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THE CORE CAN BE ACCESSED IN A BOUNDED NUMBER OF STEPS

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Abstract. This paper strengthens the result of Sengupta and Sengupta (1996). We show that for the class of TU games with non-empty cores the core can be reached via a bounded number of proposals and counterproposals. Our result is more general than this: the boundedness holds for any two imputations with an indirect dominance relation between them.

1. Introduction

The core (Kannai 1992) is the collection of agreements that once proposed are never abandoned. While this made the core popular enough it is not an irrelevant question to ask whether the core can actually be reached via a sequence of blockings. This question is answered affirmatively by Sengupta and Sengupta (1996), where they introduce a recursive algorithm that generates a sequence of imputations. This algorithm terminates with a core-imputations. They do not, however make any claims about the number of steps required to reach the core. The aim of this paper is to show that the number of steps required is bounded. We do not make use of the algorithm proposed by Sengupta and Sengupta (1996), but prove the existence of a sequence of bounded length.

Consider a 3-player gamewhere the grand coalition gets 3, pairs get 2, singletons get 0. The core of this game consists of a single payoff-vector $(1,1,1)$. Now consider an initial allocation $\left(\frac{1}{2^m}, 2-\frac{1}{2^m}, 1\right)$ and the following –rather inefficient– process:

$$\begin{aligned}
\left(\frac{1}{2^m}, 2-\frac{1}{2^m}, 1\right) &\rightarrow \left(\frac{1}{2^{m-1}}, 1, 2-\frac{1}{2^{m-1}}\right) \\
\left(\frac{1}{2^{m-1}}, 1, 2-\frac{1}{2^{m-1}}\right) &\rightarrow \left(\frac{1}{2^{m-2}}, 2-\frac{1}{2^{m-2}}, 1\right) \\
&\vdots \\
\left(\frac{1}{2}, 2-\frac{1}{2}, 1\right) \text{ or } \left(\frac{1}{2}, 1, 2-\frac{1}{2}\right) &\rightarrow (1,1,1)
\end{aligned}$$

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where the penultimate allocation depends on the parity of m. This process terminates in the core in exactly m steps. As m can be chosen arbitrarily the number of steps required has no upper bound. In this paper we show that given the result of Sengupta and Sengupta (1996) the core can be reached in a *bounded* number of steps, moreover, our proof points out where do such inefficient processes make unnecessary detours.

The result we prove is not specific to the core. We show that if an imputation a can be reached from another imputation b via a dominating sequence of imputations, then the length of this sequence is bounded. Here we will only consider paths via imputations, that is via efficient and individually rational allocations. If this is not required, as in (Sengupta and Sengupta 1994) our proof can be simplified significantly.

Regarding the process of deviations and counter-deviations as an algorithm our result shows that reaching the core is a *primitive recursive algorithm*; it can be programmed with "for" loops only (Weisstein 2002), and the running time of such a program can be set in advance. See Péter (1981) and Péter (1967) for more on primitive recursive algorithms.

The structure of the paper is as follows: First we introduce our notation and some terminology. In Section 3 we state our results. The main part of this paper is the proof of our key lemma which is presented in Section 4.

2. Preliminaries

Let (N, v) be a TU-game with player set N, and characteristic function v. Subsets of N are coalitions and $v(S)$ is the payoff for coalition S. An imputation x is a payoff-vector in $\mathbb{R}^N$ that is individually rational and efficient, that is, $x_i \geq v(\{i\})$ for all players i in N and $x(N) = v(N)$, where $x(S) = \sum_{i \in S} x^i$. Let $A(N, v)$ denote the set of imputations. The projection of x onto S is denoted by x^S; we write $x^S > y^s$ if $x^i \geq y^i$ for each $i \in S$, but $x^S \neq y^S$.

The imputation x *directly dominates* y via coalition S, or $x \succ_D^S y$ if $x^S > y^S$, $x(S) = v(S)$ and $x \in A(N, v)$. We say that x *indirectly dominates* y and we write $x \succ_I y$ if there exists a finite sequence of imputations $\left\{y_0, \ldots, y_{\lambda(\eta)}\right\}$ and a finite collection of coalitions $\left\{Y_1, \ldots, Y_{\lambda(\eta)}\right\}$, such that $x = y_{\lambda(\eta)}$, $y = y_0$, and $y_j \succ_D^{Y_j} y_{j-1}$.

Definition 1 (Path)**.** The sequence $\{(y_j, Y_j)\}_{j=1}^{\lambda(\eta)}$ is a *(dominance) path* from y to x if

- $y_{\lambda(\eta)} = x$,
- $y_j \succ_D^{Y_j} y_{j-1}$ for $j > 1$, and
- $y_1 \succ_D^{Y_1} y$,

and we denote it by η.

Paths will be denoted by Greek letters: ξ, η, ζ with the corresponding small (x, y, z), and capital letters (X, Y, Z) denoting the respective imputations and blocking coalitions. The *length* of a path η is denoted by $\lambda(\eta)$. Finally, an index 0 refers to the imputation that the path originates from, so in this case $y_0 = y$.

Definition 2 (Concatenation of paths)**.** Pats ξ and η can be concatenated if ξ is a continuation of η, formally if $x_1 \succ_D^{X_1} y_{\lambda(\eta)}$, and we write this as $\zeta = \xi \wedge \eta$ to mean that

$$\begin{aligned}\lambda(\zeta) &= \lambda(\xi) + \lambda(\eta),\\ Z_j &= \begin{cases} X_j & \text{if } j \leq \lambda(\xi) \\ Y_j & \text{otherwise,}\end{cases}\\ z_j &= \begin{cases} x_j & \text{if } j \leq \lambda(\xi) \\ y_j & \text{otherwise.}\end{cases}\end{aligned}$$

Note that in general $\xi + \eta \neq \eta + \xi$, in fact, they may or may not be defined.

Definition 3 (Class)**.** Imputations x and y belong to the same class of imputations if

(1) $x(S) \geq v(S)$ iff $y(S) \geq v(S)$ for all coalitions S and
(2) $x(R) + \sum_{i \in S \setminus R} v(\{i\}) \geq v(S)$ iff $y(R) + \sum_{i \in S \setminus R} v(\{i\}) \geq v(S)$ for all $S \subseteq N$ and $R \subseteq S$ and
(3) for all pairs S and C of coalitions such that $C \cap S \neq \emptyset$ while $C \cup S = N$ and for all partitions $\mathcal{R}$ of all coalitions $R \subset C \cap S$ we have that

$$\begin{aligned} v(S) &> v(N) - v(C) + \sum_{T \in \mathcal{R}} v(T) + x(C \cap S \setminus R) \\ \text{iff} \quad v(S) &> v(N) - v(C) + \sum_{T \in \mathcal{R}} v(T) + y(C \cap S \setminus R). \end{aligned} \tag{2.1}$$

Members of a class are safe against the same blockings (Condition 1), even after certain modifications (Conditions 2 & 3), where the initial differences are limited to a subset of the players. The finiteness of the number of classes is a trivial, but crucial property.

Definition 4 (Set of Winners)**.** *Given* a dominance path η the set of winners, $W_j(\eta)$ is the set of players who profit from each subsequent deviation, that is

$$W_j(\eta) = \bigcap_{i=j+1}^{\lambda(\eta)} Y_i.$$

Definition 5 (Composite Player)**.** A composite player is a set of players, a blocking coalition that stays together and forms part of blocking coalitions until examined, but may break up in subsequent blockings. Formally S is a composite player in path η at time j if $S = Y_h$ for some $h \leq j$ and $S \subseteq Y_i$ for all $h < i \leq j$. Should the composite player contain smaller composite players, the term will refer to the largest one.

3. Results

Lemma 6. *Let (N, v) be a game. Then there exists an integer M, such that for all a, b pairs in $A(N, v)$ such that $a \succ_I b$ there exist a dominance path π from b to a such that $\lambda(\pi) < M$.*

We give the proof of this result in the next section.

Lemma 7 (Sengupta and Sengupta (1996))**.** *Let (N, V) be a game with a non-empty core $C(N, V)$. Let a be an imputation outside $C(N, V)$. Then there exists an imputation $c \in C(N, V)$ such that $c \succ_I a$.*

This lemma, combined with Lemma 6 gives the following theorem:

Theorem 8. *Let (N, V) be a game with a non-empty core that we denote $C(N, V)$. Then there exists an integer M such that for all imputations $a \in A(N, V)$ there exists an imputation $c \in C(N, V)$ and a path π from a to c such that $\lambda(\pi) < M$.*

4. Proof of Lemma 6

4.1. **Outline.** We prove the result by contradiction.

We assume that there exists a game (N, v), such that for all $M' > M$ there exists a pair a and b in $A(N, v)$, such that $a \succ_I b$ but the shortest path from b to a has length $\lambda > M'$. We show that for M' sufficiently large there exists a shorter path, giving the required contradiction. In our argument we find two sufficiently similar imputations in the path. These two imputations divide the path into three parts. We show that the middle part can be removed, and with some modifications the end part can be reattached.

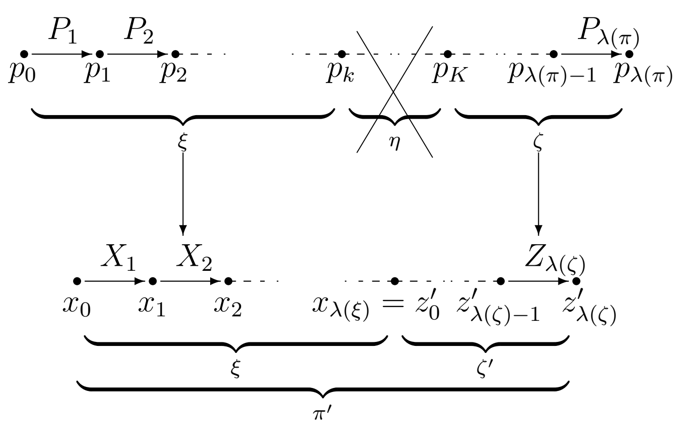

Figure 1. Sematic picture of the proof.

Formally the outline of the proof is as follows: *Given* a dominance path π there exists a break-up of the path into 3 sections, $\pi = \xi \wedge \eta \wedge \zeta$, such that there exists a modification of ζ denoted by $\widehat{\zeta}$ such that the path $\widehat{\pi} = \xi \wedge \widehat{\zeta}$ leads from b to a as well. Moreover $\widehat{\zeta}$ has the same same blocking coalitions, and hence the same length as ζ. Clearly $\lambda\,(\xi \wedge \eta \wedge \zeta) > \lambda(\xi \wedge \widehat{\zeta})$, which gives the required contradiction. Observe that ζ is a path from $y_{\lambda(\eta)}$ to a, while $\widehat{\zeta}$ is from $x_{\lambda(\xi)}$ to a. Therefore in order to be able to define $\widehat{\zeta}$, the imputations $y_{\lambda(\eta)}$ and $x_{\lambda(\xi)}$ must be sufficiently similar. What remains is to define "similar" for the two imputations, define $\widehat{\zeta}$ and show that the coalitions

$$\left\{\widehat{Z}_j\right\}_{j=1}^{\lambda(\widehat{\zeta})} = \{Z_j\}_{j=1}^{\lambda(\widehat{\zeta})}$$

are indeed blocking.

4.2. **Existence of two similar imputations.** Let k and K be such that the corresponding imputations p_k and p_K

(1) belong to the same class, and
(2) satisfy that $W_k(\pi) = W_K(\pi)$.

The number of classes is finite, and there are also a finite number of possible winner sets. Thus for sufficiently high M', that is, in a sufficiently long path π the existence of such two imputations is guaranteed. Setting $x_{\lambda(\xi)} = p_k$, and $y_{\lambda(\eta)} = p_K$ we get a suitable division. Outcomes p_k and p_K break the path into 3 sections as follows:

$$p_j \Longrightarrow \begin{cases} x_j & \text{if } j \leq k \\ y_{j-k} & \text{if } k < j \leq K \\ z_{j-K} & \text{if } K < j. \end{cases}$$

In the rest of the proof we attach the ζ path to the ξ path, cut η, thereby shortening the path. However, since $x_{\lambda(\xi)}$, and $y_{\lambda(\eta)}$ are not identical the path ζ has to be adjusted. We keep the deviating coalitions, but define a new path.

4.3. **The new path.**

4.3.1. *Creating the new imputations.* Let $\widehat{z}_0 = x_{\lambda(\xi)}$ and $\widehat{z}_{\lambda(\zeta)} = a$. Given the imputations up to $\widehat{z}_{j-1}$, where $0 < j < \lambda(\zeta)$ we define $\widehat{z}_j$. In the definition we have to take care about the following:

- Blockings are profitable, so blocking players cannot be worse off than prior to the blocking.
- Players who are blocking in the next round must have a sufficiently low payoff so that the blocking, which is fixed in advance gives them an improvement.
- Winners increase their payoffs monotonically, so it cannot exceed the end-payoff.
- Efficiency for N and for the blocking coalition and individual rationality for all players must hold.

The imputations we propose satisfy these conditions. Note that non-blocking players' payoffs have their effects only later, so they can be chosen with little restriction.

(1) If $Z_j \cup Z_{j+1} = N$ all players have blocked or will block next.
    (a) If $Z_j \cap Z_{j+1} = \emptyset$ then none of the blocking players block in the next round, thus their payoffs can be set freely. We select the payoffs for the others (who are all blocking in the next round) so that their blocking is profitable.

$$\widehat{z}_{j+1}^i = \begin{cases} \widehat{z}_j^i + \delta^i & \text{if } i \in Z_j \\ z_{j+1}^i & \text{otherwise.} \end{cases}$$

where $\delta \in \mathbb{R}^{Z_j}$ is the profit-division vector such that $\delta^i > 0$ for all $i \in Z_j$ and $\delta(Z_j) = v(Z_j) - \widehat{z}_j(Z_j)$.

(b) otherwise ($Z_j \cap Z_{j+1} \neq \emptyset$) we make sure that the blocking players who are also blocking in the next round get the minimum possible: the surplus is divided among the rest of the players[1].

$$\widehat{z}^i_{j+1} = \begin{cases} \widehat{z}^i_j & \text{if } i \in Z_j \cap Z_{j+1} \\ \widehat{z}^i_j + \delta^i & \text{if } i \in Z_j \setminus Z_{j+1} \\ z^i_{j+1} & \text{otherwise.} \end{cases}$$

where $\delta \in \mathbb{R}^{Z_j \setminus Z_{j-1}}$ is the profit-division vector such that $\delta^i > 0$ for all $i \in Z_j \setminus Z_{j-1}$ and $\delta(Z_j \setminus Z_{j+1}) = v(Z_j) - \widehat{z}_j(Z_j)$.

(2) Otherwise (if $Z_j \cup Z_{j+1} \neq N$) there exist players outside Z_j who do not block in the next round either, so we can give the soon-blocking outsiders a minimum (individually rational) payoff and the rest can share the remaining payoff. Note that in z_{j+1}, which is an imputation, they share no less, so there exists a distribution where each get at least its individually rational payoff.

(a) If all blocking players are winners we have to take special care of their payoffs:

$$\widehat{z}^i_{j+1} = \begin{cases} \widehat{z}^i_j + \delta^i & \text{if } i \in Z_j \\ v(\{i\}) & \text{if } i \in (N \setminus Z_j) \cap Z_{j+1} \\ v(\{i\}) + \gamma^i & \text{otherwise.} \end{cases}$$

where $\delta \in \mathbb{R}^{Z_j \setminus Z^*}$ is the profit-division vector such that $0 < \delta^i < z^i_{j+1} - \widehat{z}^i_j$ for all $i \in Z_j$ and $\delta(Z_j) = v(Z_j) - \widehat{z}_j(Z_j)$. Moreover $\gamma \in \mathbb{R}^{N \setminus (Z_j \cup Z_{j+1})}$ is a vector to divide the left-over such that $\gamma^i > 0$ for all $i \notin Z_j \cup Z_{j+1}$ and $\gamma(N \setminus (Z_j \cup Z_{j+1})) = v(N) - vZ_j - \sum_{i \notin Z_j} v(\{i\})$.

(b) Otherwise at some point some of the blocking players will not block. It is particularly interesting if the coalition breaks up. If this happens then we want the blocking players to have the minimal (non-increasing) payoff and the rest having the whole profit. Let $Z^* = Z_h$ where h is the smallest integer such that $h > j$, and $Z_j \nsubseteq Z_h$. Z^* is well-defined.

$$\widehat{z}^i_{j+1} = \begin{cases} \widehat{z}^i_j & \text{if } i \in Z_j \cap Z^* \\ \widehat{z}^i_j + \delta^i & \text{if } i \in Z_j \setminus Z^* \\ v(\{i\}) & \text{if } i \in (N \setminus Z_j) \cap Z_{j+1} \\ v(\{i\}) + \gamma^i & \text{otherwise.} \end{cases}$$

where $\delta \in \mathbb{R}^{Z_j \setminus Z^*}$ is the profit-division vector such that $\delta^i > 0$ for all $i \in Z_j$ and $\delta(Z_j \setminus Z^*) = v(Z_j) - \widehat{z}_j(Z_j)$ and $\gamma \in \mathbb{R}^{N \setminus (Z_j \cup Z_{j+1})}$ is a vector to divide the left-over such that $\gamma^i > 0$ for all $i \notin Z_j \cup Z_{j+1}$ and $\gamma(N \setminus (Z_j \cup Z_{j+1})) = v(N) - vZ_j - \sum_{i \notin Z_j} v(\{i\})$.

[1] Such “rest” will always exist as the grand coalition cannot be blocking.

4.3.2. *Profitability of deviations.* It remains to prove that the blocking coalitions Z_j (which we have fixed in advance) are indeed blocking, that is, none of their members lose in the blocking and at least one of them gets better off. Before starting our proof we give a simple definition and state some simple properties.

Proposition 9. *In the path constructed gains from a blocking are temporary, that is,* $\widehat{z}^i_{j+1} \not> \widehat{z}^i_{j-1}$ *except for two cases:*

(1) *If player i belongs to the just-created composite player Y_j.*
(2) *If $\widehat{z}^i_{j-1} = v(\{i\})$ and $\widehat{z}^i_{j+1} = z^i_{j+1}$.*

Proof. By construction the profit is shared among those players who leave the coalition next. In the next step, however, these players get either an individually rational payoff or their payoff in the ζ path, so they loose the surplus. The only exception is when the coalition stays together *and* keeps on blocking: if it is a composite player. □

Corollary 10. *The pre-blocking payoff of a blocking player i is either*

- *its starting payoff in the $\widehat{\zeta}$-path, $\widehat{z}^i_0$, or*
- *its individually rational payoff $v(\{i\})$, or*
- *its payoff in the ζ-path, z^i_j or*
- *player i belongs to a composite player.*

Corollary 11. *The pre-blocking payoff of a player (composite or individual) is less in the new path than in the original, $\widehat{z}^i_h > z^i_h$ unless $z^i_h = z^i_0$.*

In the following part the four types of blockings are reexamined from the point of view of profitability. This part of the proof is by induction.
Step 1. First blocking. The blocking by P_{K+1} is profitable in the path π, so $v(P_{K+1}) \geq p_K(P_{K+1})$. We have assumed that $Z_1 = P_{K+1}$. The imputations $p_k = z_0$ and p_K are in the same class, so by Cond. (1) in Defn. 3 $v(Z_1) \geq z_0(Z_1)$ and the first blocking is profitable.
Step 2. Inductive assumption. We assume that the imputation $\widehat{z}_{j-1}$ has been reached via successive profitable blockings.
Step 3. The blocking by Z_j. We discuss four cases depending on the blocking by Z_{j-1}.

1(a): By construction $\widehat{z}^{Z_j}_{j-1} = z^{Z_j}_{j-1}$, so $v(Z_j) > z_{j-1}(Z_j)$ implies $v(Z_j) > \widehat{z}_{j-1}(Z_j)$.

1(b): We look at the players in $Z_j \cap Z_{j-1}$. By Corollary 10

$$\widehat{z}_{j-1}(Z_j) = v(N) - v(Z_{j-1}) + \sum_{Z \in \mathcal{R}} v(Z) + \widehat{z}_0(Z_j \cap Z_{j-1} \setminus R), \tag{4.1}$$

where R is the set of players in Z_{j-1} who do not have their original payoffs, and $\mathcal{R}$ is a partition of this set into composite players including singleton "composite" players, that is, players who have their individually rational payoffs. Corollary 11 combined this with the profitability of the blocking Z_j gives

$$v(Z_j) > z_j(Z_j) > v(N) - v(Z_{j-1}) + \sum_{Z \in \mathcal{R}} v(Z) + z_0(Z_j \cap Z_{j-1} \setminus R). \tag{4.2}$$

Since $z_0 = p_K$ and $\widehat{z}_0 = p_k$ are in the same class, by 2.1 we have $v(Z_j) > \widehat{z}_j(Z_j)$.

2(a): By Corollary 11 we focus on the set R of players with their initial payoffs. Such a player $i \in R$ had a "winning streak", that is $i \in \bigcap_{h=1}^{j-1} Z_h$. Since in this case we assume that $W_{j-1}(\zeta) = Z_j$, the player i belongs to $W_0(\zeta)$ as well. Going back to π this means that $i \in W_K(\pi)$. By assumption then $i \in W_k(\pi)$. Due to the monotonicity of the payoffs of winning players $p_k^i \leq p_K^i$, and this argument holds for all $i \in R$. Then by the profitability of the blocking Z_j in path ζ we have

$$v(Z_j) > v(N) - v(Z_{j-1}) + \sum_{Z \in \mathcal{R}} v(Z) + z_0(Z_j \cap Z_{j-1} \setminus R),$$

where $\mathcal{R}$ is a partition of this set into composite players including singletons, that is, players with their individually rational payoffs. Our argument combined with Corollary 11 and the fact that z_0 and $\widehat{z}_0$ belong to the same class gives

$$\begin{aligned} & v(N) - v(Z_{j-1}) + \sum_{Z \in \mathcal{R}} v(Z) + z_0(Z_j \cap Z_{j-1} \setminus R) \\ > \;\; & v(N) - v(Z_{j-1}) + \sum_{Z \in \mathcal{R}} v(Z) + \widehat{z}_0(Z_j \cap Z_{j-1} \setminus R). \end{aligned}$$

But we have

$$v(N) - v(Z_{j-1}) + \sum_{Z \in \mathcal{R}} v(Z) + \widehat{z}_0(Z_j \cap Z_{j-1} \setminus R) = \widehat{z}_{j-1}(Z_j), \tag{4.3}$$

hence we can conclude $v(Z_j) > \widehat{z}_{j-1}(Z_j)$.

2(b): This case combines the previous ones, so we will not repeat our arguments. By Corollary 11 troubles are only caused by players having a too high initial payoff. By the fact that z_0 and $\widehat{z}_0$ belong to the same class Condition (3) of Definition 3 proves the profitability of Z_j.

4.3.3. *The final touch.* The only part that remains perhaps unclear is that $\widehat{z}_{\lambda(\widehat{\zeta})} = a$. Observe however, that the last blocking is of type 2(a) from the point of view of the blocking, since all the blocking players are winners. Indeed, our condition on δ ensures that all blocking players get the right payoff *provided* their payoffs are not too high. Our argument as in 2(a) shows that this condition holds. Since a is itself an imputation the payoffs for the non-blocking players can be chosen so that $\widehat{z}_\lambda^{N \setminus Z_\lambda} = a^{N \setminus Z_\lambda}$.

This completes our proof.

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