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a Dealer Market and a Crossing Network

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**DISCUSSION  
PAPER**

# Dynamic Order Submission Strategies with Competition between a Dealer Market and a Crossing Network<sup>1</sup>

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## **Abstract**

We present a dynamic microstructure model where a dealer market (DM) and a crossing network (CN) interact. We consider sequentially arriving agents having different valuations for an asset. Agents maximize their profits by either trading at a DM or by submitting an order for (possibly) uncertain execution at a CN. We develop the analysis for three different informational settings: transparency, “complete” opaqueness of all order flow, and “partial” opaqueness (with observable DM trades). We find that a CN and a DM cater for different types of traders. Investors with a high eagerness to trade are more likely to prefer a DM. The introduction of a CN increases overall order flow by attracting traders who would not otherwise submit orders (“order creation”). It also diverts trades from the DM. The transparency and “partial” opaqueness settings generate systematic patterns in order flow. With transparency, the probability of observing a CN order at the same side of the market is smaller after such an order than if it was not. Buy (sell) orders at a CN are also less likely to attract subsequent sell (buy) orders at the DM.

**JEL Codes:** G10, G20

**Keywords:** Dynamic Order Submission, Alternative Trading Systems, Dealer Market, Crossing Network, Order Flow, Market Microstructure.

# 1 Introduction

In today’s financial markets securities are simultaneously traded on a diversity of trading platforms. Different trading systems therefore compete for order flow. A well-documented example is the competition between Electronic Communication Networks (ECNs) and the Nasdaq dealer market (see e.g. Huang (2002)). A recent eye-catching combination concerns traditional continuous markets and batch-type crossing networks (CNs). CNs are defined by the SEC (1998) as “systems that allow participants to enter unpriced orders to buy and sell securities. Orders are crossed at a specified time at a price derived from another market (i.e. the continuous market)”. When faced with the choice among these trading platforms, investors can opt for the continuous market or for the CN. Despite the prevalence of CNs next to continuous markets, the dynamic aspects of the coexistence of these systems have not been well explored yet.

In this paper we investigate the interaction of a CN and a continuous (one-tick) dealer market (DM) by analyzing the impact on the composition and the dynamics of the order flow on both systems. We develop the analysis for three different informational settings: transparency, “complete” opaqueness, and “partial” opaqueness. The benchmark transparency case reflects that traders are fully informed about past order flow and hence observe the prevailing state of the CN’s order book before determining their order choice. In reality, however, CNs are rather opaque. We incorporate this informational environment by analyzing two different degrees of opaqueness. When “partially” opaque, traders observe previous trades at the DM, while “complete” opaqueness implies that traders are uninformed on both past CN and DM order flow.

Our model adapts that of Parlour (1998). While she focuses on the choice between limit and market orders on an auction market, we deal with the choice between two trading venues. Traders are assumed to arrive randomly and sequentially. When both trading systems coexist, depending upon their valuations, traders can obtain guaranteed execution in the DM, opt for cheaper but (possibly) uncertain execution on the CN, or refrain from trading. The transaction price on a CN is typically determined on another market: in our case we take the midprice of the DM.<sup>1</sup> This implies that CNs do not actively contribute to

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<sup>1</sup>This is in line with actual business practice: CNs cross at the mid-price derived from another market (see e.g. websites ITG Posit and E-crossnet).

price discovery. Order flow to the CN is gathered in an order book where time priority is assumed, i.e. orders arriving earlier receive priority in execution over their successors on the same market side. The implication is that at the cross, the last submitted orders at the excess market side do not obtain execution.

Common to the three informational settings, we do find that an increase in the DM's relative spread augments the CN's order flow. In general, a CN and a DM cater to different types of traders. Investors with a low patience to trade are more likely to trade at a DM. The existence of a CN results in "*order creation*": investors with a high patience to trade submit orders to a CN whereas they would never trade at a DM. The transparency and "partial" opaqueness settings also generate systematic patterns in order flow, a result reminiscent of Parlour (1998). These theoretical insights all point at time-varying order flow at a CN and trade flow at a DM. Our results therefore highlight that it is important to take into account the interaction between trading systems when measuring "normal" order flow. For example, when looking at an individual trading system, some order or trade flow sequences could wrongly be interpreted as being driven by information events, whereas they are actually caused by the interaction of trading systems.

Our paper is closely related to two recent strands of research. First, a number of papers have developed dynamic microstructure models for an auction market.<sup>2</sup> Parlour (1998) looks at the price dynamics in a one-tick limit order market. She shows that, even in the absence of informed trading, systematic patterns in transaction prices result because traders condition their behavior on the state of the limit order book and on their expectations for future traders' behavior.<sup>3</sup> Foucault (1999) also investigates the choice between market and limit orders and focuses on the latter orders' risk of non-execution and the winner's curse problem. He derives empirical predictions on the cross-sectional behavior of the mix between market and limit orders. Goettler, Parlour and Rajan (2004) model a dynamic limit order market as a stochastic sequential game and demonstrate order flow persistence, even in the absence of changes in the consensus value of the asset. Foucault, Kadan and Kandel (2003) endogenize the auction market's spread (the number of ticks) and study the resiliency of the limit order book

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<sup>2</sup>Note that static equilibrium models of the limit order book are much more common. Examples include Glosten (1994), Chakravarty and Holden (1995), Rock (1996) and Seppi (1997).

<sup>3</sup>Thus, a trader explicitly takes into account how her current order affects the future traders' choice to submit market or limit orders.

when introducing heterogeneity in traders' patience. In their model, however, arriving limit order traders are required to undercut existing quotes. Rosu (2004) relaxes this assumption as he considers a continuous time version of the latter model with endogenous undercutting. In contrast to the previous models, he also allows for strategic cancellation of limit orders. Our paper contributes to this line of research as we consider a dynamic microstructure model to study (partly, at least) endogenous liquidity supply by looking at the competition between two different trading venues. This is in contrast to the previously mentioned papers that restrict themselves to only one market, i.e. an auction market.

A second line of recent work applies static models to analyze the competition between a CN and a DM.<sup>4</sup> Hendershott and Mendelson (2000) start from a model where traders simultaneously decide to submit orders to one of both markets. They find that a CN is characterized by two externalities: a positive (liquidity) externality, as an increase in the CN's trading volume benefits all traders, and a negative (crowding) externality, as low-liquidity preference traders compete with higher liquidity preference traders on the same side of the market. Expanding on this paper, Dönges and Heineman (2001) focus on some game theoretic refinements to reduce the multiplicity of equilibria in the coordination game. We contribute to this line of recent work as we explicitly introduce dynamics into the analysis. These dynamics are important: a typical characteristic of a CN is that it "matches" orders at a specified time during the trading day, while the other market simultaneously operates in a continuous fashion. In particular, traders arrive sequentially and both the state of the CN's order book (when transparent) and their expectation on the behavior of future traders until the cross determine their submission strategy.

There is by now a substantial amount of empirical papers analyzing the interaction between trading systems (for an overview see Biais, Glosten and Spatt (2002)). The number of papers empirically investigating the impact of a CN on other trading systems, however, is rather limited. Gresse (2002) studies the

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<sup>4</sup>This strand of research is part of a comprehensive literature on the competition between trading systems. Examples include Glosten (1998), Foucault and Parlour (2001), Shy and Tarkka (2001), Santos and Scheinkman (2001), Di Noia (2001), Viswanathan and Wang (2002), Chemmanur and Fulghieri (2003) and Parlour and Seppi (2003).

impact of the POSIT CN on the DM segment of the London Stock Exchange. She finds that POSIT has a share of total trading volume of about one to two percent in these stocks, but that its probability of execution is still low (2-4%). Moreover, she reports that activity at POSIT does not have a detrimental effect on the liquidity at the considered DM. Næs and Ødegaard (2004) focus on trades of the Norwegian Government Petroleum fund. In a study of 4200 orders that are first sent to a CN and then, in case of non-execution, to brokers, they find that execution costs of crossed trades are lower. Conrad, Johnson and Wahal (2003) use proprietary data of 59 institutional investors in the US who choose between trading platforms. They find that realized execution costs are generally lower on alternative trading systems (including CNs). Fong, Madhavan and Swang (2004) focus on the impact of block trades on different trading venues, i.e. a limit order book, a CN and an upstairs market. They find that competition from the two latter markets imposes no adverse effect on the liquidity of the limit order book.

This paper proceeds as follows. Section 2 presents the setup of the transparency benchmark model. Section 3 provides an analysis of its equilibrium. We first deal with the markets in isolation, next we study their interaction. In Section 4, we implement two degrees of opaqueness, i.e. partial and complete opaqueness. Section 5 offers a discussion of a number of possible extensions to our model. Finally, Section 6 concludes. All proofs are relegated to Appendix A.

## 2 Setup of the Model

The model we develop is based on the setup in Parlour (1998). While her model discusses the traders' choice between market and limit orders in a continuous order driven market, we adapt it to analyze competition between two trading systems. In our economy, there are two days. Agents decide upon consumption on day 1 and day 2, denoted by  $C_1$  and  $C_2$ . Agents are risk neutral and differ with respect to their preferences over consumption on these two days. These preferences are given by the following utility function:

$$U(C_1, C_2; \beta) = C_1 + \beta C_2$$

with  $\beta$  the subjective preference or type of the agent. Next to these two "goods"  $C_1$  and  $C_2$ , an asset exists that on day 2 pays out  $V$  units of  $C_2$  per share.

As we are investigating the short-term interactions between both markets, the assumption of no uncertainty in  $V$  is a reasonable starting point.<sup>5</sup> During the first day, the trading day, claims to the asset can be exchanged for  $C_1$ . Prices in the market are exchange ratios  $C_1/C_2$ . Agents can then construct their preferred consumption path by trading claims to this asset. The trading day consists of  $T$  periods, indexed by  $t = 1, \dots, T$ . Each period exactly one agent (also referred to as trader) arrives in the market, and each agent arrives at most once. The arriving agent at time  $t$  is characterized by two elements. First, her initial endowments determine her trading orientation. With probability  $\pi_B$ , she is a buyer and has one unit of the asset she can buy in exchange for  $C_1$ , which we denote by 1. With probability  $\pi_S = 1 - \pi_B$ , she is a seller and has one unit of the asset she can sell,  $-1$ . Secondly, the agent has a type  $\beta_t$ , which is drawn from an i.i.d. continuous distribution  $F(\cdot)$  with support  $[\underline{\beta}, \bar{\beta}]$ , where we assume  $0 \leq \underline{\beta} \leq 1 \leq \bar{\beta}$ . This  $\beta_t$  captures the trader's personal trade-off between current and future consumption in the utility function above. In this way, it also determines her degree of patience for trade. In particular, if the trader is a buyer, she will be more eager to buy if she has a high beta than if her beta is low. A seller on the other hand, will be more eager to sell if she has a low  $\beta$ . In order to see this, assume that the arriving trader is a buyer. Buying the asset yields  $\beta_t V$ . She compares this value with the price in the market and performs the buy if the price is lower than the value she attaches to the asset. If  $\beta_t$  is high, and she thus attaches more weight to consumption on the second day, she will be more eager to trade than if  $\beta_t$  is low. The reasoning is that the trading gains are higher in the former case. Similarly, a seller with a low beta will be more eager to sell since she prefers consumption on the first day.

Traders can choose between submitting an order to a dealer market (DM), to a crossing network (CN), or not to submit an order. Competition between dealers on the DM is sufficiently harsh such that the spread is one tick, that is  $A - B = 1$ , with  $B$  the bid price and  $A$  the ask price. This assumption is as in Parlour (1998). The implication is that buyers can always buy at a price  $A$ , the price at which a dealer is willing to sell. Sellers looking for immediacy in the dealer market obtain  $B$ .

Next to a DM, we also introduce a CN. We assume that the matching of

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<sup>5</sup>For example, POSIT organizes up to 15 daily “crosses” for each stock. In Section 5, we will discuss uncertainty in  $V$ .



orders (the “cross”) takes place at the end of the trading day, hence after period  $T$  (we mean after the action of the agent arriving in period  $T$ ). The price of the cross is derived from the bid and ask in the dealer market and equals the midquote  $(A + B) / 2$ . Given our assumptions, executed orders at the CN face no price uncertainty.<sup>6</sup> Orders submitted to the CN are stored in the book  $c_t$ , which is a pair  $(c_t^b, c_t^s)$  where  $c_t^b > 0$  ( $c_t^s < 0$ ) represents the cumulative amount of buy (sell) orders at the CN before the order at time  $t$ .<sup>7</sup> After the action of the trader at time  $t$ , there are three possible evolutions of the CN’s order book:

$$(c_{t+1}^b, c_{t+1}^s) = \begin{cases} (c_t^b + 1, c_t^s) & \text{trader } t \text{ submits a buy order to CN} \\ (c_t^b, c_t^s - 1) & \text{trader } t \text{ submits a sell order to CN} \\ (c_t^b, c_t^s) & \text{trader } t \text{ submits no order to CN} \end{cases} .$$

The first two evolutions describe a buy and sell order, respectively. The last case, where the CN’s order book remains unchanged, stems from a trade at a dealer or not trading at all. Once submitted, orders cannot be modified or cancelled. This means that orders remain in the CN’s order book until the cross.<sup>8</sup> Order execution is determined by the imbalance between the queue of buy orders and the queue of sell orders. If  $c_T^b = |c_T^s|$ , meaning no imbalance, then all orders are executed. If  $c_T^b < |c_T^s|$ , some sell orders cannot be executed. We assume time priority such that the first  $c_T^b$  sell orders submitted are executed. If  $c_T^b > |c_T^s|$ , the first  $|c_T^s|$  buy orders are executed. It goes without saying that time priority influences the order submission strategies of the traders. Trading at the “DM as last resort” upon non-execution at the CN is not a possible strategy: such “opportunistic trading” is assumed to be sufficiently costly.<sup>9</sup> Furthermore, we

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<sup>6</sup>As Section 5 shows, introducing uncertainty in  $V$  will alter this.

<sup>7</sup>The assumption that there is only one cross during the trading day is not restrictive. Suppose that there would be multiple crosses during the trading day, i.e. crosses at time  $0 < T_1 < T_2 < \dots < T$ . If unfilled orders remain in the CN-book after an intermediate cross, the results of our model do not change. Since there is no waiting cost within the trading day ( $\beta$  applies between trading days), a trader is indifferent about at which cross precisely her order executes. Hence, what is relevant to a trader is the probability of execution until the last cross at time  $T$ . This means that the trader solves exactly the exact same model as we describe. If unfilled orders would not remain in the book after intermediate crosses, our model then captures the period from one cross to another. In that case, what is relevant to a trader, is the time until the next cross and the state of the book at the time she arrives. So, our model can be applied first to the period  $[0, T_1]$ , then to  $(T_1, T_2]$  and so on.

<sup>8</sup>This is in contrast with a limit order market where stored limit orders disappear when they are hit by a market order.

<sup>9</sup>A CN order is a free option when trading at the “DM as last resort” implies no additional

need to distinguish the case where the order book is transparent versus opaque. Section 3 discusses the case where the CN's order book is fully transparent. Arriving traders at  $t$  are able to observe both queues and base their order submission strategy on the resulting order imbalance. Herein lies one of the major differences with the limit order market model of Parlour (1998). In her model, the trader looks at both sides of the queue because the other side influences the decision of a potential counterparty. In our model with full transparency, only the order imbalance at the CN will determine the order submission strategies, not their individual length. In Section 4, we introduce different degrees of opaqueness.

### 3 Equilibrium under Transparency

In this section, we characterize the equilibrium order submission strategies when the CN's order book is fully transparent. As a first step, we consider successively a DM and a CN in isolation. This approach allows to gain intuition about the model and the structure and functioning of each market. Subsequently, we determine the equilibrium when both markets coexist. The methodology is identical in all cases. We calculate for a trader arriving at time  $t$  a cutoff  $\beta_t$  at which she is indifferent between two actions, taking execution probabilities as given.

#### 3.1 Dealer Market in Isolation

Suppose for now that there only exists a DM. In this case, a trader can choose between submitting an order to the DM or not to trade at all. She will submit an order to the dealer (DM order) as long as this yields a positive profit, otherwise she prefers not to trade. When the profit is zero, she is indifferent. Before deciding, she observes the bid and ask in the market, and her  $\beta_t$ . The profit of a buy order to the dealer is the difference between the valuation of the trader  $\beta_t V$  and the price paid, which is the ask, i.e. the profit is  $\beta_t V - A$ . Similarly, for a sell order, the profit is  $B - \beta_t V$ . From these profits, the cutoff values for  $\beta_t$  are

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cost relative to an immediate DM trade.

computed:

$$\beta_t^{b,DM} = \frac{A}{V}$$

$$\beta_t^{s,DM} = \frac{B}{V},$$

where the first superscript refers to buy (*b*) or sell (*s*), and the second to DM. The interpretation is as follows. A buyer arriving at time  $t$  who has a  $\beta_t$  higher than  $A/V$  will submit a DM buy order, all others will not. When the trader at time  $t$  is a seller, she will submit a DM sell order if her  $\beta_t$  is smaller than  $B/V$ . The order submission strategies are depicted in Figure 1. Note that traders having a  $\beta_t$  between  $B/V$  and  $A/V$  will never submit an order, regardless of whether they are a buyer or a seller.

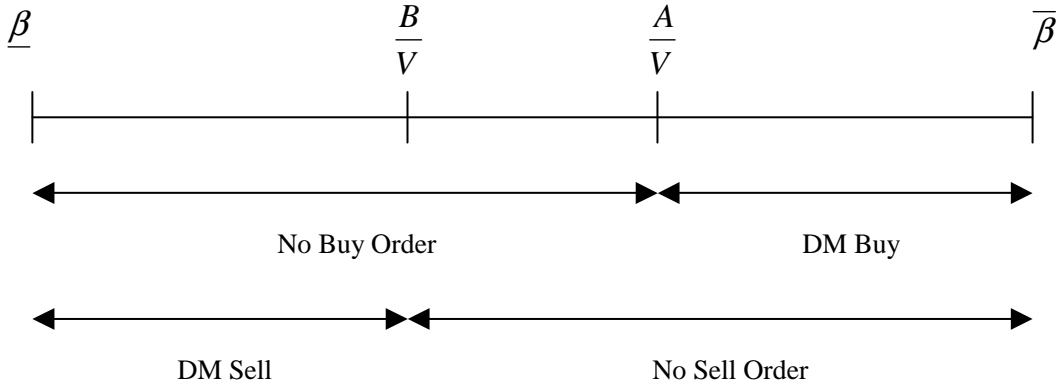


Figure 1: **Order Submission Strategies with Dealer Market in Isolation**

### 3.2 Crossing Network in Isolation

In this subsection, only a crossing network exists (and no dealer market). To compare the different settings, we assume that the price at which a cross will take place is the midquote as if a dealer market would exist:  $(A + B)/2$ . When arriving at time  $t$ , a trader also observes her  $\beta_t$  and the state of the CN's order book. A trader will submit a CN order as long as this results in a positive *expected* profit. We need to consider expected profits, since in contrast with an order to a

DM, the execution of a CN order may not be certain. If the order executes, the profit for the trader is the difference between the trader's valuation and the price paid (the midquote). When taking into account the uncertainty about execution, the expected profit of a CN buy order is  $p_t^b(\beta_t V - (A + B)/2)$ , where  $p_t^b$  denotes the expected probability of execution of a CN buy order submitted at time  $t$ . For a CN sell order, the expected profit is  $p_t^s((A + B)/2 - \beta_t V)$ , with  $p_t^s$  the probability of execution of a sell order submitted at time  $t$ . These probabilities depend on the state of the book in the CN, and the time left until the end of the trading day:  $p_t^b(c_t, T - t)$  and  $p_t^s(c_t, T - t)$ , but for notational convenience we suppress this dependence. The reasoning for this dependence is that if a trader joins the longer queue, enough future orders need to be submitted to the shorter side of the CN's order book to obtain execution. This is more likely earlier in the trading day, when there are still a lot of periods to come. Finally, when the expected profit of a CN order is negative, the trader chooses to abstain, leaving her zero profits.

A trader's strategy whether or not to submit a CN order is determined by the expected profits of this action. Solving for  $\beta_t$ , we find the following cutoff value for a buyer and a seller respectively:

$$\begin{aligned}\beta_t^{b,CN} &= \frac{A+B}{2V} \\ \beta_t^{s,CN} &= \frac{A+B}{2V}.\end{aligned}$$

In words, a trader arriving at  $t$  will submit a CN buy (sell) order if her  $\beta$  is higher (lower) than  $\beta_t^{b,CN}$  ( $\beta_t^{s,CN}$ ). To be complete, these cutoff values hold if the execution probability is strictly positive. If it is zero, a trader is always indifferent between a CN order and no order, since both yield zero profit. If this occurs, we assume that traders prefer to abstain. The order submission strategies are summarized in Figure 2. Note that in contrast with a DM in isolation, there is no "gap", i.e. there is no range of betas where neither a buyer nor a seller submits an order. The reasoning is that a CN has no spread whereas a DM is characterized by a one-tick spread.

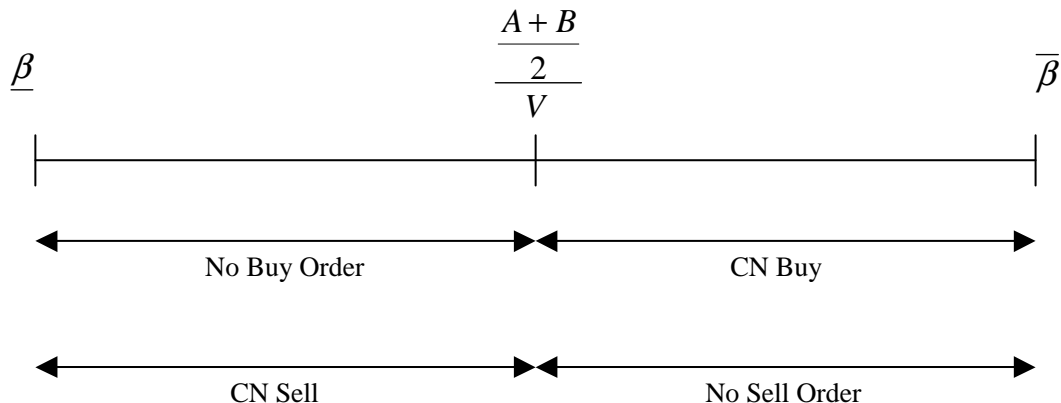


Figure 2: **Order Submission Strategies with a CN in Isolation**

### 3.3 Interaction between CN and DM

After having discussed the two trading systems in isolation, we now turn to the full model and characterize the choice problem faced by a trader arriving in the market at time  $t$ . She can choose between submitting an order to the DM, the CN, or no trade at all. Upon her arrival, she knows whether she is a buyer or a seller, observes the bid and ask price of the dealer, the state of the CN's order book  $c_t$  and her  $\beta_t$ . Recall that the CN crosses at the midprice of the dealer's bid and ask. Moreover, she knows the time remaining to the cross, the distribution of buyers and sellers and their willingness to trade. Based on this information, she chooses between three possible actions. First, she can initiate a trade at the dealer. Such an order has a guaranteed immediate execution. Secondly, she can opt for submitting an order to the CN. This would yield a better price as it allows the trader to save the half-spread, which in our model is equal to half a tick. With this order, however, she might face the risk of non-execution. Execution is certain when upon arrival she is able to join the shorter queue (due to time priority in the CN). In all other cases, the execution probability is lower than one. Thirdly, she can refrain from trading when it yields a negative (expected) profit.

Denote the strategy of a buyer arriving at time  $t$  by  $\phi_t^b(c_t, \beta_t)$  and of a seller by  $\phi_t^s(c_t, \beta_t)$  where the notation stresses that the strategy depends on the state of the CN's order book at time  $t$ ,  $c_t$  and the trader's type  $\beta_t$ . Important to note is that these strategies depend on time; in other words, they are nonstationary.

Most relevant is the number of periods left until  $T$ , the end of the trading day. The setup of this model can be seen as a stochastic sequential game. Moreover, due to the recursive nature of the game, an equilibrium is guaranteed to exist and this equilibrium is unique (since traders are indifferent between choices with zero probability).

We apply the approach introduced above to solve the trader's choice problem. Thus we again determine cutoff values for  $\beta_t$ , i.e. levels of indifference between different actions for given execution probabilities. A first cutoff value represents indifference between a trade in the DM and a CN order, a second between a CN order and no order. As will be shown in the proof of Proposition 1, no other cutoff values need to be considered. Define  $\bar{\beta}_t^b(p_t^b)$  as the value  $\beta_t$  of a buyer that is indifferent between an order to the CN and an order to the DM. It is given by:

$$\bar{\beta}_t^b(p_t^b) = \min \left[ \frac{\frac{A+B}{2}}{V} + \frac{1/2}{V(1-p_t^b)}, \bar{\beta} \right].$$

Define  $\underline{\beta}_t^b(p_t^b)$  as the  $\beta$  at which a trader is indifferent between a CN buy order and no order. It is equal to:

$$\underline{\beta}_t^b(p_t^b) = \begin{cases} \frac{\frac{A+B}{2}}{V} & \text{if } p_t^b > 0 \\ \frac{A}{V} & \text{otherwise} \end{cases}.$$

Similarly,  $\underline{\beta}_t^s(p_t^s)$  is the  $\beta_t$  of a seller that is indifferent between a CN order and an order to the dealer:

$$\underline{\beta}_t^s(p_t^s) = \max \left[ \frac{\frac{A+B}{2}}{V} - \frac{1/2}{V(1-p_t^s)}, \underline{\beta} \right],$$

whereas  $\bar{\beta}_t^s(p_t^s)$  holds for a seller at  $t$  who is indifferent between a CN order and no order, with

$$\bar{\beta}_t^s(p_t^s) = \begin{cases} \frac{\frac{A+B}{2}}{V} & \text{if } p_t^s > 0 \\ \frac{B}{V} & \text{otherwise} \end{cases}.$$

Furthermore, denote a buy at the DM by “ $1^{DM}$ ” (which transacts at the ask), and a sell at the DM by “ $-1^{DM}$ ” (transacting at the bid). Similarly, “ $1^{CN}$ ” and “ $-1^{CN}$ ” stand for a buy and sell order to the CN respectively. Proposition 1 then states the equilibrium strategies of a trader arriving at  $t$ .

**Proposition 1** *If the time  $t$  trader is a buyer, there exist cutoff values such that*

$$\beta_t \in \begin{cases} \left[ \underline{\beta}, \underline{\beta}_t^b(p_t^b) \right] & \phi_t^b(c_t, \beta_t) = 0 & (\text{no order}) \\ \left( \underline{\beta}_t^b(p_t^b), \bar{\beta}_t^b(p_t^b) \right] & \phi_t^b(c_t, \beta_t) = 1^{CN} & (\text{buy order to CN}) \\ \left[ \bar{\beta}_t^b(p_t^b), \bar{\beta} \right] & \phi_t^b(c_t, \beta_t) = 1^{DM} & (\text{buy at DM}) \end{cases} .$$

*Similarly, if the time  $t$  trader is a seller, there exist cutoff values such that*

$$\beta_t \in \begin{cases} \left[ \underline{\beta}, \underline{\beta}_t^s(p_t^s) \right] & \phi_t^s(c_t, \beta_t) = -1^{DM} & (\text{sell at DM}) \\ \left( \underline{\beta}_t^s(p_t^s), \bar{\beta}_t^s(p_t^s) \right) & \phi_t^s(c_t, \beta_t) = -1^{CN} & (\text{sell order to CN}) \\ \left[ \bar{\beta}_t^s(p_t^s), \bar{\beta} \right] & \phi_t^s(c_t, \beta_t) = 0 & (\text{no order}) \end{cases} .$$

**Proof.** See Appendix A ■

The equilibrium order submission strategies are summarized in Figure 3. Comparing this graph with Figures 1 and 2, there are some notable differences. The most important one is that the cutoff values are *dynamic* and change every period  $t$ . For the markets in isolation, this was not the case. Moreover, although the range of  $\beta$ 's at which no buy or sell order is submitted is the same, the ranges at which DM and CN orders are submitted, are in general different from the isolation cases. Compared to the DM in isolation, there is *order creation*: traders with intermediate  $\beta$ 's now submit orders to the CN which allows them to avoid paying the half-spread. The CN also introduces competition for the DM as it may divert *trades* away from the DM. Remark that this could lead to overall trade creation but also to overall trade reduction. The reasoning for this potential trade reduction is that some of the investors choosing to trade at the DM if in isolation may now opt for the CN at which their order may not execute.

Next, we derive some properties of this equilibrium. Lemma 1 shows that for higher execution probabilities, the range of  $\beta$ 's of traders who submit a CN order becomes wider, and complementary, the range of trader types who opt for an order to the dealer becomes smaller. In other words, if the execution probability of an order at the CN is larger, an arriving trader is more likely to choose such a CN order.

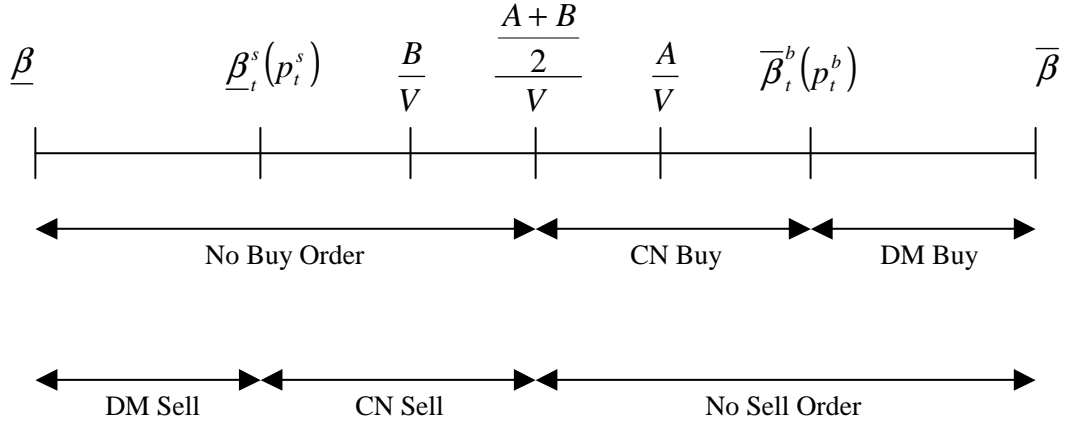


Figure 3: **Order Submission Strategies with Dealer Market and Crossing Network**

**Lemma 1** *The higher the probability of execution on the CN, the more trader types prefer to submit CN orders over market orders. That is,*

$$\frac{d\underline{\beta}_t^s(p_t^s)}{dp_t^s} \leq 0$$

$$\frac{d\bar{\beta}_t^b(p_t^b)}{dp_t^b} \geq 0.$$

**Proof.** Immediate. ■

A crucial element in the choice between a CN order and a DM trade is the execution probability at the CN, since this determines expected profits. When trader  $t$  submits a CN order, she changes the imbalance in the CN. This affects the execution probabilities of future CN orders and hence also the strategies chosen by future traders. When determining her optimal strategy, trader  $t$  must take these effects of her order into account. Proposition 2 shows how the length of the queues (and the imbalance) influences execution probabilities.

**Proposition 2** *In equilibrium, at any time  $t$ , if the CN's order book at the buy side is one unit thicker, then the probability of execution of a buy (sell) order will be lower (higher). If the CN's order book at the sell side is one unit thicker, then the probability of execution of a buy (sell) order will be higher (lower). If the book*



is one unit thicker at the buy side and one unit thicker at the sell side, then the probability of execution for both order types remains constant. Hence,  $\forall c_t, t$ ,

$$\begin{aligned}
(i) \quad p_t^b(c_t^b, c_t^s) &\leq p_t^b(c_t^b - 1, c_t^s) & \bar{\beta}_t^b(c_t^b, c_t^s) &\leq \bar{\beta}_t^b(c_t^b - 1, c_t^s) \\
(ii) \quad p_t^b(c_t^b, c_t^s) &\leq p_t^b(c_t^b, c_t^s - 1) & \bar{\beta}_t^b(c_t^b, c_t^s) &\leq \bar{\beta}_t^b(c_t^b, c_t^s - 1) \\
(iii) \quad p_t^b(c_t^b, c_t^s) &= p_t^b(c_t^b + 1, c_t^s - 1) & \bar{\beta}_t^b(c_t^b, c_t^s) &= \bar{\beta}_t^b(c_t^b + 1, c_t^s - 1) \\
(iv) \quad p_t^s(c_t^b, c_t^s) &\leq p_t^s(c_t^b, c_t^s + 1) & \underline{\beta}_t^s(c_t^b, c_t^s) &\geq \underline{\beta}_t^s(c_t^b, c_t^s + 1) \\
(v) \quad p_t^s(c_t^b, c_t^s) &\leq p_t^s(c_t^b + 1, c_t^s) & \underline{\beta}_t^s(c_t^b, c_t^s) &\geq \underline{\beta}_t^s(c_t^b + 1, c_t^s) \\
(vi) \quad p_t^s(c_t^b, c_t^s) &= p_t^s(c_t^b + 1, c_t^s - 1) & \underline{\beta}_t^s(c_t^b, c_t^s) &= \underline{\beta}_t^s(c_t^b + 1, c_t^s - 1)
\end{aligned}$$

(both formulations, in terms of probabilities and in terms of betas, are equivalent).

**Proof.** See Appendix A ■

Intuitively, Proposition 2 argues that when the queue at one side of the market is longer when a trader arrives on that side of the market, the execution probabilities of a CN order are lower relative to when the queue is shorter (parts (i), (iv), and by symmetry (ii) and (iv)). That is, traders face intertemporal competition with traders of their own type. The reasoning is as follows. Suppose that a buyer arrives at time  $\tau$  and  $c_\tau^b \geq |c_\tau^s|$ .<sup>10</sup> Then if the book is one unit thicker at the buy side, an additional CN order at the sell side must arrive in order to obtain execution. This lowers the execution probability compared to the case when the buy queue is one unit shorter (meaning also a smaller imbalance). Only when both queues are one unit thicker (parts (iii) and (vi)), execution probabilities are not affected since the imbalance remains the same. This is in contrast with an auction market as in Parlour (1998). In such a market, execution probabilities are influenced when both queues become one unit longer. This proves that in a CN the imbalance between both queues matters, while in a limit order market the individual length of both queues is relevant.

### 3.4 Empirical Predictions on Order Flow Dynamics

Having determined and characterized the equilibrium order submission strategies of traders, we now investigate whether systematic patterns can be predicted in

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<sup>10</sup>If  $c_\tau^b - 1 < |c_\tau^s|$ , the execution probability with a book of  $(c_\tau^b, c_\tau^s)$  and  $(c_\tau^b + 1, c_\tau^s)$  are both one.

transaction and order flow data. Parlour (1998) shows that this is the case in the context of an auction market with limit and market orders. The reasoning in her model is that the presence of the limit order book and the length of the queues create a relation between past and current order flow. In our model, the relation between past and current order flow is driven by the order imbalance in the CN's order book and not the length of the buy and sell orders in that order book. This means that the dynamics of the market might (and will, as we shall see) differ in our model from those of Parlour (1998). The systematic patterns in order flow are analyzed in four propositions. In all cases, we start from a given state of the book in the CN,  $c_t$ , and from a specific order, a DM order in the two first propositions and a CN order in the two latter, and investigate the effect on the order flow to the DM and the CN in the subsequent period.

Thus, we first assume that the previous order (at time  $t$ ) was a DM trade and investigate the patterns in subsequent order submissions. Suppose the trader at time  $t + 1$  is a buyer. Proposition 3 then states that the probability of observing a DM buy does not hinge on whether the previous transaction was a DM buy or a DM sell.

**Proposition 3** *The probability of a DM buy at time  $t + 1$  is independent of whether the order at time  $t$  was a DM buy or a DM sell:*

$$\begin{aligned} & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{DM} | \phi_t^b (c_t, \beta_t) = 1^{DM}, c_t] \\ &= \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{DM} | \phi_t^s (c_t, \beta_t) = -1^{DM}, c_t]. \end{aligned}$$

*A symmetric result holds for the other side of the market.*

**Proof.** Contained in the discussion below. ■

A similar result holds for CN orders following a DM order. Proposition 4 shows that the probability that the current order is a CN order (buy or sell depending on the trader that arrives) is independent of whether the previous order was a DM sell or a DM buy.

**Proposition 4** *The probability of a CN buy order at time  $t + 1$  is equal, whether the order at time  $t$  was a DM buy or a DM sell:*

$$\begin{aligned} & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN} | \phi_t^b (c_t, \beta_t) = 1^{DM}, c_t] \\ = & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN} | \phi_t^s (c_t, \beta_t) = -1^{DM}, c_t]. \end{aligned}$$

*A symmetric result holds for the other side of the market.*

**Proof.** Contained in the discussion below. ■

The results in Propositions 3 and 4 are driven by the same intuition. Recall that a trader chooses between submitting to the DM or the CN (or not trading) on the basis of three elements: her impatience as given by  $\beta_t$ , the state of the CN's order book, and the time left until the end of the trading day. The latter two factors determine the probability of execution of a CN order and hence the expected profit from choosing that strategy. If the previous order is a DM trade, none of these three elements is influenced, hence it is irrelevant whether the order is a DM buy or a DM sell. These results are in sharp contrast with Parlour (1998). In her model, market orders do influence the probabilities of subsequent orders as they change the depth in the limit order book and hence the execution probabilities of subsequent limit orders.

However, the conclusions alter when we assume that the order at time  $t$  was a CN order instead of a DM order. In this case, we obtain systematic patterns in order flow despite the fact that buyers and sellers arrive randomly. Proposition 5 shows that when the trader at time  $t$  has chosen a CN buy order, it is less likely that a buyer at  $t + 1$  will do the same, compared to when trader  $t$  did not submit a CN buy order.

**Proposition 5** *The probability of a CN buy order at time  $t + 1$  is smaller if the order at time  $t$  was a CN buy order than if it was a DM trade (buy or sell). This in turn is smaller than the probability of observing a CN buy order, conditional upon the previous order being a CN sell order:*

$$\begin{aligned} & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN} | \phi_t^b (c_t, \beta_t) = 1^{CN}, c_t] \\ \leq & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN} | \phi_t^b (c_t, \beta_t) = 1^{DM} \text{ or } \phi_t^s (c_t, \beta_t) = -1^{DM}, c_t] \\ \leq & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN} | \phi_t^s (c_t, \beta_t) = -1^{CN}, c_t]. \end{aligned}$$

*A symmetric result holds for the other side of the market.*

**Proof.** See Appendix A ■

Complementary to Proposition 5, Proposition 6 shows that it becomes more likely that the current buyer submits a DM buy, if the previous order was a CN buy order, compared to if it was another type of order.

**Proposition 6** *The probability of a DM buy at time  $t + 1$  is larger if the order at time  $t$  was a CN buy order than if it was a DM trade (buy or sell). This in turn is larger than the probability of observing a DM buy, conditional upon the previous order being a CN sell order:*

$$\begin{aligned} & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{DM} | \phi_t^b (c_t, \beta_t) = 1^{CN}, c_t] \\ \geq & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{DM} | \phi_t^b (c_t, \beta_t) = 1^{DM} \text{ or } \phi_t^s (c_t, \beta_t) = -1^{DM}, c_t] \\ \geq & \Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{DM} | \phi_t^s (c_t, \beta_t) = -1^{CN}, c_t]. \end{aligned}$$

*A symmetric result holds for the other side of the market.*

**Proof.** See Appendix A ■

The intuition behind both propositions is as follows. Assume that the time  $t + 1$  trader is a buyer. If the queue of buy orders at time  $t + 1$  is shorter than the sell queue, the equality sign applies, since the execution probability of a submitted CN buy order is one. In this case, the type of the previous order is irrelevant for the current order flow. In contrast, if after the order of the trader at  $t$  the buy queue is longer than the sell queue, i.e. when there is an imbalance, the type of the previous order does matter. Given her  $\beta_{t+1}$ , the current trader will be more likely to submit a CN buy order if the previous order increased the execution probability. This is the case when the imbalance in the CN's order book decreased, which happens after a CN sell order in the previous period. A DM trade (be it buy or sell) does not alter the imbalance, while a CN buy order at  $t$  increases the imbalance. Symmetrically, if trader  $t + 1$  is less likely to submit a CN order, she will be more likely to opt for an order to the dealer.

It is worth stressing again that although the patterns outlined in Propositions 5 and 6 are similar to the case of a limit order market in Parlour (1998), the

underlying dynamics are very different. In the case of a limit order market, the length of the queues at bid and ask are important and both market and limit orders have an effect. In our model, with a DM and a CN, it is the imbalance between buy and sell queues in the CN that is relevant, and this imbalance is influenced only by CN orders, not by DM orders.

## 4 Equilibrium under Opaqueness

With transparency, traders condition their strategies on an information set containing general information on the distribution of  $\beta$ , their individual  $\beta_t$ , the time left until the cross, traders' distributions at both market sides, past order flow and the resulting CN's order imbalance. As argued in the introduction, however, most CNs are rather opaque. For example, CNs do not actively disseminate information on the state of their order book. The implication of opaqueness is that traders are unable to base their strategies on the current imbalance in the CN's order book. In this section, we adapt our model to capture opaqueness. We deal with two different degrees of opaqueness: "complete" and "partial". "Partial" opaqueness implies that traders do observe previous DM trades but do not have information on past order flow to the CN. Therefore they can only partially infer the current state of the CN's order book. "Complete" opaqueness refers to the case where traders also do not observe past DM trades. Introducing opaqueness renders our analysis more complicated. For tractability, we will restrict the trading stage to contain two periods. We also develop a two-period model for the benchmark transparency case to compare the opaqueness results with those of transparency. To conclude we will illustrate and contrast the various informational settings, and briefly discuss the reasoning and intuition for the more general  $T$ -period case.

### 4.1 Analysis of the Two-Period Model

The determination of the equilibrium proceeds along the same lines as before. For each informational setting, four cutoff beta values (for  $t = 1, 2$ ) characterizing traders indifferent between two strategies are relevant:  $\underline{\beta}_t^b, \bar{\beta}_t^b, \bar{\beta}_t^s, \underline{\beta}_t^s$ . The cutoff value at which indifference between no order and a CN buy order holds at  $t$ ,  $\underline{\beta}_t^b$ ,

is determined by equating expected profits for both strategies:<sup>11</sup>

$$p_t^b \left( \underline{\beta}_t^b V - \frac{A+B}{2} \right) = 0.$$

Similarly, the buyer's cutoff value between trading on a DM or submitting a CN order,  $\bar{\beta}_t^b$ , stems from:

$$\bar{\beta}_t^b V - A = p_t^b \left( \bar{\beta}_t^b V - \frac{A+B}{2} \right).$$

Equivalently, for a seller, we derive the appropriate beta values from:

$$p_t^s \left( \frac{A+B}{2} - \bar{\beta}_t^s V \right) = 0$$

$$B - \underline{\beta}_t^s V = p_t^s \left( \frac{A+B}{2} - \underline{\beta}_t^s V \right).$$

Hence, as before, to determine their choice between a CN or a DM, arriving traders need to calculate the appropriate execution probability when submitting their order to the CN. For the transparency case, this was the probability that enough counterparty traders would arrive before the time of the cross to offset the created order imbalance. For the opaqueness cases, traders have no information at all on past CN order flow, and the resulting state of the CN's order book. Traders will now have to make predictions on the created imbalance, while also accounting for the behavior of future traders.

From the above, it is clear that when the respective execution probabilities are strictly positive, in each period it holds that:

$$\underline{\beta}_t^b = \bar{\beta}_t^s = \frac{A+B}{2V}.$$

When the (expected) execution probabilities are zero, as in our model in Section 3:

$$\underline{\beta}_t^b = \frac{A}{V} \text{ and } \bar{\beta}_t^s = \frac{B}{V}.$$

The cutoff values for the choice between CN or DM,  $\bar{\beta}_t^b$  and  $\underline{\beta}_t^s$  for  $t = 1, 2$ ,

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<sup>11</sup>As before,  $p_t^b$  is the execution probability of a buy order submitted at time  $t$ .

are the solutions to the following system of four equations:

$$\begin{aligned}
\bar{\beta}_1^b V - A &= p_1^b \left( \bar{\beta}_1^b V - \frac{A+B}{2} \right) \\
B - \underline{\beta}_1^s V &= p_2^s \left( \frac{A+B}{2} - \underline{\beta}_1^s V \right) \\
\bar{\beta}_2^b V - A &= p_2^b \left( \bar{\beta}_2^b V - \frac{A+B}{2} \right) \\
B - \underline{\beta}_2^s V &= p_2^s \left( \frac{A+B}{2} - \underline{\beta}_2^s V \right).
\end{aligned} \tag{1}$$

The first two equations represent the indifference equations for traders arriving in period 1 at both market sides. The last two equations offer the same for period 2. As argued before, the solution to this set of equations will depend on the assumptions on the degree of transparency of the CN's order book. In the next subsection, we first determine the solution for the benchmark transparency case. Afterwards we turn to the complete and to the partial opaqueness cases, respectively. For all cases we assume that the starting order imbalance in the CN's order book is zero.

## 4.2 Equilibrium

### 4.2.1 Transparency

In case of transparency the CN's order book is fully observable, as in Section 3. Solving (1) for this two-period example requires the determination of all four relevant execution probabilities. First of all, we present the case of a CN buy order submitted in period 1. For this order to be executed, the trader arriving in the final period 2 must be a seller who submits a CN sell order. The range of period 2 sellers who are likely to submit such an order is equal to  $[\underline{\beta}, \frac{A+B}{2V}]$ . Hence, no sellers will engage in DM trading in period 2. All sellers with a positive value to trading will submit their order to the CN as this system now guarantees certainty of execution due to the created order imbalance (see also Figure 3 for the intuition). Hence, for a buyer arriving in period 1:

$$p_{1,t}^b = \pi^s \left[ F \left( \frac{A+B}{2V} \right) - F(\underline{\beta}) \right],$$

where the second subscript  $t$  refers to transparency. Similarly, for a first period seller:

$$p_{1,t}^s = \pi^b \left[ F(\bar{\beta}) - F\left(\frac{A+B}{V}\right) \right].$$

Buyers and sellers arriving in period 2 observe the order book imbalance and determine their strategy based on this information. Hence, their execution probabilities are discrete:

$$p_{2,t}^b = \begin{cases} 1 & \text{if } c_2 = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_{2,t}^s = \begin{cases} 1 & \text{if } c_2 = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Substituting these probabilities in (1) allows us to determine  $\bar{\beta}_{1,t}^b$ ,  $\underline{\beta}_{1,t}^s$ ,  $\bar{\beta}_{2,t}^b$ , and  $\underline{\beta}_{2,t}^s$ . Clearly, these are now “path dependent” as they hinge on the observed CN’s order book imbalance. In other words, with transparency, the cutoff betas of traders depend on the state of the book at that time and can only be computed once all previous traders have chosen their strategy (i.e. in period 2).

Our two-period analysis for transparency shows that either all period 2 traders go to the DM or all go to the CN (when submitting an order). In contrast, a range of traders in period 1 will opt to go to the DM, whereas a complementary range will opt for the CN.

#### 4.2.2 Complete Opaqueness

Complete opaqueness implies that traders cannot observe the state of the CN’s order book, as well as previous order flow to both the DM and the CN. We start our two-period analysis by determining the execution probability of a CN buy order submitted in period 1. To obtain execution of this order, the trader arriving in the final period 2 must be a seller who submits a CN sell order. The range of period 2 sellers who would submit such an order is equal to  $\left[\underline{\beta}_{2,o}^s, \frac{A+B}{2V}\right]$ , where  $o$  refers to complete opaqueness. Hence, for a buyer arriving in period 1:

$$p_{1,o}^b = \pi^s \left[ F\left(\frac{A+B}{V}\right) - F\left(\underline{\beta}_{2,o}^s\right) \right].$$



Similarly, a CN sell order in period 1 is executed if the trader in period 2 is a buyer with a beta in the range  $\left[\bar{\beta}_{1,o}^b, \frac{A+B}{2V}\right]$ , yielding an execution probability:

$$p_{1,o}^s = \pi^b \left[ F\left(\bar{\beta}_{2,o}^b\right) - F\left(\frac{A+B}{2V}\right) \right].$$

In a similar fashion, we derive the execution probabilities for the trader arriving at time 2. Buyers in period 2 account for the possibility that the trader that arrived in the previous period was a seller that submitted a CN order to determine their probability of execution:

$$p_{2,o}^b = \pi^s \left[ F\left(\frac{A+B}{2V}\right) - F\left(\underline{\beta}_{1,o}^s\right) \right].$$

Similarly, the execution probability of a CN sell order in the second period is equal to the probability that the previously arriving trader was a buyer submitting a CN order:

$$p_{2,o}^s = \pi^b \left[ F\left(\bar{\beta}_{1,o}^b\right) - F\left(\frac{A+B}{2V}\right) \right].$$

Substituting these expressions for  $p_{t,o}^b$  and  $p_{t,o}^s$  for both periods ( $t = 1, 2$ ) into the system of indifference equations (1) renders a (nonlinear) system of four equations in four unknowns  $\left(\bar{\beta}_{1,o}^b, \bar{\beta}_{2,o}^b, \underline{\beta}_{1,o}^s, \underline{\beta}_{2,o}^s\right)$ . These equations allow us to determine the cutoff beta values and hence traders' order submission decisions for both periods. Important to note is that due to the complete opaqueness assumption, all cutoff betas become independent from the actual order decisions as these are unobserved, and hence from past types and orientations of traders. Therefore, each arriving trader decides using her individual  $\beta$  and general predictions on past and future traders' behavior and the resulting expected CN order book. The result is that, in contrast to the transparency case, any form of path-dependency is absent.

### 4.2.3 Partial Opaqueness

With partial opaqueness, traders observe past DM trades. This implies that traders condition their choice between CN, DM, or no order on their individual  $\beta$ , as well as on whether DM trades have been observed. In our two-period

framework, this implies that trader 2's information set becomes larger as compared to the complete opaqueness case. For the first period, she either observes a DM trade, or she does not observe any order at all. In the former case, she knows that the book is still empty and that the execution probability of any CN order is zero. When not observing a first-period DM trade, she knows the first trader submitted a CN buy or sell order, or no order. This implies that she is able to compute execution probabilities more precisely ruling out the possibility of a DM trade. Trader 1 now takes into account the impact her choice has on the subsequent trader's information set.

Again, our goal is to derive explicit equations for the execution probabilities under this assumption, which in turn could be substituted in (1) to determine the cutoff beta values  $(\bar{\beta}_{1,p}^b, \bar{\beta}_{2,p}^b, \underline{\beta}_{1,p}^s$  and  $\underline{\beta}_{2,p}^s)$ . As for the complete opaqueness case, trader 1's execution probabilities amount to:

$$p_{1,p}^b = \pi^s \left[ F \left( \frac{A+B}{V} \right) - F \left( \underline{\beta}_{2,p}^s \right) \right]$$

$$p_{1,p}^s = \pi^b \left[ F \left( \bar{\beta}_{2,p}^b \right) - F \left( \frac{A+B}{V} \right) \right].$$

Note that although the equations are the same as for complete opaqueness, the probabilities will be different as the cutoff betas reflect trader 2's behavior. Assume that trader 2 is a buyer, and that she does not observe a DM trade in the previous period. Then, her execution probability equals the probability that the previous trader has submitted a CN sell order, conditional upon the information that no DM trade is observed. However, if she had observed a DM trade in period 1, she would know that the CN book is currently empty and that the execution probability of a CN buy order is zero. Formally:

$$p_{2,p}^b = \begin{cases} 0 & \text{if DM trade at 1 is observed} \\ p(\text{CN sell at 1} | \text{no DM trade at 1}) & \text{otherwise} \end{cases}.$$

Similarly,

$$p_{2,p}^s = \begin{cases} 0 & \text{if DM trade at 1 is observed} \\ p(\text{CN buy at 1} | \text{no DM order at 1}) & \text{otherwise} \end{cases}$$

where “no DM order at 1” refers to not observing any DM order in period 1. Using Bayes’ rule, we find that:

$$= \frac{p(\text{CN sell at 1}|\text{no DM order at 1})}{p(\text{no DM order at 1})} p(\text{CN sell at 1}).$$

We now turn to the computation of each of these probabilities. Trivially, it holds that:

$$p(\text{no DM order at 1}|\text{CN sell at 1}) = 1.$$

The unconditional probability of a CN sell order occurring at 1 is:

$$p(\text{CN sell at } t_1) = \pi^s \left[ F\left(\frac{A+B}{2V}\right) - F\left(\underline{\beta}_{1,p}^s\right) \right]$$

or the probability that a seller submitting a CN order arrives (i.e. all types in the range  $\left[\underline{\beta}_{1,p}^s, \frac{A+B}{2V}\right]$ ). Finally, the unconditional probability that no DM order occurs at 1 equals:

$$p(\text{no DM order at 1}) = \pi^b F\left[\bar{\beta}_{1,p}^b\right] + \pi^s \left[ F(\bar{\beta}) - F\left(\underline{\beta}_{1,p}^s\right) \right]$$

which equals the complement of the DM order segment on both market sides.

Hence, both execution probabilities for traders arriving in period 2 can be obtained. Substituting both periods’ execution probabilities in system (1) results in four (nonlinear) equations in four unknowns. Solving this system again renders the necessary cutoff values to determine both traders’ order submission strategies. In contrast to the complete opaqueness case, however, now these values are path dependent, which is a result similar to the transparency outcome. In the next subsection, we provide simulations which allow us to compare the three different informational settings.

### 4.3 Illustration and Comparison

In this section, we illustrate the implications of the differences in informational settings. More specifically, we look at the differences in order flow and cutoff betas for transparency, complete opaqueness and partial opaqueness in a two-

period setting. To this end, we parameterize our model by making assumptions on bid and ask prices, as well as on the continuous distribution  $F(\cdot)$  with support  $[\underline{\beta}, \bar{\beta}]$ .

For our illustration, we set the bid  $B = 10$  and the ask  $A = 11$ , such that we have a one-tick market. The level of the bid and ask price determines the relative value of the half spread, which is the amount a trader potentially saves if she submits an order to the CN instead of the DM. The execution price  $P$  on the CN is the midprice from the DM,  $0.5(A + B) = 10.5$ . We also set  $V$  equal to this midprice. We assume 2 periods in the trading day and an empty CN's order book at the beginning of the trading day. A trader is characterized by whether she is a buyer or a seller, which happens with equal probability, hence  $\pi_b = \pi_s = 0.5$ . To obtain closed form solutions for our system of indifference equations (1) we impose a uniform distribution for  $\beta_t$  with support  $[\underline{\beta}, \bar{\beta}] = [0.7, 1.3]$ . Finally, we generate 100 000 paths of two sequentially arriving traders by drawing their orientation and type from the above mentioned distributions. Using our model, we then determine their optimal strategies for the three informational settings.

Tables 1 and 2 present the results of this simulation, averaged across all simulated paths. Table 1 presents the cumulative order flow to the DM and to the CN for both periods 1 and 2. In other words, the numbers in the rows for the second period 2 are the sum of order flow in both periods. Important to note as well is that the numbers report order flow, not actual trades. Table 2 shows the average cutoff betas of the traders in both periods.

The results in both tables show that transparency produces most order flow to the CN in the first period. The reasoning is that first-period traders anticipate that their CN order will be observed by second-period traders, making a CN order relatively more attractive than a DM trade. Lower transparency, i.e. complete and partial opaqueness, induces first period traders to select a DM more often. Second period order flow, however, shows opposite results. With transparency, second period orders are only submitted to the CN when an opportunity exists. Otherwise traders go to the DM or do not trade at all. With complete opaqueness, traders do not observe the exact situation of the CN's order book. This implies that they may submit an order to the CN whereas ex-post the CN's order book may have been empty or unfavorable. This is reflected in the fact that second period order flow to the CN is highest for the complete opaqueness case. The

cumulative order flow to both CN and DM is highest with complete opaqueness. This result stems from two forces. First, complete opaqueness implies that the CN receives orders that would not occur with transparency or partial opaqueness, as it would be clear that these orders are unsuccessful. Second, the DM is more competitive with complete opaqueness as traders anticipate a CN order is less likely to be hit in the second period.

Partial relative to complete opaqueness results in slightly more order flow to the CN in the first period. This result follows from traders anticipating that a CN order now attracts more CN orders in the second period. The cumulative order flow to the CN, however, is lower. Observed first-period DM trades result in no second-period CN orders.

Table 1: **Order Flow**

Note: This table presents the cumulative order flow resulting from the illustration of the model.

	DM		CN	
	Sell	Buy	Sell	Buy
	Transparency			
1	-0.22632	0.22648	-0.02286	0.02264
2	-0.45726	0.45908	-0.02549	0.02521
	Complete opaqueness			
1	-0.23073	0.23164	-0.01845	0.01748
2	-0.46325	0.46545	-0.03564	0.03524
	Partial opaqueness			
1	-0.23070	0.23164	-0.01848	0.01748
2	-0.46284	0.46538	-0.03227	0.03137

#### 4.4 The General Case

Our illustration and our theoretical analysis for opaqueness focused on a two-period model. The question then arises whether the insights from this two-period model apply to a general case with  $T$  periods. The system of equations in (1) can easily be extended to more than two periods. The computation of the different

Table 2: **Cutoff Betas**

Note: This table presents the average cutoff betas of the illustration of the model.  $\underline{\beta}_{t,i}^s$  and  $\bar{\beta}_{t,i}^s$ , with  $t = 1, 2$  and  $i = t, o, p$  (transparency, opaqueness and partial opaqueness) are the types of the seller who is indifferent between no order and a CN order, and a CN and DM order, respectively.  $\underline{\beta}_{t,i}^b$  and  $\bar{\beta}_{t,i}^b$ , with  $t = 1, 2$  and  $i = t, o, p$  are the types of the buyer who is indifferent between no order and a CN order, and a CN and DM order, respectively.

	Sell		Buy	
	$\underline{\beta}_{t,i}^s$	$\bar{\beta}_{t,i}^s$	$\underline{\beta}_{t,i}^b$	$\bar{\beta}_{t,i}^b$
	Transparency			
1	0.93651	1	1	1.06349
2	0.94667	0.95346	1.04653	1.05338
	Complete Opaqueness			
1	0.95032	1	1	1.04968
2	0.95032	1	1	1.04968
	Partial Opaqueness			
1	0.95026	1	1	1.04974
2	0.94958	0.98897	1.01103	1.05042

execution probabilities, however, becomes far more complex. Consider the impact of many trading periods relative to the two-period model for the three different cases. In all cases, the first trader assesses the probability that at least one of the subsequent traders will be a “good” one. With transparency, this probability is fairly high as the CN order becomes visible and dominates subsequent DM trades. With complete opaqueness, a CN order also becomes more attractive if the number of periods increases. The reasoning is that it becomes a dominant strategy to submit a CN order for at least those traders who would not want to trade at a DM but may gain from a trade at a CN. This increases the likelihood that at least one “good” order will arrive, which may even induce some trade diversion from the DM.

A general analysis, however, becomes quite involved, which is illustrated by the following example. Suppose that subsequently a CN sell and a CN buy order have been submitted, and that the CN’s order book cannot be observed. If the third trader is again a buyer, her CN buy order will not be executed against the sell order in the CN’s order book since an earlier buy order was submitted. In determining the expected state of the CN’s order book, she needs to take this possibility into account in computing her expected execution probability.

To sum up, the system (1) could well be extended to a  $T$ -period model. However, the computation of the execution probabilities immediately becomes much more complicated in case of complete or partial opaqueness. Nevertheless, the main insights for our two-period model apply to a more general case.

## 5 Further Extensions

In this section, we discuss a number of possible extensions of our model. A first extension considers uncertainty in  $V$ , the final payoff of the asset on day 2. The analysis presented in Sections 3 and 4 assumed that  $V$  is fixed. However, the arrival of news in between trading periods may render  $V$  to vary over time. We therefore introduce  $V_t$ , where the subscript  $t$  highlights that  $V$  is time varying. The distribution of these news innovations can be continuous or discrete, and is known to all traders. In each trading period  $t$ , traders and dealers observe the current value  $V_t$ . In line with the analysis in Sections 3 and 4, competition between dealers forces them to set the ask  $A_t$  and the bid  $B_t$  at the first possible

price on the discrete pricing grid, above and below  $V_t$ , respectively, implying a one-tick market. The price of the cross is the midquote at time  $T$ ,  $(A_T + B_T)/2$ , which is unknown at  $t$ . In contrast, the prices at which the agent can trade in the DM at time  $t$  ( $A_t$  and  $B_t$ ) are known. Note, however, that in both cases, traders remain uncertain about the final payoff of the asset on day 2. The model with stochastic  $V_t$  can be solved in the same way as before. In Appendix B, we provide an illustration for the  $T$  period model for the transparency case. In general, the strategies chosen by traders are influenced by the uncertainty in  $V$  and will depend both on its value at time  $t$ , which determines the quoted prices in the DM at that time, and on the value at  $T$ , which is the payoff of the asset and also determines the price of the cross. Moreover, also the execution probabilities are influenced by the uncertainty. The reasoning is that the trader in period  $t$  needs to take into account the behavior of traders arriving in future periods, whose decisions are also influenced by the level of  $V$  upon their arrival. As a consequence, the cutoff betas of the model will become more time varying than without uncertainty. However, these cutoff values can still be computed in the same way. Therefore, the results and intuitions of Sections 3 and 4 remain valid, meaning that systematic patterns in order flow are still likely to be observed.

The setup in Sections 3 and 4 assumed a one-tick spread in the DM. However, competition between dealers and a CN may not always produce a one-tick spread. Multiple-tick spreads are more likely for stocks where the tick size is small relative to the value of one share. Hendershott and Mendelson (2000) endogenize the spread in a static model with simultaneous order submissions. In a dynamic model such as ours, a multiplicity of interactions between DM, CN and traders arise with potentially multiple-tick spreads. We briefly highlight a number of issues related to multiple ticks. First, the introduction of multiple-tick spreads allows dealers to strategically decide on the size of the spread in each period. Our “transparency” model of Section 3 showed that order submission to the CN is especially attractive when many periods away from the cross. The reasoning is that it is quite likely that a “good” counterparty arrives before the cross takes place. This is less likely when “close” to the cross. The implication is that with multiple ticks, dealers will tend to narrow their spread in the beginning of the trading day and widen the spread when close to the cross. As usual, dealers face a trade-off in setting a wider spread. On the one hand, they make a larger profit per



trade executed. On the other hand, since a wider spread also increases the value of the half spread traders can save by going to the CN, the dealer will attract less order flow. In our dynamic model, dealers' current quotes also shape the dynamics of the future order flow and trades, as a lower current spread decreases the likelihood that future orders will go to the CN. Second, multiple-tick spreads might lead to an "asymmetric" spread around the value  $V$ , and to "uncertainty" of the price of the cross. Symmetric changes in the spread (e.g. from a one-tick around  $V$  to a three-tick spread, where both bid and ask are changed by one tick) do not influence the price at the cross. There are a number of reasons why "asymmetric" spreads around  $V$  could occur. Dealers might change their quotes due to inventory reasons. Suppose that in the past a large number of buyers have bought at the DM. Then the dealer's inventory position is likely to be low. Consequently, he will be less willing to sell and more eager to buy to rebuild his inventory position, to this end increasing his quoted ask and bid price. Important is that the price of the cross is determined by the quotes of dealers in the final period. Traders need to take into account the incentives faced by dealers in that final period, as it determines the price at which their CN trade will execute. To avoid manipulation of the price of the cross by dealers, the time of the cross could be made uncertain, a feature indeed present in many real life crossing networks.

Traders might show heterogeneity in order size. A third possible extension therefore is the introduction of multiple unit orders. In practice, next to a "regular" trading venue, CNs can also function as an alternative to an upstairs market where large orders (blocks) can be executed. Our model could be extended to include this property. A trader then faces the problem of choosing to submit a large, multi-unit order either partly or completely to the dealer, or to the CN. With a fixed spread, submitting large orders to the CN might become less attractive when already at the long side of the CN book. The reason is that it requires more future "good" counterparties for the order to be executed. If the dealer can set quotes for several sizes, also the price impact of an order should be taken into account by traders.

Our model considers liquidity traders' choice between a CN or a DM. In reality, informed trading might take place. Informed trading next to liquidity trading adds an interesting trade-off to the model. Informed traders might prefer to remain anonymous. Crossing networks allow for anonymous trading whereas

this is more difficult at a dealer market. Informed traders also prefer immediacy (Easley and O’Hara (1987)). Immediacy is more often guaranteed at a dealer market. Moreover, informed traders could opt to submit orders to both trading venues, such that they “exploit” their informational advantage as much as possible. The impact of informed trading on the interaction between CN and DM will hinge on the longevity of the informational advantage. When information becomes public before the next cross, the CN is no longer an option for informed traders. The reasoning is that the price of the cross will reflect this information. With long-lived information advantages, the trade-off between immediacy and anonymity will also shape competition between a CN and a DM.

## 6 Conclusion

We presented a dynamic microstructure model to study the interaction between a crossing network (CN) and a dealer market (DM). We compared three different informational environments. A transparency case is contrasted to two opaqueness settings (complete and partial) which are more in line with actual practice as CNs often prevent traders from observing order flows. We find that the addition of a CN to a DM setting generates two effects on the composition of order flow. First, it leads to “order creation” as the CN attracts agents who would refrain from trading in the absence of a CN. This “order creation” effect is particularly important with complete opaqueness: these agents always submit a CN order since they are unaware of an unfavorable imbalance in the CN’s order book. In contrast, transparency or partial opaqueness might reveal such imbalance. Second, some orders by relatively low eagerness to trade agents are diverted from the DM to the CN.

We also show that the execution probability at a CN is endogenous. It depends on the state of the CN’s order book (if transparent), the observed order flow, and the expectation of past and future orders. Thus, although we start from dealers willing to provide liquidity at exogenously given bid and ask prices, we partly endogenize liquidity supply and demand by looking at traders submitting orders for potential execution at a CN. The transparency and “partial” opaqueness settings produce systematic patterns in order flow. In particular, with transparency, we find that the probability of observing a CN order at the

same side of the market is smaller after such an order than if it was not. Also, the probability of observing a sell at the DM decreases and the probability of a buyer trading on the DM increases when the previous order was a CN buy.

## Appendix A: Proofs

**Proof of Proposition 1.** Suppose first that the trader arriving at time  $t$  is a buyer. She selects her strategy to maximize her profits:

$$\max [\beta_t V - A, p_t^b (\beta_t V - (A + B) / 2), 0]$$

(this is the profit of a buy order to the dealer, the CN and no order respectively).

Now define

$$\bar{\beta}_t^b(p_t^b) = \begin{cases} \bar{\beta} & \text{if } p_t^b \geq \frac{\bar{\beta}V - A}{\bar{\beta}V - (A+B)/2} \\ \text{solves } p_t^b \left( \bar{\beta}_t^b(p_t^b) V - (A + B) / 2 \right) = \bar{\beta}_t^b(p_t^b) V - A & \text{otherwise} \end{cases}.$$

This implies that  $\bar{\beta}_t^b(p_t^b)$  is an upper bound on CN buying because in the second case  $p_t^b \left( (A + B) / 2 - \bar{\beta}_t^b(p_t^b) V \right)$  increases in  $\beta$  at rate  $p_t^b V$ , whereas  $\bar{\beta}_t^b(p_t^b) V - A$  increases at rate  $V$ . The condition

$$p_t^b \geq \frac{\bar{\beta}V - A}{\bar{\beta}V - (A + B) / 2}$$

can be interpreted as follows. If this condition is fulfilled, then  $p_t^b (\bar{\beta}V - (A + B) / 2) \geq \bar{\beta}V - A$  implying that even for  $\bar{\beta}$  the profit of an order to the CN is higher than the profit of a DM order. In that case, traders always choose to submit a CN order and the region of  $\beta$ 's for which traders submit DM orders is empty. After solving and some rewriting, we find that

$$\bar{\beta}_t^b(p_t^b) = \min \left[ \frac{\frac{A+B}{2}}{V} + \frac{1/2}{V(1-p_t^b)}, \bar{\beta} \right].$$

The cutoff  $\beta$ 's between submitting an order to the CN and remaining out of the market are determined by how large the trader's valuation of the asset is, relative to its price. The lowest  $\beta$ -type who would CN buy is the one who values the asset at  $\frac{(A+B)/2}{V}$ . Hence, define

$$\underline{\beta}_t^b(p_t^b) = \begin{cases} \frac{\frac{A+B}{2}}{V} & \text{if } p_t^b > 0 \\ \frac{A}{V} & \text{otherwise} \end{cases}.$$

Consequently, also  $\bar{\beta}_t^b(p_t^b) \geq \underline{\beta}_t^b(p_t^b)$ .

Suppose that the trader arriving at time  $t$  is a seller. She chooses her strategy to maximize  $\max [B - \beta_t V, p_t^s ((A + B) / 2 - \beta_t V), 0]$ . Define

$$\underline{\beta}_t^s(p_t^s) = \begin{cases} \underline{\beta} & \text{if } p_t^s \geq \frac{B - \underline{\beta}V}{(A+B)/2 - \underline{\beta}V} \\ \text{solves } p_t^s \left( (A + B) / 2 - \underline{\beta}_t^s(p_t^s) V \right) = B - \underline{\beta}_t^s(p_t^s) V & \text{otherwise} \end{cases}.$$

$\underline{\beta}_t^s$  is a lower bound on CN selling because in the second case  $p_t^s \left( (A + B) / 2 - \underline{\beta}_t^s(p_t^s) V \right)$  decreases in  $\beta$  at rate  $p_t^s V$ , whereas  $B - \underline{\beta}_t^s(p_t^s) V$  decreases at rate  $V$ .

The condition

$$p_t^s \geq \frac{B - \underline{\beta}V}{(A + B) / 2 - \underline{\beta}V}$$

can be understood as follows. If this condition is fulfilled, then  $p_t^s \left( (A + B) / 2 - \underline{\beta}V \right) \geq B - \underline{\beta}V$  meaning that even for  $\underline{\beta}$  the profit of an order to the CN is higher than the profit of a DM trade. In that case, traders will always choose to submit a CN order and the region of  $\beta$ 's for which traders submit DM orders is empty. Rewriting the cutoff value gives:

$$\underline{\beta}_t^s(p_t^s) = \max \left[ \frac{\frac{A+B}{2}}{V} - \frac{1/2}{V(1 - p_t^s)}, \underline{\beta} \right].$$

The cutoff  $\beta$ 's between submitting an order to the CN and remaining out of the market are determined by how large the trader's valuation of the asset is, relative to its price. The highest  $\beta$ -type who would CN sell is the one who values the asset at  $\frac{(A+B)/2}{V}$ . Define

$$\bar{\beta}_t^s(p_t^s) = \begin{cases} \frac{(A+B)/2}{V} & \text{if } p_t^s > 0 \\ \frac{B}{V} & \text{otherwise} \end{cases},$$

so we find that  $\underline{\beta}_t^s(p_t^s) \leq \bar{\beta}_t^s(p_t^s)$ . ■

**Proof of Proposition 2.** We will prove the proposition in a recursive way and by contradiction. As a starting point, it can be seen that the proposition holds for the terminal period  $T$ . At time  $T$ , the execution probability of a CN order is either one (if a trader can join the strictly shortest queue in the CN) or

zero otherwise. Then the proposition holds since:

$$\begin{aligned}\bar{\beta}_T^b &= \underline{\beta}_T^b = \frac{A+B}{2V} & \text{or} & & \bar{\beta}_T^b &= \underline{\beta}_T^b = \frac{A}{V} \\ \underline{\beta}_T^s &= \bar{\beta}_T^s = \frac{A+B}{2V} & \text{or} & & \underline{\beta}_T^s &= \bar{\beta}_T^s = \frac{B}{V}.\end{aligned}$$

Suppose now that the proposition is false. Since it is, however, true at  $T$ , there must exist a period  $\tau$  such that for  $t > \tau$ , all parts of the proposition hold, but at  $\tau$  at least one part does not hold.

Suppose that (iv) does not hold at  $\tau$ . It must then be the case that:

$$\underline{\beta}_\tau^s (c_\tau^b, c_\tau^s) < \underline{\beta}_\tau^s (c_\tau^b, c_\tau^s + 1).$$

This means that a seller having a  $\beta_\tau \in \left[ \underline{\beta}_\tau^s (c_\tau^b, c_\tau^s), \underline{\beta}_\tau^s (c_\tau^b, c_\tau^s + 1) \right)$  will submit a DM order when the CN's order book is  $(c_\tau^b, c_\tau^s + 1)$  and a CN order when the CN's order book is  $(c_\tau^b, c_\tau^s)$ . In contrast, suppose that the trader would opt for a CN order in the former case, so when the CN's order book is  $(c_\tau^b, c_\tau^s + 1)$ . The CN's order book at  $\tau + 1$  then becomes  $(c_\tau^b, c_\tau^s)$ . If the CN's order book at  $\tau$  is  $(c_\tau^b, c_\tau^s)$ , the trader submits a CN sell order resulting in the CN's order book  $\tau + 1$  being  $(c_\tau^b, c_\tau^s - 1)$ . The next trader, arriving at time  $\tau + 1$ , can be either a buyer or a seller.

*case a: A seller arrives at  $\tau + 1$ .*

In this case, we know that by assumption (iv) holds for all periods  $t > \tau$ . Therefore:

$$p_{\tau+1}^s (c_\tau^b, c_\tau^s - 1) \leq p_{\tau+1}^s (c_\tau^b, c_\tau^s).$$

Moreover, due to time priority at the CN, an order that has been submitted in  $\tau + 1$  will only be executed if the previous order in the queue has been executed. This means that conditional on a seller arriving at  $\tau + 1$ :

$$p_\tau^s ((c_\tau^b, c_\tau^s) | \text{seller arrives at } \tau + 1) \leq p_\tau^s ((c_\tau^b, c_\tau^s + 1) | \text{seller arrives at } \tau + 1).$$

Then it follows that

$$\begin{aligned}
& \left( \frac{A+B}{2} - \beta_\tau V \right) p_\tau^s ((c_\tau^b, c_\tau^s + 1) | \text{seller arrives at } \tau + 1) \\
& \geq \left( \frac{A+B}{2} - \beta_\tau V \right) p_\tau^s ((c_\tau^b, c_\tau^s) | \text{seller arrives at } \tau + 1) \\
& \geq B - \beta_\tau V.
\end{aligned}$$

Hence, if the trader at time  $\tau + 1$  is a seller, the payoff of a CN sell order is higher when the sell side of the CN's order book in the crossing network is thinner in period  $\tau$ . Hence, it cannot be optimal for a trader to submit a DM order when the queue at the sell side is shorter.

*case b: A buyer arrives at  $\tau + 1$ .*

We know that by assumption (ii) is true at  $\tau + 1$ . This means that either traders do not change their behavior or traders with:

$$\beta_{\tau+1} \in \left[ \bar{\beta}_{\tau+1}^b (c_\tau^b, c_\tau^s), \bar{\beta}_{\tau+1}^b (c_\tau^b, c_\tau^s - 1) \right]$$

submit a DM order when the CN's order book is  $(c_\tau^b, c_\tau^s)$ , which results in a CN's order book at time  $\tau + 2$  of  $(c_\tau^b, c_\tau^s)$ , and submit a CN order when the CN's order book is  $(c_\tau^b, c_\tau^s - 1)$  giving a book at  $\tau + 2$  of  $(c_\tau^b + 1, c_\tau^s - 1)$ . Since (vi) holds at  $\tau + 2$  (vi), for a seller arriving in this period:

$$p_{\tau+2}^s (c_\tau^b, c_\tau^s) = p_{\tau+2}^s (c_\tau^b + 1, c_\tau^s - 1).$$

Similarly, because (iv) is true:

$$p_{\tau+2}^s (c_\tau^b, c_\tau^s + 1) \geq p_{\tau+2}^s (c_\tau^b, c_\tau^s).$$

Since an order submitted at  $\tau + 2$  can only be executed if an order submitted at time  $\tau$  has been executed, it follows that:

$$\begin{aligned}
& \left( \frac{A+B}{2} - \beta_\tau V \right) p_\tau^s ((c_\tau^b, c_\tau^s + 1) | \text{buyer arrives at } \tau + 1) \\
& \geq \left( \frac{A+B}{2} - \beta_\tau V \right) p_\tau^s ((c_\tau^b, c_\tau^s) | \text{buyer arrives at } \tau + 1) \\
& \geq B - \beta_\tau V.
\end{aligned}$$

Hence, conditional upon a buyer arriving at time  $\tau + 1$ , there is a contradiction.

Statement (iv) is therefore true.

A symmetric proof can be constructed for (i). Along the same lines as above, the other parts of the proposition can be proven. ■

**Proof of Proposition 5.** The time  $t$  probability of observing a CN buy at time  $t + 1$  is

$$\Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN}] = \pi_B \left[ F \left( \bar{\beta}_{t+1}^b (c_{t+1}) \right) - F \left( \underline{\beta}_{t+1}^b (c_{t+1}) \right) \right].$$

Suppose that the order imbalance in the CN  $c_t^b - |c_t^s| < T - t - 1$ , such that the probability of execution of a CN buy is not zero. Then  $\underline{\beta}_{t+1}^b (c_{t+1})$  is independent of the CN's order book. If the order at time  $t$  was a CN buy order, then the CN's order book at time  $t + 1$  is  $(c_t^b + 1, c_t^s)$ , if it was a CN sell the CN's order book becomes  $(c_t^b, c_t^s - 1)$  and if the order was a market order (buy or sell) the CN's order book does not change:  $c_t = c_{t+1}$ . From Proposition 2, we know that

$$\bar{\beta}_{t+1}^b (c_t^b + 1, c_t^s) \leq \bar{\beta}_{t+1}^b (c_t^b, c_t^s) \leq \bar{\beta}_{t+1}^b (c_t^b, c_t^s - 1).$$

Since  $F(\cdot)$  is monotonically nondecreasing in  $\beta$ , the result follows.

Suppose now that  $c_t^b - |c_t^s| \geq T - t - 1$ , this means that either no CN orders are submitted, in which case the proposition holds trivially, or at time  $t$  the extra CN order submitted changes the execution probability to zero. In this case  $\bar{\beta}_{t+1}^b (c_t^b + 1, c_t^s) = A/V$ , hence the result follows since also  $\underline{\beta}_{t+1}^b (c_{t+1}) = A/V$ .

A similar proof can be constructed for CN sell orders. ■

**Proof of Proposition 6.** The time  $t$  probability of observing a CN buy at time  $t + 1$  is

$$\Pr [\phi_{t+1}^b (c_{t+1}, \beta_{t+1}) = 1^{CN}] = \pi_B \left[ F(\bar{\beta}) - F \left( \underline{\beta}_{t+1}^b (c_{t+1}) \right) \right].$$

$F(\bar{\beta})$  is fixed and independent of the CN's order book. If the order at time  $t$  was a CN buy order, then the CN's order book at time  $t + 1$  is  $(c_t^b + 1, c_t^s)$ , if it was a CN sell the CN's order book becomes  $(c_t^b, c_t^s - 1)$  and if the order was a



market order (buy or sell) the CN's order book does not change:  $c_t = c_{t+1}$ . From Proposition 2, we know that

$$\bar{\beta}_{t+1}^b (c_t^b + 1, c_t^s) \leq \bar{\beta}_{t+1}^b (c_t^b, c_t^s) \leq \bar{\beta}_{t+1}^b (c_t^b, c_t^s - 1).$$

Since  $F(\cdot)$  is monotonically nondecreasing in  $\beta$ , the result follows.

A similar proof can be constructed for CN sell orders. ■

## Appendix B: Uncertainty

In this appendix, we introduce uncertainty into the model by making  $V$  a stochastic variable  $V_t$  which varies over time. More specifically, we assume that  $V_t$  follows a random walk.<sup>12</sup>

$$V_t = V_{t-1} + \varepsilon_t$$

where  $\varepsilon_t$  is identically and independently distributed with mean zero and variance  $\sigma_\varepsilon^2$ . Its distribution can be continuous, e.g. the normal distribution, or discrete and is known to all traders. The noise variable can be interpreted as the occurrence of a news shock in the market in between periods during the trading day. These shocks influence the underlying value of the asset. In each period  $t$ , traders and dealers observe the current value  $V_t$ . As the value of the asset changes over time, dealers are allowed to adjust their quotes. More specifically, before the trader at  $t$  arrives, dealers set their bid and ask prices around  $V_t$ . Competition is assumed to be harsh such that they set the ask  $A_t$  and the bid  $B_t$  at the first possible price on the discrete pricing grid, respectively above and below  $V_t$ . In this way, we retain a one-tick market. Note that this assumption also implies that quotes in future periods will remain at their current levels as long as future values of  $V$  remain between  $A_t$  and  $B_t$ . Dealer prices only change when  $V$  exceeds the ask in the previous period or falls below the previous bid.

Traders condition their optimal strategies on a number of variables. Some of these variables are, as in our baseline model, known with certainty. In particular, we assume that the trader arriving at time  $t$  observes the state of the order book in the CN  $c_t$  (so we assume transparency), the time left until the cross  $T - t$ , the proportions of buyers and sellers  $\pi_b$  and  $\pi_s$  and her type  $\beta_t$ , as well as the overall distribution of  $\beta$ . She also observes  $V_t$ , the current realization of the value of the asset, and the resulting dealer market quotes  $A_t$  and  $B_t$ . However, she is uncertain about the final value of the asset  $V_T$  and the resulting price of the cross  $(A_T + B_T)/2$ . Therefore, relying on her knowledge of the underlying stochastic process of  $V$ , she needs to determine expected values for these variables. Consequently, her submission strategy will depend on these expectations.

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<sup>12</sup>This is without loss of generality. Our methodology can be easily adapted for alternative assumptions on the data generating process of  $V_t$ .

Suppose that the trader that arrives at time  $t$  is a buyer. The expected profit of submitting a buy order to the dealer is:

$$E_t \left( \Pi_t^{b,DM} \right) = \beta_t E_t (V_T) - A_t$$

where  $E_t$  denotes the expectations operator, based on the information set at time  $t$ .

Alternatively, submitting a CN buy order will give her the following expected profit:

$$E_t \left( \Pi_t^{b,CN} \right) = p_t^b \left( \beta_t E_t (V_T) - E_t \left( \frac{A_T + B_T}{2} \right) \right).$$

As possible execution of this order only occurs at the end of the trading day, she needs to form expectations on  $V_T$  and the associated price of the cross. Note that the latter is in contrast with a DM buy order, where the trader is uncertain about the value  $V_T$  but knows the current buy price  $A_t$ . Due to the random walk assumption for  $V_t$ , it holds that all expectations equal the current values. Hence,  $E_t (V_T) = V_t$  and  $E_t ((A_T + B_T) / 2) = (A_t + B_t) / 2$ . Given that she knows the distributions of  $\varepsilon_t$  and  $\beta_t$ , as well as the state of the order book  $c_t$  and the remaining number of periods  $T - t$ , she can compute the expected execution probability of a CN buy order,  $p_t^b$ . If she is able to join the shortest queue  $p_t^b$  reaches unity. Otherwise  $p_t^b$  equals the probability that enough traders that are willing to join the market's sell side queue will arrive, hence counterbalancing the imbalance that would be created by her order.

The profit of submitting no order remains zero.

The cutoff values for  $\beta_t$  at which traders are indifferent between two strategies could be calculated as in Section 3.3.<sup>13</sup> A buyer at  $t$  is indifferent between submitting a CN order or a DM trade at the following cutoff value:

$$\bar{\beta}_t^b (p_t^b, V_t) = \min \left[ \frac{\frac{A_t + B_t}{2}}{V_t} + \frac{1/2}{V_t (1 - p_t^b)}, \bar{\beta} \right].$$

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<sup>13</sup>As our traders are considered to be risk neutral, they are indifferent between two strategies yielding the same expected profit.

Similarly, she is indifferent between submitting a CN order or no order at:

$$\underline{\beta}_t^b(p_t^b, V_t) = \begin{cases} \frac{\frac{A_t+B_t}{2}}{V_t} & \text{if } p_t^b > 0 \\ \frac{A_t}{V_t} & \text{otherwise} \end{cases} .$$

Deriving the seller's cutoff values is completely symmetric. Comparing these values with the ones found in Section 3.3, at first sight they appear to be very similar. However, some important differences do exist.

First, as traders are uncertain about the final underlying asset value  $V_T$ , the cutoff betas now depend on their expectation of this value and of the resulting surrounding quotes. Due to our random walk assumption, these forecasts equal the current value of the variables (i.e.  $V_t$ ,  $A_t$  and  $B_t$ ), implying that current variable levels will determine traders' expected profits and hence their trading strategy. Therefore, if news arrives in the market altering the underlying value of the asset, the expectation for  $V_T$  (and depending on the position in the pricing grid possibly for  $A_T$  and  $B_T$ ) will also vary. Consequently, as compared to our baseline model, the cutoff betas are more dynamic and also depend on the current state of the market.<sup>14</sup> Moreover, note that if  $V_t$  changes, but remains between the previous bid and ask prices (such that these do not change), this will still affect the order submission strategy of the trader since the ratio of the (expected) price of the cross and the (expected) value of the asset changes.

Secondly, as discussed above, introducing uncertainty in the model also has repercussions for the computation of the probability of execution,  $p_t^b$ . As in our baseline model, to determine  $p_t^b$  trader  $t$  will account for the behavior of traders arriving in future periods. However, these traders' decisions will now also be influenced by the level of  $V$  at that point in time. The current trader takes this effect into account while determining her submission strategy.

To sum up, we have shown that it is possible to calculate cutoff values which appear to be similar to those in our baseline model. Based on these values, a graph comparable to Figure 3 could be drawn. However, now traders' strategies also reflect uncertainty on future values of  $V$ . Nevertheless, it is clear that even with

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<sup>14</sup>This is obviously the case for  $\bar{\beta}_t^b$ , and it is even more apparent for  $\underline{\beta}_t^b$ . In the baseline model,  $\underline{\beta}_t^b$  is in general equal to a fixed number  $((A+B)/2/V)$ , and only changes to  $A/V$  when the execution probability of a CN buy order becomes zero. In our extended model,  $\underline{\beta}_t^b$  changes every period (unless  $V_t = V_{t-1}$ ).

these new cutoff values and the resulting strategies, propositions 3 to 6 remain valid. Hence, systematic patterns in order flow are still likely to be observed. In other words, our empirical predictions in Section 3.4 are robust to the introduction of uncertainty in the underlying value of the asset.

## References

- Biais, B., L. Glosten, C. Spatt, 2002, The Microstructure of Stock Markets, *CEPR Discussion Paper 3288*.
- Chakravarty, S., C. Holden, 1995, An Integrated Model Of Market And Limit Orders, *Journal of Financial Intermediation*, 4, pp. 213-241.
- Chemmanur, T., P Fulghieri, 2003, Competition and Co-operation Among Exchanges: a Theory of Cross Listing and Endogenous Listing Standards, *Working Paper*.
- Conrad, J., K. Johnson, S. Wahal, 2003, Institutional Trading and Alternative Trading Systems, *Journal of Financial Economics*, 70, pp. 99-134.
- Di Noia, C., 2001, Competition and Integration Among Stock Exchanges in Europe: Network Effects, Implicit Mergers and Remote Access, *European Financial Management*, 7, pp. 39-72.
- Dönges, J., F. Heinemann, 2001, Competition for Order Flow as a Coordination Game, *Working Paper*.
- Easley, D., M. O'Hara, 1987, Price, Trade Size, and Information in Securities Markets, 19, *Journal of Financial Economics*, 19, pp. 69-90.
- Fong, K., A. Madhavan, P. Swang, 2004, Are Upstairs Markets Pareto Improving?, *School of Banking and Finance, University of New South Wales Working Paper*.
- Foucault, T., 1999, Order Flow Composition and Trading Costs in a Dynamic Limit Order Market, *Journal of Financial Markets*, 2, pp. 99-134.
- Foucault, T., C. Parlour, 2001, Competition for Listings, forthcoming in *RAND Journal of Economics*.
- Foucault, T., O. Kadan, E. Kandel, 2003, Limit Order Book as a Market for Liquidity, forthcoming in *Review of Financial Studies*.
- Glosten, L., 1994, Is the Electronic Open Limit Order Book Inevitable?, *Journal of Finance*, 49, pp. 1127-1161.

- Glosten, L., 1998, Precedence Rules, *Columbia University Working Paper*.
- Goettler, R., Parlour, C., Rajan, U., 2004, Equilibrium in a Dynamic Limit Order Market, forthcoming in *Journal of Finance*.
- Gresse, C., 2002, Crossing Network Trading and the Liquidity of a Dealer Market: Cream Skimming or Risk Sharing?, *Working Paper*.
- Hendershott, T., H. Mendelson, 2000, Crossing Networks and Dealer Markets: Competition and Performance, *Journal of Finance*, 55, pp. 2071-2115.
- Huang, R., 2002, The Quality of ECN and NASDAQ Market Maker Quotes, *Journal of Finance*, 57, pp. 1285-1319.
- Næs, R., B. Ødegaard, 2004, Equity Trading by Institutional Investors. To Cross or Not to Cross?, *Working Paper*.
- Parlour, C., 1998, Price Dynamics in Limit Order Markets, *Review of Financial Studies*, 11, pp. 789-816.
- Parlour, C., D. Seppi, 2003, Liquidity-Based Competition for Order Flow, *Review of Financial Studies*, 16, pp. 301-343.
- Rock, K., 1996, The Specialist's Order Book and Price Anomalies, *Working Paper*.
- Rosu, I., 2004, A Dynamic Model of the Limit Order Book, *M.I.T. Working Paper*.
- Santos, T., J. Scheinkman, 2001, Competition Among Exchanges, *Quarterly Journal of Economics*, 116, pp. 1027-1061.
- Securities and Exchange Commission, 1998, Regulation of Exchanges and Alternative Trading Systems, *Release Number 34-40760, 8 December*.
- Seppi, D., 1997, Liquidity Provision with Limit Orders and a Strategic Specialist, *Review of Financial Studies*, 10, pp. 103-150.
- Shy, O., J. Tarkka, 2001, Stock Exchanges Alliances, Access Fees and Competition, *Bank of Finland DP 22/2001*.

Viswanathan, S., J. Wang, 2002, Market Architecture: Limit-Order Books versus Dealership Markets, *Journal of Financial Markets*, 5, pp. 127-167.