac{1}{2}\left(v_2^2 - v_2^{*2}\right), \\ \alpha_3 &= -1 + \frac{1}{2}\left(v_3^2 - v_3^{*2}\right), \end{aligned} \tag{23}$$

[6] Or, equivalent:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \left[\left(r(t) - r^*(t)\right) + \sum_{i=1}^{3}\nu_i\left(\nu_i - \nu_i^*\right) F_i(t) + \sigma_M^2\right] dt \\ &+ \sum_{i=1}^{3}\left(v_i - v_i^*\right)\sqrt{F_i\left(t\right)} dZ_i\left(t\right) + \sigma_M dV_1\left(t\right) - \sigma_M^* dV_1^*\left(t\right). \end{aligned}$$

and

$$A_V(\tau) = \tau \sum_{i=1}^{3} \theta_i (v_i - v_i^*)^2 \left(1 - \frac{1}{\tau\kappa_i}(1 - \exp\{-\kappa_i \tau\})\right) + (\sigma_M^2 + \sigma_M^{*2})\tau,$$

$$B_{V,i}(\tau) = \tau (v_i - v_i^*)^2 \frac{1}{\tau\kappa_i}(1 - \exp\{-\kappa_i \tau\}), \qquad i = \{1, 2, 3\}. \tag{24}$$

Finally, we can easily derive that the market price of risk parameters v_i and λ_i, $i = \{1, 2, 3\}$, are related to each other as:

$$v_i = \frac{\lambda_i / \sigma_i}{\rho_i}, \qquad v_i^* = \frac{\lambda_i^* / \sigma_i}{\rho_i}. \tag{25}$$

This market price of risk equality is important. It allows for the identification of the exchange rate dynamics by means of the bond market factors. It can be considered a projection on the factors of the bond market.

3 Empirical analyses of joint bond markets

3.1 Data

Exponential-affine models are considerably easier to estimate if the data consist of zero coupon yields. Such data are rarely available, except for short term maturities. Therefore, most studies use zero coupon yields that are estimated from prices of (government) coupon bonds. Recently however, researchers often use the yields as implied by LIBOR and swap rates (Piazzesi, 2000, Duffie and Singleton, 1999). Though not strictly without default risk, these may be more relevant than government bond rates, since most interest rate derivatives are priced by means of LIBOR and swap rates. Moreover, they are only minimally affected by credit risk because of their special netting features (Duffie and Huang, 1996).

Monthly observed LIBOR rates of 3, 6 and 12 month maturities are retrieved for the UK and the US (we picked every first Tuesday of each month from Datastream). Swap rates as well for maturities of 2 to 5 years. We then construct the zero-coupon LIBOR yields from the semi-annually coupon based swap rates according to Piazzesi (2000, p. 17).

We constructed 7 series of each 118 monthly observations for each country (14 series in total, $nd = 118$, $N = N^* = 7$). First observation is 07/04/1987, last observation is 02/12/1997. The as such constructed (LIBOR) yield dynamics for the selected maturity set are visualized in **figure** 1. Some descriptive statistics are reported in **table** 1. We collected the monthly exchange rate dynamics from Datastream.

There are substantial differences in the average term structure and in the dynamics of the term structures between the countries. The UK term structure is inverted in the first halve of the sample (and is on average inverted) while the US term structure has a normal inclination for the entire sample. The average yield curve is upward sloping for the US and inverted hump shaped for the UK (with the hump located near the one year maturity). All yields

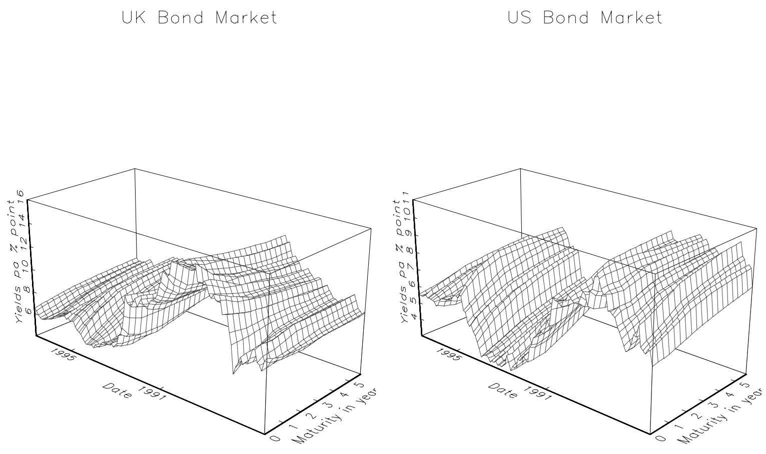

Figure 1: Constructed US-UK (LIBOR) yield data as implied by swap rates.

are very persistent, with first-order monthly autocorrelations between 99.6% and 99.8% (not reported here). The average volatility curve, as measured by the sample standard deviation, is downward sloping for both countries.

The correlations among yields of bonds with differing maturities within and between the national bond markets are also reported. It is noteworthy that the correlations within national bond markets are extremely high (ranging from a low of 89% to a high of almost 100%) and monotonically decreasing with maturity. Focussing on the cross-country correlations, we observe significant and positive correlations, but to a lesser degree and with no such clear pattern.

Given these high correlations the idea originated that a limited set of factors might drive the complete yield curve of a given market. We will report unconditional and conditional evidence to support and motivate this statement. Also, we observe that annualized monthly log exchange rate changes correlate negatively with each yield series. The in absolute value monotonically increasing correlation is weak and does not exceed 5% (10%) between the UK (US) yield series and the log exchange rate changes.

3.2 Unconditional analysis of joint bond markets

3.2.1 Factor Analysis

In order to increase intuition about the proposed conditional model, we will introduce it by estimating its unconditional counterpart. Factor analysis (FA) has been put forward in the

Table 1: **Summary statistics UK-US bond and exchange rate dynamics.**

	$^{uk}_{3m}$	$^{uk}_{6m}$	$^{uk}_{1yr}$	$^{uk}_{2yr}$	$^{uk}_{3yr}$	$^{uk}_{4yr}$	$^{uk}_{5yr}$	$^{us}_{3m}$	$^{us}_{6m}$	$^{us}_{1yr}$	$^{us}_{2yr}$	$^{us}_{3yr}$	$^{us}_{4yr}$	$^{us}_{5yr}$	Δs_{t+1}
Mean	9.33	9.21	9.04	9.09	9.19	9.25	9.30	6.15	6.20	6.34	6.85	7.15	7.39	7.57	-0.005
Std.	3.26	3.10	2.80	2.41	2.14	1.92	1.79	1.93	1.88	1.79	1.73	1.61	1.52	1.44	0.415
								correlations							
$^{uk}_{3m}$	1.00	1.00	0.99	0.96	0.95	0.94	0.94	0.72	0.70	0.67	0.68	0.70	0.71	0.72	-0.01
$^{uk}_{6m}$		1.00	1.00	0.98	0.97	0.96	0.95	0.74	0.72	0.70	0.70	0.72	0.74	0.75	-0.02
$^{uk}_{1yr}$			1.00	0.99	0.98	0.98	0.97	0.76	0.75	0.73	0.74	0.76	0.77	0.78	-0.03
$^{uk}_{2yr}$				1.00	1.00	0.99	0.99	0.79	0.78	0.77	0.79	0.81	0.82	0.83	-0.03
$^{uk}_{3yr}$					1.00	1.00	1.00	0.79	0.78	0.78	0.80	0.82	0.83	0.84	-0.03
$^{uk}_{4yr}$						1.00	1.00	0.77	0.77	0.76	0.79	0.81	0.82	0.84	-0.03
$^{uk}_{5yr}$							1.00	0.76	0.76	0.75	0.78	0.80	0.82	0.83	-0.04
$^{us}_{3m}$								1.00	1.00	0.99	0.96	0.94	0.91	0.89	-0.07
$^{us}_{6m}$									1.00	0.99	0.98	0.95	0.93	0.91	-0.08
$^{us}_{1yr}$										1.00	0.99	0.97	0.95	0.93	-0.08
$^{us}_{2yr}$											1.00	1.00	0.98	0.97	-0.08
$^{us}_{3yr}$												1.00	1.00	0.99	-0.08
$^{us}_{4yr}$													1.00	1.00	-0.08
$^{us}_{5yr}$														1.00	-0.08
Δs_{t+1}															1.00

Means and standard deviations are reported in percentages p.a.. Δs_{t+1} stands for the p.a. monthly change in log exchange rate between period t and $t+1$.

literature as a particularly suitable tool to deal with many noisy and independent (though highly correlated) variables. The technique is robust to the distribution of the underlying factors (unlike the Kalman filter) and the unconditional orthogonality among factors is ensured. The disadvantage is, of course, the sample-specific character of the result and the almost complete lack of economic meaning attached to the factors. In the following, we will tackle the latter critique by imposing economic structure while still keeping the explanatory power of the factors unchanged. The FA model in our case can be stated as follows:

$$\underbrace{Y_t - \mu}_{[2N\times 1]} = \underbrace{B'}_{[2N\times 3]} \underbrace{F}_{[3\times 1]} + \underbrace{\varepsilon_t}_{[2N\times 1]}, \quad (26)$$

where Y_t is the vector of demeaned monthly yield changes, containing the N^* yield series of the foreign country stacked underneath the N series of the domestic country (for convenience we assume $N^* = N$ henceforth), B is the so-called factor loading matrix and F is the vector of factors.

The factor decomposition produced by a factor analysis is not unique. Rotating the factors simultaneously rotates the factor loadings. However, some rotations may be more easily interpreted than the original factors produced by the factor analysis. In line with the proposed model of section 2, we want to change the interpretation of the factors F_t to a 'local UK factor' $\tilde{F}_1$, a 'local US factor' $\tilde{F}_3$ and a factor 'common' to both countries $\tilde{F}_2$. The local domestic factors are defined as having zero loadings on the instantaneous maturity foreign bonds, hence we will need to use some kind of proxy. We impose an orthonormal factor rotation that reflects the desired decomposition. The factor analysis assumptions and the orthogonal rotation employed is explained in detail in **appendix B**.

3.2.2 Discussion

Notice that we assumed in section 2.1 that merely three factors are adequate in describing the joint term structure of the UK-US dataset. This is not obvious *a priori*. We will present some simple unconditional factor analysis evidence in this subsection. Fitting an international three-factor model over the complete time period resulted in a explanatory powers of 97.4% and 91.3% respectively on average over all maturities. To gauge how much is lost by merging two term structures we estimated a three factor model on each market separately. This resulted in explanatory powers of 98.7% and 99.4% on average for the UK and US market. The loss for the UK is negligible (1.3% on average), while the loss for the US market is significantly larger, especially at the short end (8.1% on average). From **table 2**, we conclude that three factors do a fairly good job in describing the joint market dynamics both through time and across the maturity spectrum. This is due to the significant correlations within and across the two countries considered.

Figure 2 reports the cumulative percentage of explained variation in the observed data when respectively 1, 2, and 3 factors were used, averaged over all 14 maturities (a 2-year

Table 2: **Three factors per country versus an international three factor model: How much do we loose ?**

Model	UK	US	UK-US	
# Factors	3	3	3	
Maturity			UK	US
3m	99.6	99.3	96.3	75.6
6m	99.5	98.8	99.3	86.8
1yr	98.8	99.6	95.9	95.8
2yr	96.8	99.5	95.9	98.2
3yr	99.0	99.7	98.7	96.5
4yr	99.1	99.9	98.5	94.7
5yr	98.0	99.4	97.1	91.7
average	98.7	99.4	97.4	91.3

The table presents explained unconditional variance statistics for the complete sample (April 7, 1987 until December 2, 1997). We report results for different settings: 3 factors for the 7 maturity UK sample, the same for the US, and finally the 3-factor 14 maturity case for the joint UK-US market.

moving window was used, for further details see Bliss, 1997). A single factor explains at least 40% throughout the whole period under consideration. Adding two factors raises the interest rate variation explained to at least 90% overall. The ability of a few factors (not tied to any particular theory) to explain changes in interest rates is remarkably constant. From 1993 onwards, cumulative explanatory power was somewhat lower than before.

The previous findings were based on the initial factors and loadings. We want to impose more economic structure on these factors by minimizing the local factor three month loadings with respect to the foreign country, while retaining the orthogonality condition. This effectively alters the interpretation of the factors into local and common factors as discussed above. The explanatory power of the resulting factors is reported in **table 3**. It is seen that cumulative explanatory power remains the same as is to be expected (compare with table 2). The proportions of total explained variances of the foreign yields accounted for by the domestic local factor are small but non-negligible (6.5% and 2.0% respectively on average for UK and US, quasi monotonically increasing towards the long end (hump at the 4 year maturity)). The explanatory power of the common factor increases quasi monotonically towards the long end of the term structure, while the opposite evolution is noted for the local factors in their home country. We conclude that the empirical results support the theoretical specification of our factor decomposition. Also, in our eyes, the importance of the common factor in the total explanatory power of the factor model is the translation of the significant international correlations between these countries. Finally, the effect of the local factor on the foreign country term structure while being small, is non-negligible, possibly signalling

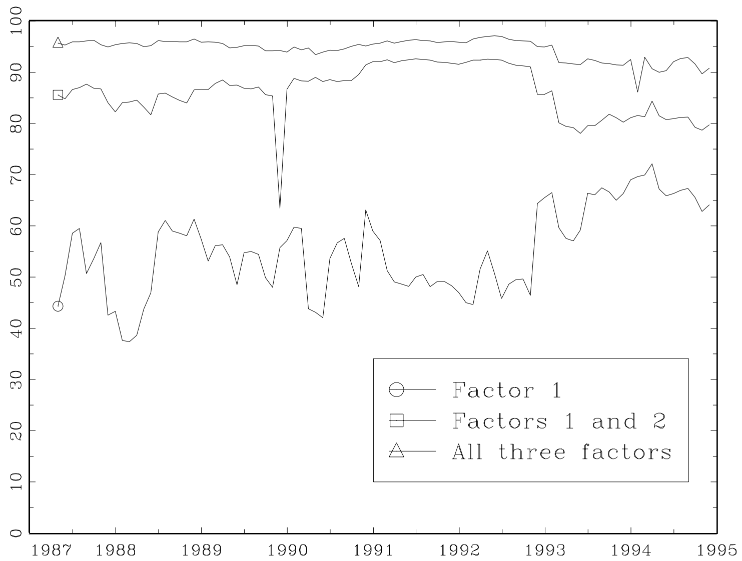

Figure 2: Cumulative variation explained by unconditional factor analysis of UK-US joint term structure (2 year window).

spill-over effects.

There is an important *caveat* to be mentioned with respect to these unconditional results, motivating the research for more formal term structure models. The factors above are possibly inconsistent with the no-arbitrage principle and serve at most as a motivating exercise. We will now report the estimation results from a consistent arbitrage-free model for the joint UK-US term structure where we explicitly allow the foreign factor to play a role in describing the domestic term structure.

3.3 Conditional analysis of joint bond markets

3.3.1 Estimation results

The model parameters have been identified using an approximate Kalman filter and quasi maximum likelihood estimation. The Kalman filter procedure is explained in detail in **appendix C**. The estimated parameters describe three factors that differ in their rates of mean reversion and volatility and combine to reproduce the shape of the joint term structure through time. We show that our three factors, though highly restricted by the nature of the framework in section 2, perform well in fitting the 14 maturity joint term structure. To this end, we report the errors made in fitting the correlations within and between bond markets.

The quasi maximum likelihood parameter estimates under the empirical probability mea-

Table 3: **Explained variance by orthogonally rotated factors.**

Maturity	United Kingdom				United States			
	Total of var. expl.	Proportion explained by			Total of var. expl.	Proportion explained by		
		UK	com	US		UK	com	US
3m	96.3	94.3	2.0	0.0	75.6	0.0	26.7	48.9
6m	99.3	90.0	0.0	0.9	86.8	0.2	34.9	51.8
1yr	95.9	75.7	2.7	1.7	95.8	1.2	44.4	50.3
2yr	95.9	26.7	41.2	9.4	98.2	2.9	51.0	44.3
3yr	98.7	18.3	53.1	9.9	96.5	3.2	52.1	41.2
4yr	98.5	13.5	59.4	12.0	94.7	3.7	52.3	38.7
5yr	97.1	11.4	61.3	11.7	91.7	3.3	52.2	36.1
average	97.4	47.1	43.7	6.5	91.3	2.0	44.8	44.0

We used the Newton Raphson algorithm with step-halving grid search and inverse of Hessian gradient calculation to find the optimal θ_1, θ_2 and θ_3. Convergence after 10 iterations, where the convergence criterion, based on the maximum absolute difference in both parameter and functional values between two successive iterations, is set to 1e-14.

sure P are presented in **table 4**.[7] The estimates characterize a rather slow mean reverting and stable common factor, in contrast to faster mean reverting and volatile local factors. Implied halving times are approximately 8 months, 18 months and 9 months respectively. The corresponding dynamics of the filtered latent state variables together with the implied instantaneous maturity interest rates for both countries are shown in **figure** 3. The θ parameter values seem excessive compared to the 3-month empirical means in **table 1** that serve as a proxy for the riskless interest rates. However, given the non-zero covariance between the factors, this simple relation does not hold anymore. Instead, when using the empirical means from the filtered factor dynamics, we get 9.51% and 5.96% respectively.

The magnitudes of the standard deviations are reasonable, given that we are capturing the information in two term structures by means of only 3 factors. The UK measurement error standard deviation magnitudes are in line with those of a typical one factor model fit, while those of the US can be compared with two factor fits (compare e.g. with tables 4 and 7 in de Jong, 2000). This difference in fit is clearly visible in **figure 6** below. Obviously, including an extra common factor will improve this fit, but this extension is left for future

[7] Technicalities**:** estimation was performed with the GAUSS© constrained maximum likelihood (CML) routine, using Newton-Raphson's algorithm with step-halving line search. The convergence criterion, based on the maximum absolute difference in both parameter and functional values between two successive iterations, is set to 1e-4. Convergence was found to be relaible for a set of diiferent starting values. The (discrete) sampling frequency considered in the estimation is monthly, i.e. $\Delta = 1/12$. When state variables turned zero, we imposed them to be zero. Thus, the Kalman filter estimator is linear only for $F_i(t) \geq 0$. Prior to estimation on our dataset, we checked a simpler version of our program for correctness by replicating the results of Duan and Simonato (1998, the Fama & Bliss dataset) in terms of parameter levels and robust standard errors.

research. The interpretation of h_s differs from a measurement error standard deviation and is discussed in **appendix C.2**.

Table 4: **QML parameter estimates for the UK-US joint bond market.**

	UK factor	Common factor	US factor
κ	1.0444* (0.2117)	0.4515* (0.0690)	0.9427* (0.3461)
θ	0.0101* (0.0022)	0.0925* (0.0027)	0.0045* (0.0021)
σ	0.1489* (0.0205)	0.0563* (0.0049)	0.0679* (0.0366)
λ	-0.1541* (0.3164)	-0.1333* (0.0352)	-0.5424* (0.2300)
λ^*	-0.4912 (0.2392)	-0.1775* (0.0360)	-0.7392* (0.3297)

	UK	US
h_{3m}	0.0020* (0.0002)	0.0037* (0.0002)
h_{6m}	0.0142* (0.0112)	0.0025* (0.0002)
h_{1yr}	0.0021* (0.0001)	0.0017* (0.0001)
h_{2yr}	0.0039* (0.0003)	0.0003* (0.0000)
h_{3yr}	0.0041* (0.0003)	0.0002* (0.0000)
h_{4yr}	0.0043* (0.0003)	0.0713* (0.0000)
h_{5yr}	0.0048* (0.0003)	0.0002* (0.0000)

h_s	0.2493* (0.0661)	ρ_1^2	0.0556 (0.0784)
x_{21}	-0.4290* (0.1395)	ρ_2^2	0.1165 (0.9104)
x_{23}	-1.4070 (1.2838)	ρ_3^2	0.3951 (8.9802)

QML estimates for our international three factor CIR model with monthly observations on 3, 6, 12, 24, 36, 48 and 60 months yield series from April 1987 until December 1997. Robust standard errors between brackets (see **appendix C.4**). A star superscript denotes significance at the 5% level. Mean loglikelihood is 80.0191 (without the constant term in the loglikelihood function and without discarding any initial observations). h_{6m}^{UK} and h_{4yr}^{US} are scaled in the table, true optimal values are 1000 times smaller.

On average, risk premia evaluated at the unconditional factor means θ are monotonically increasing and range from 0.23% for the three month bond to 1.38% for the 5 year bond for the UK and from 0.30% to 1.14% for the US. Almost all market prices of risk parameter estimates are statistically significant, λ_1 is the only exception. This implies that there is significant time variation in the term premia.

The estimated factor loadings are plotted in **figure** 4. The figure portrays the common factor as having an effect on yields that decays slower than its local counterpart as maturity is lengthened. For the UK, the effect of the domestic local factor swiftly dies out and attains less than half of the common factor effect at the five year maturity. For the US, it even turns

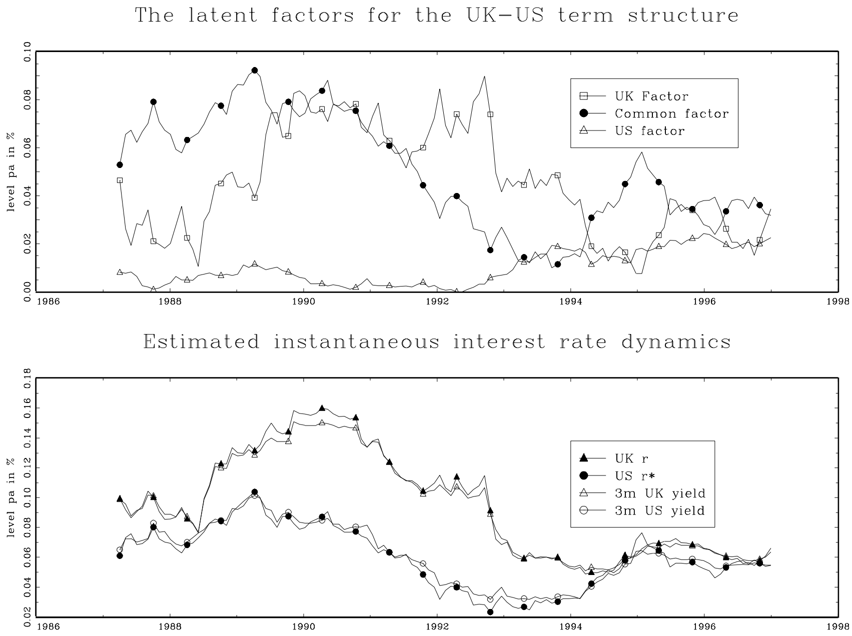

Figure 3: Filtered local and common factors and filtered instantaneous interest rates.

negative at the long end[8]. The foreign local factor, finally, has an increasing negative effect towards the long end and cannot be neglected.

We refrain from presenting a formal Lagrange multiplier test (as in Duan and Simonato, 1999). Nor do we test the model specification by testing the restrictions the model imposes between the cross sectional pricing equations and the dynamics of the factors (see de Jong, 2000). This is left for future research. However, it is by now well established that the class of ('complete') affine models is statistically soundly rejected. Given that a model test is always a joint test of the factor process specification and the market price of risk specification, this apparent rejection has been a major motivation to find more flexible parametrizations of the price of risk (see Duarte, 1999, and Duffee, 2000).

3.3.2 Implications for bond markets

As argued by Dai and Singleton (2000), it is potentially important for a multifactor affine model to permit negative correlation among the factors. We find evidence for negatively correlated factors. The covariance between our factors (σ_{21} and σ_{23}) does not allow us to neatly isolate the interpretation of our three factors however. Rather, we will conduct a standard impulse-response analysis based on the underlying orthogonal Wiener shocks $dW(t)$.

[8] The negative loadings at the long end are caused by the factor dependency. They follow as the solution to the system of ODEs that assures arbitrage free prices.

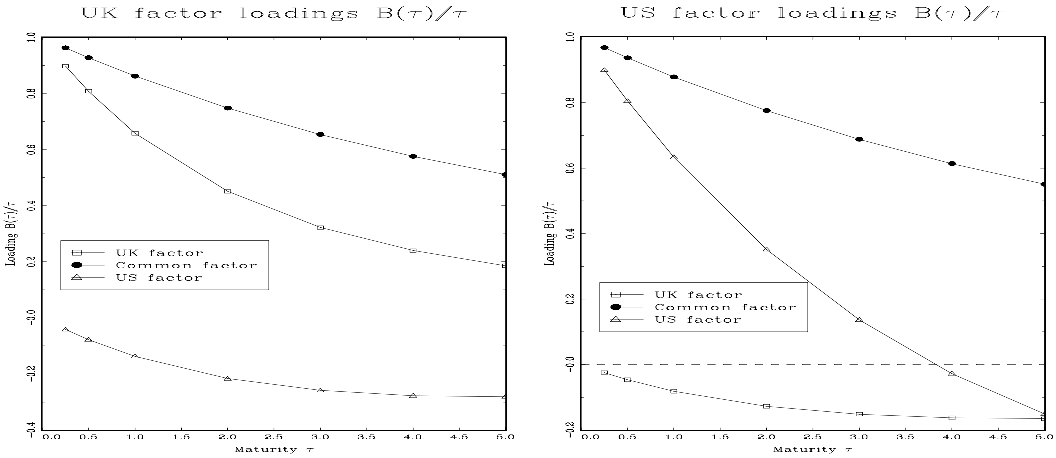

Figure 4: Factor loadings.

The effects of these Wiener shocks are visualized in **figure 5**. Roughly speaking, a Wiener 2 shock shifts both yield curves upwards and represents an international level effect. A Wiener 1 shock mainly tilts and shifts the short end of the UK term structure upwards (where the effect on the US term structure is less pronounced). A Wiener 3 shock tilts the US term structure without shifting it, while resembling something between a level effect and a tilting effect for the UK market. To summarize, an impulse response analysis shows that three types of shocks are crucial to the model: two portfolio shifting shocks and one (common) level effect shock.

Table 5: **Explained variance by factors for the conditional case (April 7, 1987 until December 2, 1997).**

Maturity	UK			US		
	Total of var. expl.	Proportion explained by: dom. & com.	for.	Total of var. expl.	Proportion explained by: dom. & com.	for.
3m	99.6	99.5	0.1	96.5	97.0	-0.5
6m	100.0	99.6	0.4	98.3	97.4	0.9
1yr	99.5	97.7	1.7	99.2	91.9	7.3
2yr	97.4	93.2	4.2	100.0	86.5	13.5
3yr	96.3	87.7	8.5	100.0	77.8	22.2
4yr	95.0	78.4	16.6	100.0	71.5	28.5
5yr	92.8	66.9	26.0	100.0	67.6	32.4
Mean	97.2	89.0	8.2	99.1	84.2	14.9

Table 5 documents the total explanatory power (R^2 statistic) of the international factor

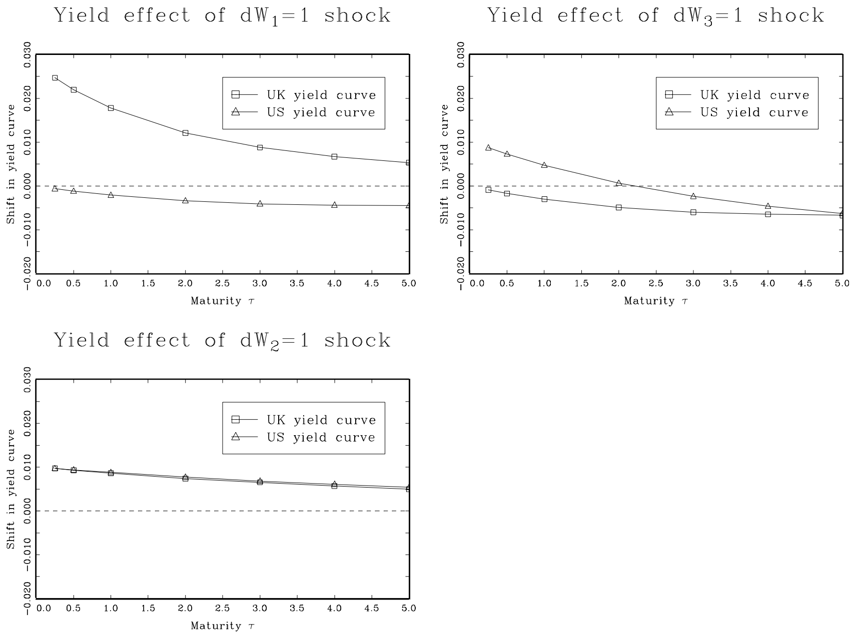

Figure 5: Impulse response analysis: the effects of Wiener shocks on the UK and US term structure.

model and the contribution of each foreign factor versus the combined contribution[9] of the remaining two factors for each country. It can be seen that the three factors together capture most of the variation of the yields in the two countries (ranging from 92.8% to 99.99%). Compared with the unconditional explanatory power in **table 3**, this is superior for the US and the short end of the UK term structure, but somewhat inferior to the long end of the UK term structure. Also, note that the foreign factor accounts for far more variation in the long end than was the case in the unconditional analysis. Remember though that the conditional estimation results are consistent with the absence of arbitrage opportunities, while the unconditional ones are not.

In **figure 6**, we plot the data dynamics together with their model fits. As can be inferred from a visual inspection, the model is able to fit the UK and US yield data rather well, though the fit deteriorates somewhat towards the long (short) end of the UK (US) term structure.

Multi-factor models are designed to model the correlations among the different bonds by a limited number of factors. If multi-factor models are to be used (for pricing, risk management or portfolio performance analysis), these factors should be able to model the observed *international* correlations as well. From **table 6** we see that our model does fairly well in fitting the correlations within, but also between bonds, certainly at the short end of the

[9] The non-zero correlations between the common and domestic factors (x_{21} and x_{23}) make it impossible to decompose the explanatory power further.

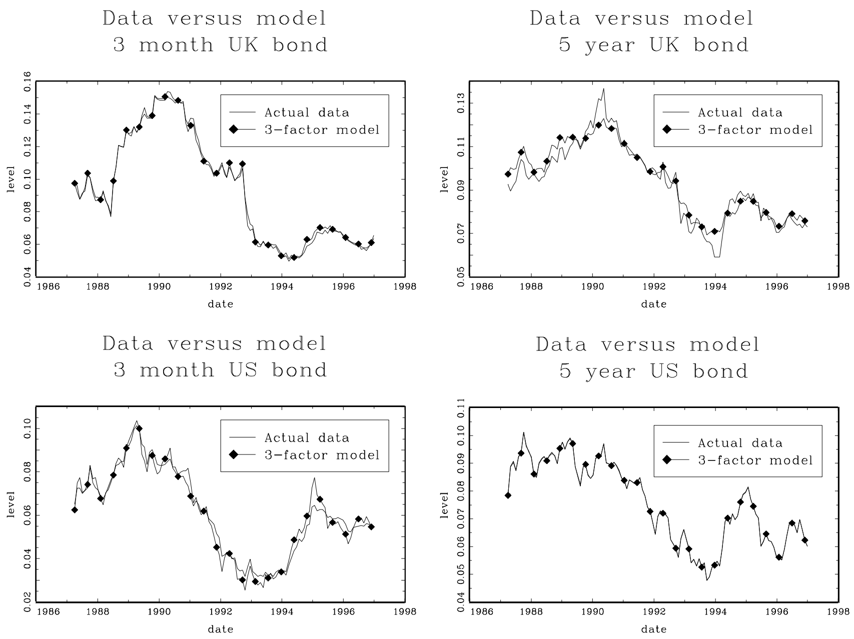

Figure 6: Examples of model fit: short and long end of the UK and US term structures.

joint term structure. In general, the fitting error does not exceed 5% in absolute differences, but is far smaller for most correlations.

3.3.3 Exchange rate risk premium and implications

We included the annualized monthly log exchange rate dynamics as the 15th series. Theoretically the model allows for a Fama beta statistic that substantially differs from the one that is implied by UIP and, hence, one that is more in line with the documented forward premium puzzle literature (Fama, 1984, Fama and Bliss, 1987, Campbell and Shiller, 1991). But what happens if we take the model to the data ? The Fama beta statistic for a τ_i horizon can be expressed as:

$$\beta_{FAMA,i} = \frac{cov\left(\frac{1}{\tau_i}\Delta s\left(t\right), y(\tau_i) - y^*(\tau_i)\right)}{var\left(y(\tau_i) - y^*(\tau_i)\right)}, \tag{27}$$

where

$$cov\left(\frac{1}{\tau_i}\Delta s\left(t\right), y(\tau_i) - y^*(\tau_i)\right) = B_s'\left(\tau_i\right) var_t\left(F(t)\right)\left(B\left(\tau_i\right) - B^*\left(\tau_i\right)\right),$$

$$var\left(y(\tau_i) - y^*(\tau_i)\right) = \left(B\left(\tau_i\right) - B^*\left(\tau_i\right)\right)' var_t\left(F(t)\right)\left(B\left(\tau_i\right) - B^*\left(\tau_i\right)\right).$$

Table 6: **Correlations within and between the UK-US bond markets together with their model fit error.**

	uk *3m*	*uk* *6m*	*uk* *1yr*	*uk* *2yr*	*uk* *3yr*	*uk* *4yr*	*uk* *5yr*	*us* *3m*	*us* *6m*	*us* *1yr*	*us* *2yr*	*us* *3yr*	*us* *4yr*	*us* *5yr*
uk *3m*	·	0.00	0.01	0.02	0.02	0.01	0.01	-0.02	0.01	0.05	0.05	0.03	0.00	-0.02
uk *6m*	1.00	·	0.00	0.01	0.01	0.01	0.00	0.00	0.00	0.04	0.04	0.02	-0.00	-0.02
uk *1yr*	0.99	1.00	·	0.01	0.01	0.00	0.00	0.00	0.00	0.03	0.03	0.02	-0.00	-0.02
uk *2yr*	0.96	0.98	0.99	·	-0.00	-0.00	-0.00	0.00	0.00	0.00	0.02	0.01	-0.01	-0.02
uk *3yr*	0.95	0.97	0.98	1.00	·	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.01	-0.00
uk *4yr*	0.94	0.96	0.98	0.99	1.00	·	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.02
uk *5yr*	0.94	0.95	0.97	0.99	1.00	1.00	·	0.00	0.00	0.00	0.00	0.00	0.00	0.04
us *3m*	0.72	0.74	0.76	0.79	0.79	0.77	0.76	·	0.00	0.01	0.01	0.00	0.00	0.00
us *6m*	0.70	0.72	0.75	0.78	0.78	0.77	0.76	1.00	·	0.00	0.00	0.00	0.00	0.00
us *1yr*	0.67	0.70	0.73	0.77	0.78	0.76	0.75	0.99	0.99	·	0.00	0.00	0.00	0.01
us *2yr*	0.68	0.70	0.74	0.79	0.80	0.79	0.78	0.96	0.98	0.99	·	0.00	0.00	0.01
us *3yr*	0.70	0.72	0.76	0.81	0.82	0.81	0.80	0.94	0.95	0.97	1.00	·	0.00	0.00
us *4yr*	0.71	0.74	0.77	0.82	0.83	0.82	0.82	0.91	0.93	0.95	0.98	1.00	·	0.00
us *5yr*	0.72	0.75	0.78	0.83	0.84	0.84	0.83	0.89	0.91	0.93	0.97	0.99	1.00	·

The lower triangular of this table is the same as the yield correlation subtable in **table 1**. The upper triangular reports the absolute errors made in fitting the correlation matrix with our international 3 factor CIR model (negative numbers imply an underestimation, and *vice versa*). Monthly observed 3, 6, 12, 24, 36, 48 and 60 months yield series from April 1987 until January 1997.

The $\beta_{FAMA,1m}$ statistic equals -0.52 for our dataset. This desirable feature of our common-local factor model cannot be mimicked by traditional factor models of the term structure. In these models, the Fama beta statistic always exceeds 1, by construction.

Indeed, equation (21) in section 2 gives a tentative rational explanation for the rejection of the uncovered interest rate parity hypothesis. Our model implies that UIP is expected to hold only in a risk-neutral world, i.e. if $\lambda_i = 0$ and $\lambda_i^* = 0$ (if all prices of risk are equal pairwise, the same statement holds). In general and in our sample, risk averse agents demand non-zero risk premia, thereby driving the Fama beta statistic away from 1. The model suggests that the interest rate differential will not contain all relevant information with respect to the expected (instantaneous) exchange rate change, since this spread is totally independent of the level of the common factor F_2.

Using equation (22), discrete time exchange rate changes may be analyzed with our factor model. From **figure 7**, it is clear that our factor model is unable to fit the volatile exchange rate return series, both in level and squared. The conclusion would be that bond markets account for little variability in the exchange rate dynamics. This is a common finding in the literature (see e.g. Ahn, 1999) and should not come as a surprise given that we are imposing the bond market factors to fit *predictions* of exchange rate changes and/or volatility and given that more factors than bond factors play a role in this. Compared to a GARCH(1,1) model prediction, our model performs worse in terms of the conditional standard deviation fit. It

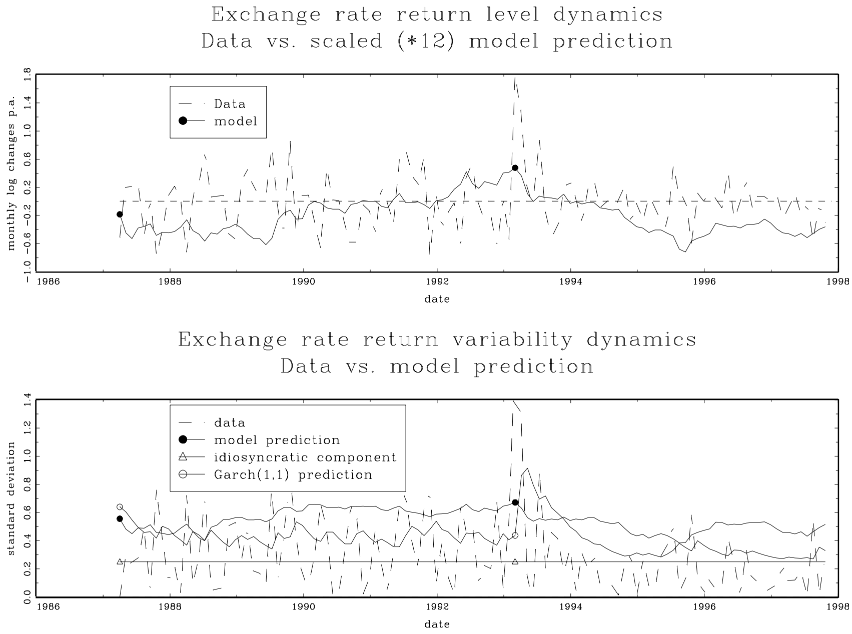

Figure 7: Exchange rate return related model fit: predicted level and standard deviation versus sample realization.

can be seen that our estimates are somewhat biased upwards. This may be due to the rigid specification of the idiosyncratic risk.

For completeness, we report endogenous term premia in **table 7** (evaluated at the unconditional factor means). We compute term premia as the negative of the covariance between the pricing kernel returns and the bond returns. The viewpoint taken here is that of a UK investor who is able to invest in either UK bonds or US bonds, where the latter are to be converted to the UK currency. In our model, the domestic currency foreign bond returns contain an extra premium to compensate for the exchange rate risk. The fact that exchange rate risk is priced is in line with the international arbitrage pricing theory as discussed in Dumas and Solnik (1995).

In general, bond premia are monotonically increasing over the maturity spectrum (there is a slight hump at the long end of the US curve). The premia are similar in size across the two countries. The UK investor that invests in the US will reap an extra premium for bearing exchange rate risk. This premium amounts to 94 basis points evaluated at the unconditional factor means. This implies that the combined reward for investing abroad is far higher compared to investing domestically. The risk of an international investment is higher as well though, and there is an idiosyncratic part to the exchange rate risk, not priced in the market. Idiosyncratic risk is defined here as the risk that is orthogonal to the bond market

Table 7: **Endogenous term premia for a UK investor with an instantaneous investment horizon.**

Maturity	Term premia UK bond investment	Term premia US bond investment		
	total (bp)	bond	exch.r	total (bp)
3m	22.6	24.9	94.2	119.1
6m	42.0	46.7	94.2	140.9
1yr	72.7	81.7	94.2	175.9
2yr	110.7	124.9	94.2	219.1
3yr	129.1	142.3	94.2	236.5
4yr	136.4	142.9	94.2	237.1
5yr	137.6	132.7	94.2	226.9

Term premia are denoted in basis points (bp).

factors (see the discussion in section 2.3). We necessarily assumed it to be constant. Since this risk remains unpriced, it will serve as a deterrent for international bond diversification. We will investigate in the next section whether the exchange rate is still sufficiently priced for an agent to place some weight on foreign bonds, when given the opportunity. Obviously, in a MVE setting, there is a trade-off between the additional spanning possibilities of foreign bonds, and the unpriced exchange rate risk.

4 Application: the gains from international diversification

Taking the viewpoint of a UK-investor who initially only invests in the UK, we study the question whether this investor can enhance the mean-variance characteristics of his portfolio by also investing in the US market, with an instantaneous investment horizon and conditional on the information that is present in both term structures. As such we can evaluate the potential gains from diversification for any term structure constellation in the home and foreign market. This conditional approach is not feasible within the traditional mean-variance type of analysis. We concentrate on the MVE frontiers that arise from dynamic investment strategies on the shortest possible time horizon. The instantaneous investment horizon is in line with the principle of dynamic trading (the cornerstone of the success of the derivative pricing theory) where investors are able to rebalance their portfolio continuously. Other instantaneous versus discrete time portfolio management issues are discussed in Nielsen and Vassalou (1998). Finally, we assume perfect markets and thus leave the incorporation of transaction costs and short sale constraints for future research.

The UK investor is allowed to invest in the 3 month, 6 month, 1, 2, 3, 4 and 5 year maturity UK and US bonds. Obviously, he will only be interested in the UK currency denominated holding period returns of these bonds. Hence, a first step in this analysis consists of generating the local currency bond returns of domestic and foreign bonds. Using Itô's lemma and the no-arbitrage condition, the instantaneous return on investment in domestic bonds with maturity τ is given by:

$$\frac{dp\left(\tau,F(t)\right)}{p\left(\tau,F(t)\right)}=\left(r\left(t\right)-B\left(\tau\right)'\Sigma_1\Lambda F(t)\right)dt-B\left(\tau\right)'\Sigma_1\Sigma_F dW\left(t\right). \tag{28}$$

Next to the investment proceeds from investing in domestic bonds, investors have access to returns obtained from investments abroad. The local currency dynamics of these investments can be derived rather straightforwardly by solving for $d\left(S\left(t\right)p^*\left(\tau,F(t)\right)\right)$ using Itô's lemma:

$$\begin{aligned}\frac{d\left(S\left(t\right)p^*\left(\tau,F(t)\right)\right)}{S\left(t\right)p^*\left(\tau,F(t)\right)}=&\left[\left(r\left(t\right)-B^*\left(\tau\right)'\Sigma_1\Lambda F(t)\right)+\sum_{i=1}^{3}\nu_i\left(\nu_i-\nu_i^*\right)F_i\left(t\right)\right]dt\\ &-B^*\left(\tau\right)'\Sigma_1\Sigma_F dW\left(t\right)+\sum_{i=1}^{3}\left(\nu_i-\nu_i^*\right)\sqrt{F_i\left(t\right)}dZ_i\left(t\right)+\sigma_M dV_1-\sigma_M^* dV_1^*.\end{aligned} \tag{29}$$

Several issues are noteworthy in this last equation. First, note that the prices of factor risks that enter in the excess returns of domestic currency denominated foreign bonds are those of the home country, Λ. This transformation of foreign prices of risk Λ^* to domestic prices of risk Λ is due to the exchange rate dynamics. The role of the exchange rate in this type of models is exactly that of changing this factor prices of risk to the one of the local currency (see the discussion in section 2). Second, exchange rate risk is priced (rewarded), as shown by the additional expected return $\sum_{i=1}^{3} v_i(v_i-v_i^*)F_i\left(t\right)dt$. This is in line with the international arbitrage pricing theory literature (see Dumas and Solnik, 1995 and the references therein). Third, the dynamics of domestic currency foreign bond prices are not spanned by the domestic bond prices since additional stochastic components, $dZ(t)$, enter. These shocks do not have perfect correlation with the shocks hitting the local bond market, $dW(t)$. In principle, then, there could be gains from diversification, depending on the prices of exchange rate risk. Fourth, the risk that remains non-priced in the model $(\sigma_M dV_1-\sigma_M^* dV_1^*)$ hampers the foreign bond investment. However, this risk can be hedged by taking opposite positions in foreign bonds of different maturities, hence the potential gain remains an open question.

In order to assess the potential gains to international diversification we analyze the difference between the MVE frontier of the local bond market and the frontier arising in the international bond market case. These MVE frontiers can be constructed easily given the above price dynamics[10]. Suppose we have a portfolio with local currency value $\mathcal{P}\left(t\right)=\omega'\bar{p}\left(t\right)$ with $\bar{p}\left(t\right)$ an $[2N\times 1]$ vector of local currency prices of domestic and foreign bonds. The price dynamics of the domestic currency portfolio can be written concisely as:

[10] Note that a direct closed form solution for the minimum-variance portfolio (Campbell, Lo and MacKinlay, 1997, pp. 184-185) turned out to be unfeasible. The highly correlated bond returns caused the portfolio variance matrix to become non-invertible. We solved the problem indirectly instead making use of the Gauss© maximum likelihood routine.

$$\frac{d\mathcal{P}(t)}{\mathcal{P}(t)} = \omega' \frac{d\bar{p}(t)}{\bar{p}(t)} = \omega' C_1 F(t)\, dt + \omega' C_2 \Sigma_1 \Sigma_F dW(t) + \omega' C_3 C_4 dZ(t) + \omega' C_5 dV_2, \tag{30}$$

where C_1, C_2, C_3, C_4 and C_5 denote auxiliary matrices, defined as follows. Noting that $r(t) = F_1(t) + F_2(t)$ we stack the results for each maturity i in the ith row of C_1, e.g. $[1,1,0] - B(\tau_i)' \Sigma_1 \Lambda$. The foreign country equivalent is stacked immediately underneath these, e.g. $[1,1,0] - B^*(\tau_i)' \Sigma_1 \Lambda + (v_1(v_1 - v_1^*), v_2(v_2 - v_2^*), v_3(v_3 - v_3^*))$. Analogously, let C_2 be an $[2N \times 3]$ matrix stacking the $-B(\tau_i)'$ rows above the $-B^*(\tau_i)'$ rows. The $[2N \times 3]$ matrix C_3 contains on the last N rows the vector $((v_1 - v_1^*), (v_2 - v_2^*), (v_3 - v_3^*))$ and zero rows otherwise. Finally, define C_4 as $diag(\sqrt{F_1(t)}, \sqrt{F_2(t)}, \sqrt{F_3(t)})$, ρ as $diag(\rho_1, \rho_2, \rho_3)$ and C_5 as the identity matrix with the idiosyncratic variance, h_s^2, on its diagonal. Finally, V_2 is just a standard Wiener process.

From this equation we can easily recover the instantaneous expected return and variance of the portfolio. More specifically:

$$E_t\left[\frac{d\mathcal{P}(t)}{\mathcal{P}(t)}\right] = \omega' C_1 F(t)\, dt$$

$$E_t\left[\left(\frac{d\mathcal{P}(t)}{\mathcal{P}(t)}\right)^2\right] = \omega' C_2 \Sigma_1 \Sigma_F \Sigma_F' \Sigma_1' C_2' \omega dt + 2\omega' C_2 \Sigma_1 \Sigma_F \rho C_4' C_3' \omega dt \tag{31}$$

$$+\omega' C_3 C_4 C_4' C_3' \omega dt + \omega' C_5 \omega dt.$$

Figure 8 shows this shift for a number of time points. In general, the graphs illustrate that gains from diversification are considerable and (though less visible) time-dependent. Note that we have used the instantaneous variance rather than the instantaneous standard deviation on the horizontal axis. The curve almost becomes a straight line when we put standard deviations on the horizontal axis, in line with what we would expect.

Obviously, there is a trade-off between the additional spanning possibilities and the incurred unpriced exchange rate risk implied by the inclusion of foreign bonds in an unhedged portfolio. In essence, the outcome will depend on whether or not the exchange rate risk is sufficiently priced in the market. The part of the exchange rate risk that is orthogonal to the bond market factors -the idiosyncratic risk- remains unpriced in our model and will serve as a deterrent for international bond diversification. Apparently, the (non-priced) idiosyncratic exchange rate risk does not outweigh the increased spanning possibilities of the inclusion of foreign bonds for an investor that cares about mean and variance. Put differently, we have found that exchange rate risk is sufficiently priced to make international bond portfolios superior to domestic ones in their trade-off between risk and return. This model implication corroborates the so-called home bias puzzle (French and Poterba, 1991). It is puzzling why portfolios are overwhelmingly dominated by domestic assets, though there are gains to be reaped from investing abroad in a MVE sense. It goes without saying that the above graphs

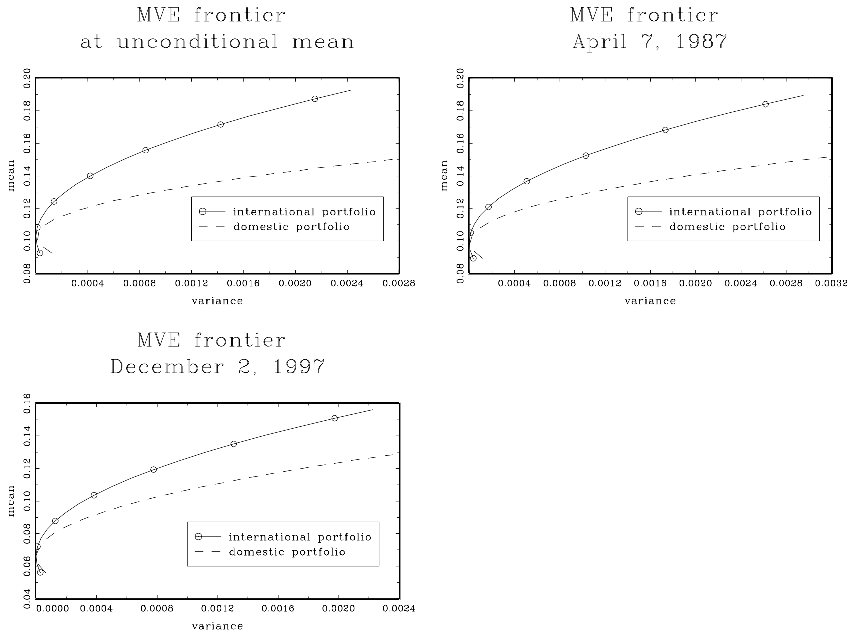

Figure 8: *Conditional* MVE frontiers from a UK investor viewpoint (instantaneous investment horizons).

are merely informal evidence for an enlarged efficient set. We leave for future research the more formal mean-variance spanning tests to analyze whether this is a significant result (in the statistical sense, see de Roon, 1997).

5 Conclusions

Why should one care about yet another affine term structure model ? First, because our model is rather successful in capturing the *joint* bond market dynamics across time and across the maturity spectrum, and this despite its tight parametrization. Second, because it presents theoretical arguments as to why rational deviations from UIP are to be expected (moreover, this theoretical argument is corroborated by direct evidence). Finally, because our model allows for (international finance) applications that condition upon the information that is contained in the term structure, and this within a consistent and arbitrage-free framework.

The model is estimated using a standard Kalman filter algorithm. The empirical results provide evidence in favor of a limited number of factors, driving the joint bond market. Both the conditional and unconditional analysis attribute more than 90 percent of the dynamics of the yield curves to three factors. The foreign local factor explains some of the dynamics of the yield curve, especially at the longer end of the spectrum.

The model reports significant statistical evidence for the existence of time-varying risk premia. The latter provide a tentative explanation for the deviation from UIP. In line with Dumas and Solnik (1995) we find that exchange rate risk is priced in international asset markets. Moreover, the remaining non-priced idiosyncratic part of the exchange rate risk seems not to outweigh the increased spanning possibilities when the portfolio of a mean-variance optimizing agent is allowed to include foreign bonds as well. In other words, we find that exchange rate risk is sufficiently priced to make international bond portfolios superior to domestic ones in their trade-off between risk and return (at least at the instantaneous investment horizon). Finally, the gains to diversification are time-dependent, pleading for an active international portfolio management, and they are in line with the reported home bias puzzle in asset allocation (French and Poterba, 1991).

Further research should present more formal tests about some of the model claims. An interesting extension is the inclusion of a second common factor. This will enable capturing less strongly correlated term structures (Germany-US e.g.). More applications may be elaborated conditional on the term structure information (obvious candidates are international portfolio conditional Value at Risk or the pricing of cross-country interest rate derivatives). Alternative model specifications concern the 'complete' and 'essential' affine case (for a definition see Duffee, 2000) or the Gaussian case with exact maximum likelihood. Finally, an important and challenging extension could be the inclusion of stocks in the information set, thereby pouring more identifying information into the model concerning the pricing kernel specification (see Bekaert and Grenadier, 1999).

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APPENDICES

Appendix A: Exchange Rate Calculations

Here we derive the result that the expected discrete-time exchange rate change retains the linear factor structure within the CIR-class of models. We start from the instantaneous exchange rate dynamics, i.e. equation (21) in the main text and take conditional expectations:

$$E_t\left[ds\left(t\right)\right] = \left[\left(F_1\left(t\right) - F_3\left(t\right)\right) + \frac{1}{2}\sum_{i=1}^{3}\left(v_i^2 - v_i^{*2}\right)F_i\left(t\right) + \frac{1}{2}\left(\sigma_M^2 - \sigma_M^{*2}\right)\right]dt. \qquad (32)$$

Defining

$$\begin{aligned} \alpha_1 &= 1 + \frac{1}{2}\left(v_1^2 - v_1^{*2}\right) \\ \alpha_2 &= \frac{1}{2}\left(v_2^2 - v_2^{*2}\right) \\ \alpha_3 &= -1 + \frac{1}{2}\left(v_3^2 - v_3^{*2}\right) \end{aligned} \qquad (33)$$

we can rewrite the above conditional expectation as:

$$E_t\left[ds\left(t\right)\right] = \sum_{i=1}^{3}\alpha_i F_i\left(t\right)dt + \frac{1}{2}\left(\sigma_M^2 - \sigma_M^{*2}\right)dt. \qquad (34)$$

By definition, $E_t[s\left(t+\tau\right) - s\left(t\right)] = \int_t^{t+\tau} E_t\left[ds\left(u\right)\right]$, such that plugging in

$$E_t[s\left(t+\tau\right) - s\left(t\right)] \equiv \int_t^{t+\tau} E_t\left[ds\left(u\right)\right] = \sum_{i=1}^{3}\alpha_i \int_t^{t+\tau} E_t\left[F_i\left(u\right)\right]du + \frac{1}{2}\left(\sigma_M^2 - \sigma_M^{*2}\right)\tau. \qquad (35)$$

Note that there are easy closed form expressions for the conditional expectations of the factors under the assumption of square root dynamics for the factors (see CIR, 1985):

$$E_t\left[F_i\left(t+\tau\right)\right] = \theta_i + \exp\left\{-\kappa_i\tau\right\}\left(F_i\left(t\right) - \theta_i\right). \qquad (36)$$

Plugging in these conditional expectations and integrating subsequently results in the desired linear factor structure, as stated in the upper panel of equation (22). ■

An analogous reasoning yields the lower panel of equation (22).

Appendix B: Factor Analysis and Orthogonal Rotation

The FA model in our case can be stated as follows:

$$\underbrace{Y_t - \mu}_{[2N \times 1]} = \underbrace{B'}_{[2N \times 3]} \underbrace{F}_{[3 \times 1]} + \underbrace{\varepsilon_t}_{[2N \times 1]}, \tag{37}$$

where Y_t is the vector of demeaned monthly yield changes, containing the N^* yield series of the foreign country stacked underneath the N series of the domestic country (for convenience we assume $N^* = N$ henceforth), B is the so-called factor loading matrix and F is the vector of factors.

The random vectors $Y_{t,}$ F and ε_t are assumed to satisfy :

$$\begin{aligned} E(F) &= 0, & Cov(F) &= I, \\ E(Y_t) &= \mu, & Cov(Y_t) &= \Sigma \\ E(\varepsilon_t) &= 0, & Cov(\varepsilon_t) &= \Pi, \end{aligned} \tag{38}$$

where Π is assumed to be diagonal (it should always be verified that the off diagonal elements are indeed negligible). Spectral decomposition states that any real symmetric matrix, such as the yield covariance matrix Σ, can be diagonalized into the form:

$$\Sigma = V\Lambda V', \tag{39}$$

where Λ is the diagonal matrix of ordered eigenvalues of Σ such that $\Lambda_{1,1} > \Lambda_{2,2} > ... > \Lambda_{2N,2N}$ and the columns of V are the corresponding $[2N \times 1]$ orthonormal eigenvectors of Σ. For our purposes, we take out the three biggest eigenvalues together with their corresponding eigenvectors V_1, V_2 and V_3. Then the factor loading matrix equals:

$$B = \left(\sqrt{\Lambda_{1,1}}V_1, \sqrt{\Lambda_{2,2}}V_2, \sqrt{\Lambda_{3,3}}V_3\right)'. \tag{40}$$

In the context of factor models, the factors are typically portfolios of traded securities. Here they are just linear combinations of yields, with weighting vectors $V_1/\sqrt{\Lambda_{1,1}}$, $V_2/\sqrt{\Lambda_{2,2}}$, $V_3/\sqrt{\Lambda_{3,3}}$ respectively:

$$F_t = \left(\frac{V_1}{\sqrt{\Lambda_{1,1}}}, \frac{V_2}{\sqrt{\Lambda_{2,2}}}, \frac{V_3}{\sqrt{\Lambda_{3,3}}}\right)' \cdot \mathbf{Y}_t. \tag{41}$$

The rotation matrix T has 9 elements but the orthogonality assumption reduces the free parameters to three (Bliss, 1997). We built T from three orthogonal rotation matrices, each leaving one column of B unchanged and recombining the two remaining columns, while preserving the orthogonal structure of the resulting $\tilde{B}$. Defining:

$$T_1(\theta_1) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix},$$
$$T_3(\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix}, \qquad T(\theta)=T_1(\theta_1)T_2(\theta_2)T_3(\theta_3) \tag{42}$$

then $T(\theta)$ is an orthogonal matrix with free parameters $\theta = (\theta_1, \theta_2, \theta_3)'$. A nonlinear optimizer was used to search over feasible values of $\theta_1, \theta_2, \theta_3$ (in the range of $-\pi$ to $+\pi$ radials) to find the values that minimize the absolute value of the domestic factor loadings on the foreign local factor, *id est* elements $\tilde{B}_{N+1,1}$ and $\tilde{B}_{1,3}$. The orthogonality ensures that our rotated factors keep explaining as much variability as the original ones, while economic structure is imposed on the $\tilde{B}$. Since $T \cdot T' = I$, it follows that

$$B'F = B'\left(TT'\right)F = \tilde{B}'\tilde{F}, \tag{43}$$

or, the *rotated* factor loadings $\tilde{B}' = B'T$ and factors $\tilde{F} = T'F$ explain as much as the original ones.

By definition, a local factor of one country should have no direct effect on the instantaneous maturity bond of the other country. The instantaneous maturity bond will be proxied here in the unconditional analysis by means of the three month maturity bond (this is not the case in the proposed model). We assume the first rotated factor to be the UK local factor, the second the common and the third the US local factor. To implement this decomposition, we solve the following simple optimization problem:

$$\underset{\theta}{Min}\ \ \tilde{B}^2_{N+1,1} + \tilde{B}^2_{1,3} \tag{44}$$

$$s.t. \tag{45}$$

$$\tilde{B} = B'T(\theta).$$

In other words, we are trying to find economic meaningful factors and loadings while preserving the covariance matrix (and hence the cumulative explanatory power). Note that the economic meaningfulness is discretionary to some extent and should be interpreted as conforming to the model in section 2. We could have chosen to minimize $\tilde{B}'_{[1:N,1]} \cdot \tilde{B}_{[1:N,1]} + \tilde{B}'_{[N+1,3]} \cdot \tilde{B}_{[N+1,3]}$ instead, implying a different definition for 'local' factors. Also, we could have minimized the variance of $\tilde{B}'_{[1:N,2]}$ and $\tilde{B}'_{[N+1:2N,2]}$, forcing the second factor to resemble a level effect. But, in order to be consistent with the model that is developed in section 2, we choose (45).

Appendix C: QML and the approximate Kalman filter algorithm

C.1 Introduction and related literature

'Data' is just another word for 'measurements'. Hence, filtering unobserved state variables from (panel) data necessitates the assumption of no measurement errors or the use of (approximate[11]) filtering techniques. The first approach has been followed by e.g. Pearson and Sun (1994) and Chen and Scott (1993) and relies on an inversion of the yield curve such that the state variables can be backed out subsequently. Obviously, the choice of maturities used in the inversion procedure is not innocuous and the results are potentially very sensitive to the particular choice made. The second approach is to assume that all the yield series are measured with some measurement (or observation) error. The presence of these measurement errors means that our estimated model is built up by a theoretical term structure model and a measurement error specification. In this case one has to rely on a filtering procedure to back out the state variables from the yield curve. The main reason for using the Kalman filter is that the factors are unobservable and must be estimated from measurements contaminated with errors. The advantages of the Kalman filter algorithm are numerous in this setting: more efficient estimation, full use of conditioning information and computational efficiency due to its recursive character. For literature to which this technique is related, see (chronologically) Harvey, 1989, Pennacchi, 1991, Chen and Scott, 1995, Santa-Clara, 1995, Jegadeesh and Pennacchi, 1996, Ball and Torous, 1996, Gong and Remolano, 1996, Gong and Remolano, 1997, Lund, 1997, Duan and Simonato, 1999, Babbs and Nowman, 1999, and de Jong, 2000).

C.2 State space representation of the statistical term structure model

The state space representation of a model consists of a measurement equation and a transition equation. The measurement equation models the cross-sectional variation in the data while the transition equation models the time series properties of the data, implied by the factor dynamics.

Model:

$$YY_t = AA + BB' F_t + \xi_t, \qquad \xi_t \sim \mathcal{N}(0, H) \tag{46}$$

$$F_{t+\Delta} = \Phi F_t + \eta_{t+\Delta}, \qquad \eta_{t+\Delta|t} \sim \mathcal{N}(0, Q_t). \tag{47}$$

Denote the $[(2N+1)\times 1]$ data matrix as $YY_t = [Y_t(\tau_1), ..., Y_t(\tau_N), Y_t^*(\tau_1), ..., Y_t^*(\tau_N), \Delta s_t]'$, where Δs_t is the annualized monthly log exchange rate change one period forward, $t =$

[11] An exact filtering algorithm is computationally infeasible (see Lund, 1997). Instead researchers rely on the linear Kalman filter and (quasi) maximum likelihood estimation, henceforth (Q)ML.

$\{1, ..., nd\}$. AA, BB, F_t and ξ_t are $[1 \times (2N+1)]$, $[3 \times (2N+1)]$, $[3 \times 1]$ and $[(2N+1) \times 1]$ matrices. AA is constructed by stacking $\frac{1}{\tau}A(\tau), \frac{1}{\tau}A^*(\tau)$ and $A_s(\tau)$ next to each other, $BB(\tau)$ analogously.

For simplicity, we assume that the measurement errors ξ_t are cross-sectionally and intertemporally uncorrelated.[12] H is a diagonal $[(2N+1) \times (2N+1)]$ matrix with parameters $h_i^2(UK)$, $h_i^2(US)$, $i = \{1, ..., N\}$, and h_e^2.[13] The assumption of differences in the diagonal elements is reasonable given that trading activity and thus the bid-ask spread is not equal across different maturities and bond markets.

All model parameters, as summarized in **table 4**, are gathered in the parameter vector ϑ.

C.3 The exact Kalman filter algorithm

Initial condition for stationary state variable vector:

$$\hat{F}_0 = E\left[F_t\right], \qquad \hat{P}_0 = var\left[F_t\right]. \tag{48}$$

Predict:

$$\hat{F}_{t+\Delta|t} = E_t\left[F_{t+\Delta}\right] = \Phi \hat{F}_t, \qquad \hat{P}_{t+\Delta|t} = var_t\left[F_{t+\Delta}\right] = \Phi \hat{P}_t \Phi' + Q_t. \tag{49}$$

Update:

$$\hat{F}_{t+\Delta} = E_{t+1}\left[F_{t+\Delta}\right] = \hat{F}_{t+\Delta|t} + K_{t+\Delta} u_{t+\Delta}, \qquad \hat{P}_{t+\Delta} = var_{t+1}\left[F_{t+\Delta}\right] = L_t \hat{P}_{t+\Delta|t}, \tag{50}$$

where

$$K_{t+\Delta} = \hat{P}_{t+\Delta|t} BB V_{t+1}^{-1}, \qquad L_{t+\Delta} = I_{3\times3} - K_{t+\Delta} BB'. \tag{51}$$

Likelihood contribution:

$$u_{t+\Delta} = YY_{t+\Delta} - AA - BB' \hat{F}_{t+\Delta|t},$$

$$V_{t+\Delta} = BB' \hat{P}_{t+\Delta|t} BB + H,$$

and

$$-2 \ln L_t = -2 l_t = \ln |V_t| + u_t' V_t^{-1} u_t. \tag{52}$$

[12] We can conceptually generalize the model to one where the errors are cross-sectionally dependent by filling up H, but we refrain from this given the computational burden of $\frac{(2N+1)(2N)}{2}$ additional parameters.

[13] The measurement error standard deviation interpretation does not hold for h_s and h_e. In this paper we define $h_e^2 = var_t\left[s(t+\tau) - s(t)\right] + h_s^2$. The error implied by the bond factor model also incorporates a prediction error. Given that this error cannot be identified by definition in our model, we aggregate it into one single source of risk, h_s.

Distribution of the estimator $\hat{\vartheta}$:

Following Monte Carlo based evidence in Duan and Simonato (1999) and de Jong (2000), we assume $\hat{\vartheta}$ to be asymptotically consistent, efficient and normally distributed (see following discussion). The robust standard errors reported in **table 4** are derived from the so-called robust covariance matrix below:

$$\sqrt{nd}\left(\hat{\vartheta}_{nd} - \vartheta_0\right) \sim \mathcal{N}\left(0, \Sigma_A^{-1} \cdot \Sigma_B \cdot \Sigma_A^{-1}\right), \tag{53}$$

where

$$\begin{cases} \Sigma_B \equiv \dfrac{1}{nd} \displaystyle\sum_{t=1}^{nd} \left(\dfrac{\partial l_t}{\partial \vartheta}\right)' \left(\dfrac{\partial l_t}{\partial \vartheta}\right) \\ \\ \Sigma_A \equiv \dfrac{1}{nd} \displaystyle\sum_{t=1}^{nd} f_t, \end{cases} \tag{54}$$

with

$$f_t = \frac{\partial \mu_t}{\partial \vartheta}' \Omega^{-1} \frac{\partial \mu_t}{\partial \vartheta} + \tfrac{1}{2} \frac{\partial \Omega'}{\partial \vartheta} \left[\Omega^{-1} \otimes \Omega^{-1}\right] \frac{\partial \Omega}{\partial \vartheta}. \tag{55}$$

where $\otimes$ stands for the Kronecker product and $\frac{\partial l_t}{\partial \vartheta}, \frac{\partial \mu_t}{\partial \vartheta}, \frac{\partial \Omega}{\partial \vartheta}$ and f_t are matrices of dimension $[1 \times np]$, $[(2N+1) \times np]$, $[(2N+1)^2 \times np]$ and $[np \times np]$ respectively (np stands for the number of model parameters in $\hat{\vartheta}$).

The Kalman filter is a recursive algorithm for computing the mathematical expectation of a hidden state vector F_t, conditional on observing a history of noisy signals on the hidden state, $E_t[F_t|YY_t, YY_{t-1}, ..., YY_0]$. Information about F_t stems from two sources: the predicted value (determined by the historic term structures and the assumed factor dynamics) and the observed term structure at time t. Both sources contain some error. The prediction because of the innovations in the state variables between t and $t+1$ as well as the uncertainty about the estimate, the observations because of potential measurement errors. The optimal linear predictor is formed by combining these two pieces of information, and the Kalman filter solves this problem. The weights given to the interest rates and to the prediction depend on the relative sizes of their covariance matrices, so if the interest rates are very noisy, the updated estimated predictor will differ only little from the ex ante predictor. To summarize, the Kalman filter procedure updates the estimation every time a new observation becomes available. The filter first forms an optimal linear predictor of the unobserved state variables, conditional on the previous estimated values. These estimates for the unobserved state variables are then updated using the information provided by the observed variables. As a by-product the filter provides fitting errors together with their conditional variance that can be used to construct the quasi-loglikelihood function, which in its turn can be used to find ML estimates of the parameters governing the dynamics of the state variables and the yield curve.

C.4 Statistical inference considerations

The intertemporal dimension of the data is modeled through the transition equation. The latter describes the dynamics of the state variables but does not rely on a complete characterization of the transition distribution (which is the product of 3 non-central χ^2 distributions in the case of independent factors). Instead, it uses a partial characterization focussing on the affine form of the first two moments only. These two moments capture all information in case we would be dealing with a multivariate Gaussian model, and hence the maximum likelihood inference with a linear Kalman filter is exact. The primary appeal of the exact maximum likelihood technique stems from the well known optimality conditions of the resulting estimator under these ideal conditions: the estimator is consistent, efficient and asymptotically normally distributed. For our non-Gaussian model a linear Kalman filter is not optimal but the maximum likelihood estimation technique based on the Gaussian density may be given a quasi maximum likelihood interpretation and is optimal within the class of all linear estimators.[14]

[14] Following Duan and Simonato (1999) and de Jong (2000), we refrain from implementing the exact but computationally unfeasible filtering recursions (Harvey, 1989, pp. 162-165) and apply a Gaussian QML approach instead, still based on a linear Kalman filter. This approach yields an asymptotically consistent and efficient estimator (see Bollerslev and Woolridge, 1992, for a proof). There are, however, two subtleties that invalidate the above and render it an approximation with unknown theoretical properties. First, the factors themselves are latent and based upon a linear prediction. These predicted state variables will enter in the conditional mean and variance imputing errors in the likelihood function. Second, the Kalman filter algorithm is only linear as long as the state variables are positive. These complications make the QML estimator asymptotically consistent and efficient only in an approximate sense.
However, to assess the quality of this approximation, Duan and Simonato (1999) and de Jong (2000) perform Monte Carlo analyses and conclude that the error implied is negligible for reasonable parameter vector constellations within the exponential-affine class of term structure models. Moreover, Duffee and Stanton (2000) advocate the use of the linear Kalman filter above EMM and an SNP auxiliary model estimation approach (see Gallant and Tauchen, 1992).
Hence we discard the potential biases and assume the approximate QML estimator to be asymptotically consistent and efficient such that it can be used for statistical inference.