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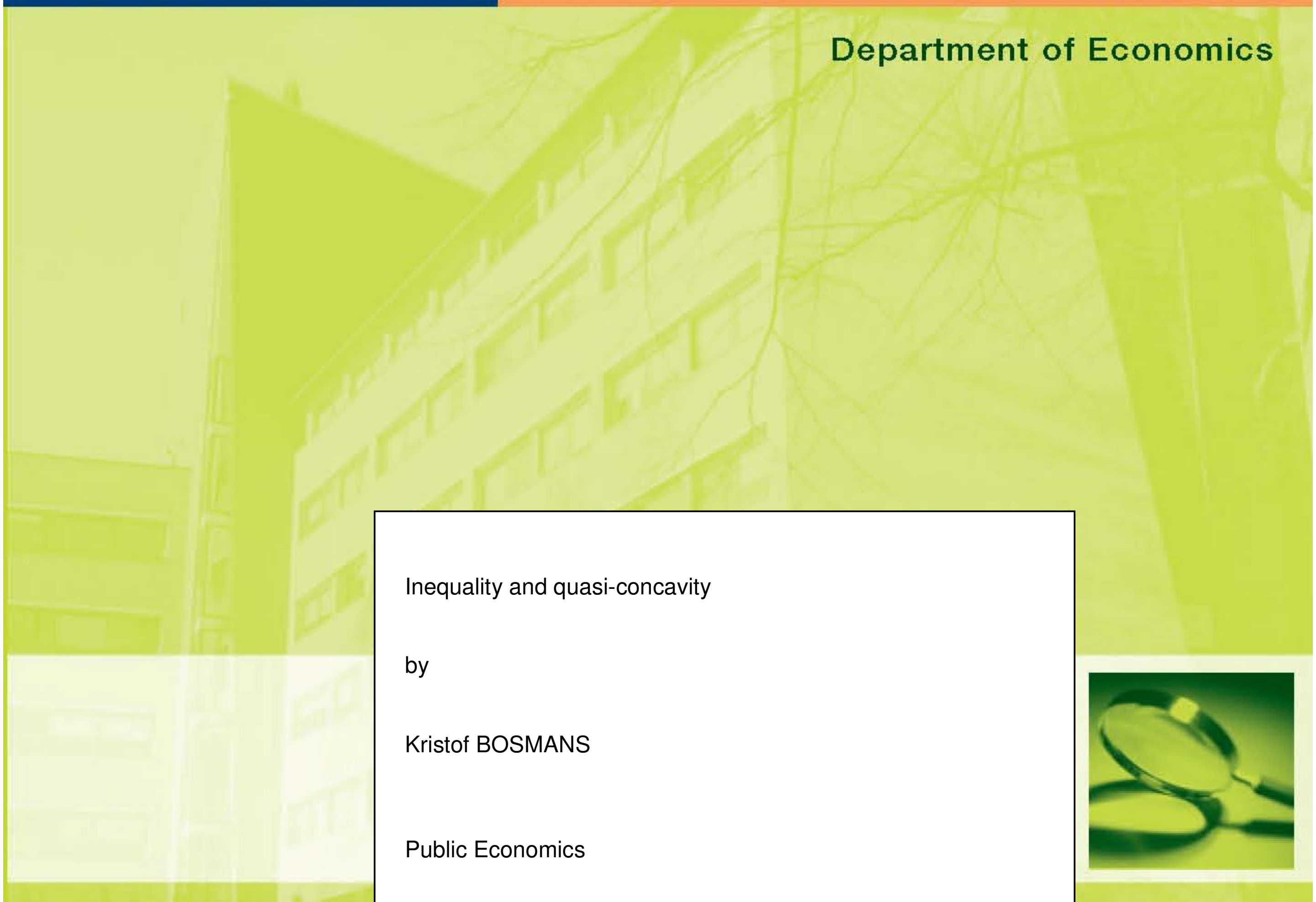

Inequality and quasi-concavity

by

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DISCUSSION
PAPER

Inequality and Quasi-Concavity*

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Abstract

We discuss a property of quasi-concavity for inequality measures. Defining income distributions as relative frequency functions, this property says that a convex combination of any two given income distributions is weakly more unequal than the least unequal income distribution of the two. The quasi-concavity property is not essential to the idea of inequality comparisons in the sense of not being implied by the fundamental, i.e., Lorenz type, axioms on their own. However, it is shown that all inequality measures considered in the literature—i.e., the class of decomposable inequality measures and the class of normative inequality measures based on a social welfare function of the rank-dependent expected utility form—satisfy the property (and even a stronger version). The quasi-concavity property is then shown to greatly reduce the possible inequality patterns over a much studied type of income growth process.

Keywords: Inequality, Quasi-Concavity, Growth, Rank-Dependent Expected Utility
JEL Classification Numbers: D31, D63

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1 Introduction

Consider two income distributions, defined as relative frequency functions, as well as a convex combination of these two income distributions. As an example, take two groups of income receivers, with income distributions f and g, that are merged. In that case, the income distribution of the merged group is a convex combination of those of the two original ones: $\alpha f(x) + (1 - \alpha)g(x)$ for all incomes x, where α is the share of the population size of the group with income distribution f in the total population size of the merged group. Can anything in general be said with respect to how a convex combination of any two given income distributions relates in terms of income inequality to each of those given two income distributions? In this paper it is shown that a general statement concerning this question is possible for the two classes of inequality measures that, together, encompass virtually the entire literature on inequality measures—viz., the class of decomposable inequality measures, and the class of normative inequality measures based on the general social welfare function of the rank-dependent expected utility form. These inequality measures turn out to satisfy a property of quasi-concavity, saying that any convex combination of two given income distributions is at least as unequal as the least unequal of the two. Moreover, the inequality measures even come close to satisfying strict quasi-concavity: if the two given income distributions have unequal means, then the convex combination is *strictly* more unequal than the least unequal of the two income distributions. We believe the result is of technical interest since it shows that simple mathematical properties, hitherto not explicitly studied in the literature on inequality measurement, hold very generally. Moreover, although the quasi-concavity properties are satisfied by all well known inequality measures, they are not necessary characteristics of the idea of inequality in the sense that they are not implied by the fundamental, i.e., Lorenz type, axioms. Indeed, the fact that, as we shall see below, some direct implications of the quasi-concavity properties have received criticism in the literature, reveals that they are not uncontroversial.

Besides being of technical interest, the result is relevant to the question of how inequality evolves during a process of income growth in a dual economy, a topic that has interested economists since Kuznets (1955). At any stage of such an income growth process the economy is characterized by an income distribution which is a convex combination of the income distributions in the first (low mean income) stage and in the final (high mean income) stage—the later the stage, the greater the weight of the final stage income distribution. The question of interest is how inequality evolves during the transition from the first to the final stage income distribution. Kakwani (1988) and Anand and Kanbur (1993) amongst others have shown that various popular inequality measures imply that inequality follows an inverted-U pattern—i.e., inequality increases at the early stages of the process and decreases afterwards. We show that this is an implication of the considered quasi-concavity properties, which are satisfied by all these inequality measures. Taking a more fundamental perspective, Fields (1987, 1993) has argued that, during a simplified income growth process, inequality should follow a different pattern, viz., a U pattern. Fields' work can be seen as a critique of some the consequences of the quasi-concavity properties. Our result enables us to generalize the findings of Kakwani and of Anand and Kanbur and reveals that it is impossible to find any inequality measure conforming to the standard approach in the literature that allows expression of Fields' view, even if this view is watered down severely.

The paper is organized as follows. Section 2 deals with notation and basic concepts. In Section 3, we show axiomatically that the quasi-concavity properties are satisfied by all inequality quasi-orderings satisfying the transfer principle, a weak invariance axiom, and decomposability. Instead of focussing exclusively on relative inequality concepts, as is common in the literature, we consider the weak invariance axiom of Bossert and Pfingsten (1990) that allows for relative and absolute inequality concepts as well as intermediate ones. We believe the given context, which is typically concerned with inequality comparisons of income distributions with unequal mean incomes, makes such a more general approach relevant. While the result of Section 3 applies to, amongst others, the inequality measures based on a social welfare function of the expected utility form, it does not apply to its rank-based alternatives, the generalized Gini indices, as these are not decomposable. Therefore, we consider in Section 4 the class of inequality measures (absolute, relative as well as intermediate cases) based on a social welfare function of the rank-dependent expected utility form, which generalizes both the class of expected utility inequality measures and the class of generalized Gini indices. Benefiting from functional representability of the given inequality orderings, it is shown that the quasi-concavity properties are also satisfied by all members of this general class of normative inequality measures. In Section 5 we spell out the implications of the results of Sections 3 and 4 for the question of how inequality evolves during a process of income growth in a dual economy. Section 6 concludes. All the proofs are contained in the Appendix.

2 Preliminaries

Suppose there is a finite number of individuals in society each having a positive income which is an element of $\mathbb{R}_{++}$. Let the relative frequency function $f : \mathbb{R}_{++} \to [0,1]$ represent the income distribution, the number $f(x)$ being the proportion of the population with income x. We denote the set of income distributions with $\mathscr{F}$. We write the support of any $f \in \mathscr{F}$ as $\left\{x_{f1}, x_{f2}, \ldots, x_{fn}\right\}$. For any income distribution $f \in \mathscr{F}$, the mean income, $\sum_{i=1}^{n} f(x_{fi})x_{fi}$, is denoted with $\mu(f)$. Inequality comparisons of income distributions are captured by a binary relation $\preceq$ ("is not more unequal than") over the elements of $\mathscr{F}$. The relation's asymmetric and symmetric factors are denoted $\prec$ and $\sim$, respectively. We assume that the relation $\preceq$ is a quasi-ordering, i.e., is reflexive and transitive. A quasi-ordering that is complete is an ordering. An inequality measure is defined as a function $I : \mathscr{F} \to \mathbb{R}$ that represents some inequality ordering.

Before we can define the axioms and properties, it is necessary to introduce three additional pieces of notation. First, an income distribution g is said to be obtained from any $f \in \mathscr{F}$ by a *mean preserving spread* if and only if, for any $x \in \mathbb{R}_{++} \setminus \{x_1, x_2, x_3, x_4\}$ such that $x_1 < x_2 \leq x_3 < x_4$, it holds that $g(x) = f(x)$, while $g(x_1) = f(x_1) + \delta$, $g(x_2) = f(x_2) - \delta$, $g(x_3) = f(x_3) - \delta$ and $g(x_4) = f(x_4) + \delta$, where $\delta > 0$ is a scalar such that $g \in \mathscr{F}$ and $\mu(f) = \mu(g)$. Informally, when g is obtained from f by a mean preserving spread, this means that g is obtained from f by a series of regressive transfers. Second, for any $f \in \mathscr{F}$, let $f_{x \mapsto \psi(x)} = f \circ \psi^{-1}$. So, for instance, $f_{x \mapsto \gamma x}$ denotes the income distribution obtained from f by multiplying each individual's income by γ. Finally, for any $f, g \in \mathscr{F}$ and any scalar α, let $\alpha f + (1-\alpha)g$ denote the function $\alpha f(x) + (1-\alpha)g(x)$ for all $x \in \mathbb{R}_{++}$.

We now define three basic axioms. The first is the well known transfer principle which says that regressive transfers increase inequality.

Axiom 1 (TP). For any $f \in \mathscr{F}$, if g is obtained from f by a mean preserving spread, then $f \prec g$.

The second axiom is a general invariance condition proposed by Bossert and Pfingsten (1990).

Axiom 2 (βINV). There is some scalar $\beta \in [0,1]$ such that the following holds. For any $f \in \mathscr{F}$ and any scalar λ such that $f_{x \mapsto x + \lambda(\beta x + 1 - \beta)} \in \mathscr{F}$ it holds that $f \sim f_{x \mapsto x + \lambda(\beta x + 1 - \beta)}$.

The axiom (βINV) generalizes the popular relative ($\beta = 1$) and absolute ($\beta = 0$) cases. Inequality relations satisfying (βINV) for $\beta \in (0,1)$ are referred to as intermediate inequality relations. In line with the literature, we consider (TP) and (βINV) to be the fundamental axioms.

Finally, a popular but certainly less compelling axiom is decomposability, which says, loosely speaking, that any transformation of the income distribution that changes only the incomes of a subgroup of the population and leaves mean income unaffected, should affect overall inequality in the same direction as it affects the income inequality in the subgroup.[1]

Axiom 3 (DEC). For any $f, g, h \in \mathscr{F}$ such that $\mu(f) = \mu(g)$ it holds that

$$f \preceq g \Leftrightarrow \alpha f + (1-\alpha)h \preceq \alpha g + (1-\alpha)h \quad \text{for any } \alpha \in (0,1).$$

The main focus of this paper are the properties quasi-concavity and strict quasi-concavity which describe a particular way in which a convex combination of two given income distributions compares in terms of inequality with these given two income distributions.

Property 1 (QC). For any $f, g \in \mathscr{F}$ it holds that

$$f \preceq g \Rightarrow f \preceq \alpha f + (1-\alpha)g \quad \text{for any } \alpha \in (0,1). \tag{1}$$

[1]For a critique of decomposability, see Sen and Foster (1997, pp. 149-163). The axiom they refer to as "subgroup consistency" is closest to our definition.

Property 2 (SQC). For any $f,g \in \mathscr{F}$ such that $f \neq g$ it holds that

$$f \preceq g \Rightarrow f \prec \alpha f + (1-\alpha)g \quad \text{for any } \alpha \in (0,1). \tag{2}$$

More relevant than (SQC), however, will turn out to be the following property which says that (2) has to hold only conditionally.

Property 3 (CSQC). For any $f,g \in \mathscr{F}$ such that $\mu(f) \neq \mu(g)$, (2) holds.

Obviously, (SQC) implies both (QC) and (CSQC), while the latter two properties are independent.

Using the minimal framework of inequality quasi-orderings, we examine in Section 3 the relationships between the three basic axioms of inequality measurement and the quasi-concavity properties. The main result of the section is that the three axioms (TP), (βINV) and (DEC) are sufficient for the properties (QC) and (CSQC) to hold. In Section 4, similar results are shown to hold for the members of an important class of inequality orderings consistent with (TP) and (βINV) but not (necessarily) with (DEC).

3 Inequality Quasi-Orderings

Before we present the main result of this section, we consider some direct links between, on the one hand, the axioms separately or in pairs, and, on the other hand, the properties (QC), (SQC) and (CSQC), and the natural counterparts of the former two, viz., quasi-convexity and strict quasi-convexity.

Property 4 (QV). For any $f,g \in \mathscr{F} \setminus \{f \in \mathscr{F} \mid f(e) = 1 \text{ for some } e \in \mathbb{R}_{++}\}$ it holds that

$$f \preceq g \Rightarrow \alpha f + (1-\alpha)g \preceq g \quad \text{for any } \alpha \in (0,1). \tag{3}$$

Property 5 (SQV). For any $f,g \in \mathscr{F} \setminus \{f \in \mathscr{F} \mid f(e) = 1 \text{ for some } e \in \mathbb{R}_{++}\}$ such that $f \neq g$ it holds that

$$f \preceq g \Rightarrow \alpha f + (1-\alpha)g \prec g \quad \text{for any } \alpha \in (0,1). \tag{4}$$

Note that the perfectly equal income distributions are excluded from the set of income distributions over which (3) and (4) are required to hold in the definitions of (QV) and (SQV). The reason for this is that the properties (QV) and (SQV) would otherwise be incompatible with the commonsense requirement that any unequal income distribution is strictly more unequal than any perfectly equal one.[2]

First, we wish to point out the relationship between the fundamental axioms (TP) and (βINV) and the five properties. Before we move on, consider the following lemma which will be useful in what follows.

[2]This can straightforwardly be seen by letting f and g in (3) or (4) both be perfectly equal income distributions (with $f \neq g$). Note, furthermore, that the "commonsense requirement" is implied by (TP) and (βINV) jointly.

Lemma 1. *Let* $\preceq$ *be an inequality quasi-ordering satisfying* (TP) *and* (βINV). *Then, for any* $f \in \mathscr{F}$ *and any scalar* λ *such that* $f_{x\mapsto x+\lambda(\beta x+1-\beta)} \in \mathscr{F}$ *and* $f \neq f_{x\mapsto x+\lambda(\beta x+1-\beta)}$*, it holds that* $f \prec \alpha f + (1-\alpha) f_{x\mapsto x+\lambda(\beta x+1-\beta)}$ *for any* $\alpha \in (0,1)$.

The following remark is an immediate corollary of Lemma 1.

Remark 1. Any inequality quasi-ordering satisfying (TP) and (βINV) does not satisfy (QV).

Although, jointly, (TP) and (βINV) rule out both (QV) and (SQV), the axioms do not imply the counterparts of these axioms (QC) or (SQC) and also not (CSQC). The following example illustrates this.

Example 1. Take the inequality measure $I(f) = AI_{GE}^{10}(f) + I_{GE}^{-9}(f)$, where A is a positive and finite scalar and where I_{GE}^{θ} is the generalized entropy inequality measure, given by

$$I_{GE}^{\theta}(f) = \frac{1}{\theta^2 - \theta} \sum_{i=1}^{n} f(x_{fi}) \left[\left(\frac{x_{fi}}{\mu(f)} \right)^{\theta} - 1 \right],$$

where $\theta > 0$. Since the generalized entropy inequality measure satisfies (TP) and (βINV) (for $\beta = 1$), I obviously satisfies the axioms as well. Contrarily, I_{GE}^{θ} satisfies (DEC), while I does not. Now consider three income distributions f, g and h such that $(f(10), f(50)) = (0.4, 0.6)$, $(g(10), g(50)) = (0.9, 0.1)$ and $(h(10), h(50)) = (0.65, 0.35)$. Let, moreover, $A = \frac{I_{GE}^{-9}(f) - I_{GE}^{-9}(g)}{I_{GE}^{10}(g) - I_{GE}^{10}(f)}$. It holds that

$$23.370 = I(h) = I\left(\frac{1}{2}f + \frac{1}{2}g\right) < I(f) = I(g) = 270.061,$$

which implies that I violates (QC), (SQC) and (CSQC). Furthermore, it can be shown that (4) holds for the chosen f and g.

The example shows that not all reasonable inequality quasi-orderings, in the sense of consistency with the fundamental axioms (TP) and (βINV), must satisfy (QC) and (CSQC). However, in this paper it is shown that all inequality quasi-orderings commonly considered in the literature do so.

Second, we consider the relationship between the (DEC) axiom and the five properties. The axiom is clearly related to both (QC) and (QV) as the following remark shows.

Remark 2. Let $\preceq$ be an inequality quasi-ordering satisfying (DEC). Then, (1) and (3) hold for any $f, g \in \mathscr{F}$ such that $\mu(f) = \mu(g)$.

Remark 2 reveals that (DEC) imposes linearity over sets of income distributions with equal means—i.e., a convex combination of any two given income distributions with equal means must lie, in terms of inequality, between those two given income distributions. Clearly, then, on its own (DEC) implies a bias to neither (QC) nor (QV). Note that, for all pairs of income distributions f,g such that $\mu(f) = \mu(g)$ and $f \sim g$, (2) and (4) fail to hold if (DEC) is satisfied. So, given the weak assumption (which would be implied, for instance, by continuity) that at least one such pair f,g exists, (DEC) is incompatible with both (SQC) and (SQV). Also note that, for all pairs of income distributions f,g such that $\mu(f) = \mu(g)$ and $f \prec g$, (2) and (4) are implied by (DEC). Consequently, (CSQC) is in a sense as far as one can go in the direction of (SQC) while still satisfying (DEC): for all pairs of income distributions f,g for which (SQC) implies (2) while (CSQC) does not, either (DEC) already implies (2), or (DEC) is inconsistent with (2).

To summarize, we have seen that (TP), (βINV) and (DEC) rule out (QV), (SQV) and, typically, (SQC) and that, moreover, the fundamental axioms (TP) and (βINV) are not sufficient for (QC) or (CSQC). The following result shows that any inequality quasi-ordering satisfying (TP), (βINV) and (DEC), must satisfy (QC) as well as (CSQC), i.e., comes as close to satisfying (SQC) as allowed by (DEC).

Proposition 1. *Any inequality quasi-ordering satisfying* (TP), (βINV) *and* (DEC) *satisfies* (QC) *and* (CSQC).

Proposition 1 has implications that are relevant in the context of the study of the evolution of inequality in a developing country. We postpone the discussion of these implications until Section 5, but consider here relevant results concerning this context by Kakwani (1988) and Anand and Kanbur (1993) that are generalized in Proposition 1. Anand and Kanbur present results that imply that the inequality orderings represented by the following relative inequality measures satisfy (CSQC): the first and second Theil inequality measures, the coefficient of variation, the entire class of Atkinson inequality measures, and the Gini index in the case of non-overlapping income distributions f and g.[3] The same has been shown by Kakwani for the entire class of generalized entropy inequality measures, thus generalizing the results pertaining to all measures considered by Anand and Kanbur except the Gini index.[4] Proposition 1 shows that neither the demand that inequality be a relative concept nor even completeness or continuity are essential in obtaining the result. Examples of absolute inequality measures covered by Proposition 1 are the variance and the entire class of Kolm inequality measures. Notable inequality measures that Proposition 1 does not deal with—because they do not satisfy (DEC)—are the Gini index in the general, possibly overlapping, case, as well as its rank-based generalizations. In the next section, we show that a similar result as

[3]Moreover, the logarithmic variance, also considered by Anand and Kanbur, can be added to the list. Anand and Kanbur borrow the result concerning this inequality measure from Robinson (1976). We do not consider the logarithmic variance as it does not satisfy (TP).

[4]Kakwani (1988, pp. 210-213) mistakenly believes to have proved the result only for the generalized entropy inequality measures for which $\theta \geq 1$ and $\theta = 0$. However, he proves the result also for the entire Atkinson class which is ordinally equivalent to the generalized entropy class in the case where $\theta < 1$. Therefore, the ordinal nature of the property (CSQC) implies that the result applies to the entire generalized entropy class.

Proposition 1 holds for a class of normative inequality measures that encompasses both the well known classes of decomposable normative inequality measures (the Atkinson and Kolm inequality measures) and the generalized Gini indices.

4 Normative Inequality Orderings

Normative inequality measures are based on some conception of social ethics, captured by a social welfare function $W : \mathscr{F} \to \mathbb{R}$. Define the equally distributed equivalent income for any income distribution $f \in \mathscr{F}$, $\xi(f)$, as the per capita income which, if distributed equally, yields the same level of social welfare as f. That is, $\xi(f)$ for any income distribution $f \in \mathscr{F}$ is defined as $\xi(f) = e$, where $e \in \mathbb{R}_{++}$ is such that there is a $g \in \mathscr{F}$ such that $g(e) = 1$ and $W(f) = W(g)$. It is common to define relative normative inequality measure using

$$I(f) = 1 - \frac{\xi(f)}{\mu(f)}, \tag{5}$$

and absolute normative inequality measures using

$$I(f) = \mu(f) - \xi(f). \tag{6}$$

We assume throughout this section that, for any $f \in \mathscr{F}$, the elements of the support, $\{x_{f1}, x_{f2}, \ldots, x_{fn}\}$, are indexed such that $x_{f1} < x_{f2} < \cdots < x_{fn}$.

The well known Atkinson and Kolm inequality measures are based on a social welfare function of the expected utility (EU) form,

$$W(f) = \sum_{i=1}^{n} f(x_{fi})\, u(x_{fi}), \tag{7}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function. The relative (Atkinson) case, which we denote with $I_{EU}^{1,\varepsilon}$, is obtained as (5) with W as in (7) and

$$u(x) = \frac{1}{1-\varepsilon} x^{1-\varepsilon}, \tag{8}$$

where $\varepsilon > 0$. Following Bossert and Pfingsten (1990), the general class of relative and intermediate EU inequality measures—more specifically, those satisfying (βINV) for $0 < \beta \leq 1$—is defined as $I_{EU}^{\beta,\varepsilon}(f) = \frac{1}{\beta} I_{EU}^{1,\varepsilon}\left(f_{x \mapsto \beta x + 1 - \beta}\right)$ for all $f \in \mathscr{F}$. Finally, the absolute (Kolm) family of EU inequality measures (corresponding to $\beta = 0$), denoted with $I_{EU}^{0,\gamma}$, is obtained as (6) with W as in (7) and

$$u(x) = -\exp(-\gamma x), \tag{9}$$

where $\gamma > 0$. All members of the relative, intermediate and absolute families of EU inequality measures satisfy (TP), (βINV) and (DEC) and thus are covered by Proposition 1.

The generalized Gini indices are based on a social welfare function of the Yaari form (Yaari, 1987),

$$W(f)=\sum_{i=1}^{n}\pi\left(f(x_{fi}),\sum_{k=1}^{i}f\left(x_{fk}\right)\right)x_{fi}, \tag{10}$$

where, for all $i=1,2,\ldots,(n-1)$,

$$\pi\left(f(x_{fi}),\sum_{k=1}^{i}f\left(x_{fk}\right)\right)=\phi\left(\sum_{j=i}^{n}f(x_{fj})\right)-\phi\left(\sum_{j=i+1}^{n}f(x_{fj})\right),$$

$\pi(f(x_{fn}),1)=\phi(f(x_{fn}))$ and $\phi:[0,1]\to[0,1]$ is a continuous and strictly increasing function with $\phi(0)=0$ and $\phi(1)=1$. The relative case, denoted with $I^{1,\phi}_{Yaari}$, is given by (5) with W as in (10). Again, following Bossert and Pfingsten (1990), the complete class of relative, intermediate and absolute Yaari (or generalized Gini) inequality measures is obtained as $I^{\beta,\phi}_{Yaari}(f)=\frac{1}{\beta}I^{1,\phi}_{Yaari}\left(f_{x\mapsto\beta x+1-\beta}\right)$ for all $f\in\mathscr{F}$.[5] In order for the inequality measure to satisfy (TP), ϕ has to be strictly convex. A well known subclass of the generalized Gini indices is that of the S-Gini indices for which $\phi(p)=p^{\rho}$, where $\rho>1$, with as a notable special case the Gini index ($\rho=2$).

Both the complete class of EU inequality measures and the complete class of Yaari inequality measures are encompassed by a more general class of inequality measures based on a social welfare function of the rank-dependent expected utility (RDEU) form,

$$W(f)=\sum_{i=1}^{n}\pi\left(f(x_{fi}),\sum_{k=1}^{i}f\left(x_{fk}\right)\right)u(x_{fi}), \tag{11}$$

where, for all $i=1,2,\ldots,(n-1)$,

$$\pi\left(f(x_{fi}),\sum_{k=1}^{i}f\left(x_{fk}\right)\right)=\phi\left(\sum_{j=i}^{n}f(x_{fj})\right)-\phi\left(\sum_{j=i+1}^{n}f(x_{fj})\right),$$

$\pi(f(x_{fn}),1)=\phi(f(x_{fn}))$ and $\phi:[0,1]\to[0,1]$ and $u:\mathbb{R}\to\mathbb{R}$ are continuous and strictly increasing functions with $\phi(0)=0$ and $\phi(1)=1$. The relative case, denoted with $I^{1,\varepsilon,\phi}_{RDEU}$, is given by (5) with W as in (11) and either u as in (8) or u the identity function. Again, following Bossert and Pfingsten (1990), we obtain the relative, intermediate and part of the absolute RDEU inequality measures as $I^{\beta,\varepsilon,\phi}_{RDEU}(f)=\frac{1}{\beta}I^{1,\varepsilon,\phi}_{RDEU}\left(f_{x\mapsto\beta x+1-\beta}\right)$ for all $f\in\mathscr{F}$. This class includes the cases where $0<\beta\leq 1$ if u is not the identity function and the cases where $0\leq\beta\leq 1$ if u is the identity function. Hence,

$$I^{\beta,\varepsilon,\phi}_{RDEU}(f)=\frac{1}{\beta}\left[1-\frac{\left(\sum_{i=1}^{n}\pi\left(f(x_{fi}),\sum_{k=1}^{i}f\left(x_{fk}\right)\right)\left(x_{fi}+\frac{1-\beta}{\beta}\right)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}}{\mu(f)+\frac{1-\beta}{\beta}}\right], \tag{12}$$

[5] The absolute case is obtained by taking the limit $\beta\to 0$.

where $0 < \beta \leq 1$ if $\varepsilon > 0$ and $0 \leq \beta \leq 1$ if $\varepsilon = 0$, and where π is defined as above. The absolute RDEU inequality measures not given by $I_{RDEU}^{\beta,\varepsilon,\phi}$ are the cases where u is not the identity function. These are obtained as (6) with W as in (11) and u as in (9) and are denoted with $I_{RDEU}^{0,\gamma,\phi}$. Hence,

$$I_{RDEU}^{0,\gamma,\phi}(f) = \mu(f) + \frac{1}{\gamma} \ln \left[\sum_{i=1}^{n} \pi \left(f(x_{fi}), \sum_{k=1}^{i} f(x_{fk}) \right) (\exp(-\gamma x_i)) \right], \tag{13}$$

where $\gamma > 0$ and π is defined as above. RDEU inequality measures satisfy (TP) if and only if ϕ is strictly convex when u is the identity function and convex otherwise.[6] It is straightforward to see that all members of the EU and Yaari classes of inequality measures are members of the class of RDEU inequality measures given by (12) or (13): whenever ϕ is the identity function, the class of RDEU inequality measures reduces to the class of EU inequality measures, and whenever u is the identity function ($\varepsilon = 0$ in (12)), the inequality measures are of the generalized Gini type. The only inequality measures in the RDEU class that satisfy (DEC) are those that also belong to the EU class. RDEU inequality measures for which neither ϕ nor u is the identity function, and which, consequently, belong to neither the EU class nor the Yaari class, have been studied by Ebert (1988) and Chateauneuf et al. (2002). Ebert (1988) shows that, in general, RDEU inequality measures also incorporate a decomposability idea, obviously a weaker one than (DEC).

To prove a result similar to Proposition 1 for the entire RDEU class, we require the following well known result which we state formally and prove for the sake of completeness.

Lemma 2. *Let W be a social welfare function of the RDEU form, given by* (11). *Then, the following two statements hold:*

(i) *If ϕ is convex, then W is convex, i.e., for any $f, g \in \mathscr{F}$ it holds that*

$$W(\alpha f + (1-\alpha)g) \leq \alpha W(f) + (1-\alpha)W(g) \quad \textit{for any } \alpha \in (0,1).$$

(ii) *If ϕ is strictly convex, then W is strictly convex, i.e., for any $f, g \in \mathscr{F}$ such that $f \neq g$ it holds that*

$$W(\alpha f + (1-\alpha)g) < \alpha W(f) + (1-\alpha)W(g) \quad \textit{for any } \alpha \in (0,1).$$

The following proposition says that the properties (QC) and (CSQC) are satisfied for inequality orderings representable by any member of the class of RDEU inequality measures.

Proposition 2. *Let $\preceq$ be an inequality ordering that satisfies* (TP) *and is representable by any RDEU inequality measure, given by* (12) *or* (13). *Then, the following two statements hold:*

[6] See Chew, Karni and Safra (1987).

(i) *If ϕ is convex, then $\preceq$ satisfies* (QC) *and* (CSQC).

(ii) *If ϕ is strictly convex, then $\preceq$ satisfies* (SQC).

Proposition 2 says that (SQC) is satisfied if the weighting function corresponding to the RDEU inequality measure is strictly convex, while this need not be the case if the weighting function is convex. The following remark makes it clear that there is a general positive relationship between the degree of convexity of the weighting function and the degree to which full (SQC) is approached.

Remark 3. Let $\preceq_1$ and $\preceq_2$ be two inequality orderings that both satisfy (TP) and are representable by any RDEU inequality measures, given by (12) or (13). Let, moreover, the respective weighting functions ϕ_1 and ϕ_2 be such that ϕ_1 is "at least as convex as" ϕ_2, i.e., there exists a convex function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\phi_1 = \chi \circ \phi_2$. Then, for any $f, g \in \mathscr{F}$ and any $\alpha \in (0,1)$, $f \prec_2 \alpha f + (1-\alpha) g$ implies $f \prec_1 \alpha f + (1-\alpha) g$, while the converse does not hold.

The proof of Remark 3 is implicit in the proof of Proposition 2 and is therefore omitted.

5 Inequality and Income Growth in a Dual Economy

We shall now examine how the properties (QC) and (CSQC) are related to the question of how inequality changes during a process of income growth in a dual economy. In such an economy, the population is distributed over a traditional sector and a modern one. We denote the income distribution associated to the traditional sector with g and that associated to the modern sector with f and assume $\mu(f) > \mu(g)$. In the first stage of the income growth process, the economy is completely traditional and hence is characterized by the income distribution g, while in the final stage it is completely modern and hence characterized by the income distribution f. During the transition from a traditional to a modern economy, it is assumed that population shifts from the traditional sector to the modern sector in a gradual fashion. By consequence, at any stage strictly in between the first and the final, the income distribution of the economy as a whole is $\alpha f + (1-\alpha) g$, where $\alpha \in (0,1)$ is the percentage of the population in the modern sector. The question is how inequality evolves during this growth process, i.e., as α rises over the interval $(0,1)$. As we have seen in the previous sections, all well known inequality concepts satisfy the properties (QC) and (CSQC). As the following proposition shows, these properties turn out to greatly reduce the number of allowed patterns that describe inequality evolution during the considered growth process. The proposition focusses on (CSQC) which has the stronger implications of the two properties, and, for convenience, restricts attention to inequality orderings.

Proposition 3. *Let $\preceq$ be an inequality ordering for which* (CSQC) *holds. Consider, moreover, a dual economy income growth process with as the first and final stage income distributions respectively any $g \in \mathscr{F}$ and $f \in \mathscr{F}$ such that $\mu(f) > \mu(g)$.*

Then, only the following three patterns describing the evolution of inequality during the income growth process are possible:

(i) *An inverted-U pattern, i.e., there exists an* $\alpha^* \in (0,1)$ *such that, for any* $\alpha, \alpha' \in (0, \alpha^*]$, *if* $\alpha > \alpha'$ *then*

$$\alpha' f + \left(1-\alpha'\right) g \prec \alpha f + (1-\alpha)\, g,$$

and, for any $\alpha, \alpha' \in [\alpha^*, 1)$, *if* $\alpha > \alpha'$ *then*

$$\alpha f + (1-\alpha)\, g \prec \alpha' f + \left(1-\alpha'\right) g.$$

(ii) *A strictly increasing pattern, i.e., for any* $\alpha, \alpha' \in (0,1)$, *if* $\alpha > \alpha'$ *then*

$$\alpha' f + \left(1-\alpha'\right) g \prec \alpha f + (1-\alpha)\, g.$$

(iii) *A strictly decreasing pattern, i.e., for any* $\alpha, \alpha' \in (0,1)$, *if* $\alpha > \alpha'$ *then*

$$\alpha f + (1-\alpha)\, g \prec \alpha' f + \left(1-\alpha'\right) g.$$

Fields (1987, 1993) has studied the considered income growth process in its most simple case, i.e., that where the first and final stage income distributions are both perfectly equal. He notes that, on the one hand, the popular inequality measures—by which he means those studied by Anand and Kanbur (1993) (see our Section 3)—all imply an inverted-U pattern of inequality in this case, while, on the other hand, different patterns are also plausible. In his own work, Fields defends a U pattern on the basis of the notions "elitism of the rich" and "isolation of the poor."[7] Loosely speaking, elitism of the rich says that, for relatively high values of α, increases in α lead to greater inequality because the "rich" (i.e., those in the modern sector) then attain a more elite position. Similarly, isolation of the poor says that, for relatively low values of α, decreases in α cause inequality to increase because the "poor" (i.e., those in the traditional sector) then become more isolated.[8]

Proposition 3 allows us to generalize some of Fields' observations. A first implication of the proposition is that continuous inequality orderings that satisfy (CSQC) always imply an inverted-U pattern, not only in the simple case considered by Fields, but also in the more general case where the perfectly equal first and final stage income distributions are replaced by any two income distributions

[7] See especially Fields (1993).

[8] The simple case of the income growth process has also been considered by Temkin (1986) and by Amiel and Cowell (1994). Using his own framework for inequality measurement, the philosopher Temkin gives justifications for the three patterns dealt with in Proposition 3 as well as for a pattern of constant inequality during the entire process. Amiel and Cowell provide questionnaire results showing that respondents support several patterns amongst which the U pattern proposed by Fields is quite popular.

that are equally unequal.[9] By Proposition 1 and 2, this holds for all continuous inequality measures considered in the literature, not only for the inequality measures dealt with by Anand and Kanbur. Note that, for the case with first and final stage income distributions that are equally unequal, the patterns (ii) and (iii) in Proposition 3 are only possible for noncontinuous inequality orderings since these patterns involve a discontinuity at $\alpha = 0$ or at $\alpha = 1$.[10] A second implication of Proposition 3 is that the U pattern proposed by Fields cannot occur for any inequality ordering satisfying (CSQC), not even when the condition that the initial and final stage income distributions are equally unequal is dropped. Furthermore, even over part of the income growth process, i.e., as α rises over a subinterval of $(0, 1)$, a U pattern is impossible. Note, however, that, as Example 1 shows, the fundamental axioms (TP) and (βINV) do not exclude the occurrence of a U pattern over part of the income growth process for some first and final stage income distributions.

6 Conclusion

The literature on inequality measurement has focussed exclusively on the specific strategy of supplementing the fundamental axioms, (TP) and (βINV), with decomposability ideas, i.e., ideas concerning how changes in the inequality of subgroups have to relate to changes in overall inequality—directly, in the form of the (DEC) axiom, or, indirectly, by basing inequality measures on a (RDEU) social welfare function that incorporates a weak decomposability condition. It was shown in this paper that all inequality measures considered in the literature satisfy the quasi-concavity properties (QC) and (CSQC). Moreover, it was shown that the latter property implies that only three patterns describing how inequality evolves during a process of income growth in a dual economy are possible.

The results of this paper clarify and generalize a problem revealed by Fields (1987, 1993). Fields has pointed out that in a simple case of the dual economy income growth process more patterns are plausible than the three that our results show to be implied by (CSQC). Since our analysis reveals that the properties (QC) and (CSQC) limit the allowed patterns even in far more general cases and that, moreover, these properties are satisfied by virtually all inequality measures considered in the literature, it allows us to state Fields critique with more force. If it is argued that all plausible views about inequality comparisons should be expressible, then we have to conclude that the properties (QC) and (CSQC) unduly limit the scope of inequality measurement. It follows that, if we want a more satisfactory theory of inequality measurement, then we should focus on supplementing the fundamental axioms in alternative ways, rather than with decomposability ideas.

[9] Note that in the case considered by Fields, the first and final stage income distributions are also equally unequal. At least, this would follow from (βINV) or from the common sense assumption that all perfectly equal income distributions are equally unequal.

[10] An example of an inequality ordering that satisfies (TP), (βINV) and (DEC) and yet never implies an inverted-U pattern can straightforwardly be constructed on the basis of the leximin social welfare ordering.

Appendix: Proofs

For convenience, in the proofs we usually abbreviate, for any $f, g \in \mathscr{F}$ and any scalar α, the expression $\alpha f + (1-\alpha) g$ with αfg.

Proof of Lemma 1

Take any $f \in \mathscr{F}$, any scalar λ such that $g = f_{x \mapsto x + \lambda(\beta x + 1 - \beta)} \in \mathscr{F}$ and $f \neq g$, and any $\alpha \in (0,1)$. Note that, by definition, $\lambda = \frac{\mu(g) - \mu(f)}{\beta \mu(f) + 1 - \beta}$.

Consider $h = f_{x \mapsto x + \lambda'(\beta x + 1 - \beta)}$, where $\lambda' = \frac{\mu(\alpha fg) - \mu(f)}{\beta \mu(f) + 1 - \beta}$. The choice of λ' ensures that $\mu(h) = \mu(\alpha fg)$. We now prove the claim that by (TP), $h \prec \alpha fg$ holds. Note that the supports of f, g and h have the same number of elements. Now, clearly, to any element of the support of f, say x, there corresponds exactly one element in the support of g, viz., $x + \lambda(\beta x + 1 - \beta)$, which appears with frequency $f(x)$. By consequence, in αfg, there is, for any element in the support of f, a pair of incomes such that the sum of frequencies is $f(x)$ and the mean income for the group of individuals with any of these two incomes is $\alpha x + (1-\alpha)(x + \lambda(\beta x + 1 - \beta))$. Similarly, to any element in the support of f, x, there corresponds exactly one income in the support of h, $x + \lambda'(\beta x + 1 - \beta)$, which appears with frequency $f(x)$. Now, it can be checked that $x + \lambda'(\beta x + 1 - \beta) = \alpha x + (1-\alpha)(x + \lambda(\beta x + 1 - \beta))$. Therefore, αfg can be obtained from h by a sequence of mean preserving spreads. Hence, $h \prec \alpha fg$ by (TP). Since , moreover, $f \sim h$ by (βINV), $f \prec \alpha fg$ holds by transitivity. ■

Proof of Proposition 1

Suppose that any $f, g \in \mathscr{F}$ such that $f \preceq g$ and any $\alpha \in (0,1)$ are given. Since the case in which $\mu(f) = \mu(g)$ has already been dealt with in Remark 2, we only consider the case in which $\mu(f) \neq \mu(g)$.

Consider $h = f_{x \mapsto x + \lambda(\beta x + 1 - \beta)}$, where $\lambda = \frac{\mu(g) - \mu(f)}{\beta \mu(f) + 1 - \beta}$. The choice of λ ensures that $\mu(h) = \mu(g)$. Two cases are possible: either (a) $h \in \mathscr{F}$, or (b) $h \notin \mathscr{F}$.

(a) In the first case $f \sim h$ by (βINV), so that, by transitivity, $h \preceq g$. Clearly then, by (DEC), $\alpha fh \preceq \alpha fg$. By Lemma 1, $f \prec \alpha fh$, so that, by transitivity, $f \prec \alpha fg$.

(b) The second case occurs if and only if λ is such that in going from f to h, nonpositive incomes get nonzero frequency (which is only possible if $\mu(f) > \mu(g)$). Consider $\hat{f} = f_{x \mapsto x + \lambda'(\beta x + 1 - \beta)}$ and $\hat{g} = g_{x \mapsto x + \lambda'(\beta x + 1 - \beta)}$, where λ' is some scalar such that $[x^* + \lambda'(\beta x^* + 1 - \beta)] + \lambda(\beta [x^* + \lambda'(\beta x^* + 1 - \beta)] + 1 - \beta) > 0$ with x^* the lowest income to appear with nonzero frequency in f. We can then return to the beginning of this proof and prove the result for $\hat{f}$ and $\hat{g}$ without getting case (b). If the result holds for $\hat{f}$ and $\hat{g}$, then it must hold for f and g as well by (βINV) and transitivity. ■

Proof of Lemma 2

First note that (11) can be rewritten as

$$W(f) = u(x_{f1}) + \sum_{i=2}^{n} \phi\left(\sum_{j=i}^{n} f(x_{fj})\right)\left[u(x_{fi}) - u\big(x_{f(i-1)}\big)\right].$$

Take any $f, g \in \mathscr{F}$ and any scalar $\alpha \in (0,1)$ and write the support of $\alpha f + (1-\alpha)g$ as $\{x_1, x_2, \ldots, x_n\}$ with the elements indexed such that $x_1 < x_2 < \cdots < x_n$. We then have

$$\begin{aligned}
&W(\alpha f + (1-\alpha)g) \\
&\quad = u(x_1) + \sum_{i=2}^{n} \phi\left(\sum_{j=i}^{n} [\alpha f(x_j) + (1-\alpha)g(x_j)]\right)[u(x_i) - u(x_{i-1})] \\
&\quad \leq \alpha\left[u(x_1) + \sum_{i=2}^{n} \phi\left(\sum_{j=i}^{n} f(x_j)\right)[u(x_i) - u(x_{i-1})]\right] \\
&\qquad\qquad + (1-\alpha)\left[u(x_1) + \sum_{i=2}^{n} \phi\left(\sum_{j=i}^{n} g(x_j)\right)[u(x_i) - u(x_{i-1})]\right] \\
&\quad = \alpha W(f) + (1-\alpha)W(g),
\end{aligned}$$

where the inequality follows from the convexity of ϕ. The inequality holds strictly whenever ϕ is strictly convex and $f \neq g$. ■

Proof of Proposition 2

Take any $f, g \in \mathscr{F}$ such that $f \preceq g$, and any scalar $\alpha \in (0,1)$.

We first consider the case where $\preceq$ is representable by (12). Defining the function W^β as $W^\beta(f) = W\left(f_{x \mapsto x + \frac{1-\beta}{\beta}}\right)$ for all $f \in \mathscr{F}$ with W as in (11) and u as in (8), we have

$$I_{RDEU}^{\beta,\varepsilon,\phi}(\alpha fg) = \frac{1}{\beta}\left[1 - \frac{\big((1-\varepsilon)W^\beta(\alpha fg)\big)^{\frac{1}{1-\varepsilon}}}{\mu(\alpha fg) + \frac{1-\beta}{\beta}}\right]. \tag{14}$$

We have to show the following: (a) expression (14) is at least as great as (strictly greater than) $I_{RDEU}^{\beta,\varepsilon,\phi}(f)$ whenever ϕ is (strictly) convex, and (b) expression (14) is strictly greater than $I_{RDEU}^{\beta,\varepsilon,\phi}(f)$ whenever $\mu(f) \neq \mu(g)$.

Firstly, consider

$$\frac{1}{\beta}\left[1-\frac{\left((1-\varepsilon)\left[\alpha W^{\beta}(f)+(1-\alpha)W^{\beta}(g)\right]\right)^{\frac{1}{1-\varepsilon}}}{\mu(\alpha fg)+\frac{1-\beta}{\beta}}\right] \tag{15}$$

$$= \frac{1}{\beta}\left[1-\frac{1}{\mu(\alpha fg)+\frac{1-\beta}{\beta}}\left[\alpha\left[\mu(f)+\frac{1-\beta}{\beta}\right]^{1-\varepsilon}\left(\frac{\left[(1-\varepsilon)W^{\beta}(f)\right]^{\frac{1}{1-\varepsilon}}}{\left(\mu(f)+\frac{1-\beta}{\beta}\right)}\right)^{1-\varepsilon}\right.\right.$$

$$\left.\left.+(1-\alpha)\left[\mu(g)+\frac{1-\beta}{\beta}\right]^{1-\varepsilon}\left(\frac{\left[(1-\varepsilon)W^{\beta}(g)\right]^{\frac{1}{1-\varepsilon}}}{\left(\mu(g)+\frac{1-\beta}{\beta}\right)}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}\right]$$

$$= \frac{1}{\beta}\left[1-\left[1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)\right]A\right],$$

where

$$A = \frac{\left[\alpha\left[\mu(f)+\frac{1-\beta}{\beta}\right]^{1-\varepsilon}+(1-\alpha)\left[\mu(g)+\frac{1-\beta}{\beta}\right]^{1-\varepsilon}\left[\frac{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(g)}{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)}\right]^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}{\mu(\alpha fg)+\frac{1-\beta}{\beta}}$$

$$= B\times C,$$

where

$$B=\frac{\alpha\left[\mu(f)+\frac{1-\beta}{\beta}\right]+(1-\alpha)\left(\left[\mu(g)+\frac{1-\beta}{\beta}\right]\left[\frac{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(g)}{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)}\right]\right)}{\beta\left[\alpha\mu(f)+(1-\alpha)\mu(g)\right]+\frac{1-\beta}{\beta}} \tag{16}$$

and

$$C = \frac{\left[\alpha\left[\mu(f)+\frac{1-\beta}{\beta}\right]^{1-\varepsilon}+(1-\alpha)\left[\mu(g)+\frac{1-\beta}{\beta}\right]^{1-\varepsilon}\left[\frac{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(g)}{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)}\right]^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}{\alpha\left[\mu(f)+\frac{1-\beta}{\beta}\right]+(1-\alpha)\left(\left[\mu(g)+\frac{1-\beta}{\beta}\right]\left[\frac{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(g)}{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)}\right]\right)}$$

$$= \left[1-I_{EU}^{1,\varepsilon}(\alpha h_1+(1-\alpha)h_2)\right], \tag{17}$$

where $h_1, h_2 \in \mathscr{F}$ are such that $h_1(x)=1$ if $x=\mu(f)+\frac{1-\beta}{\beta}$ and $h_2(x)=1$ if $x=\left[\mu(g)+\frac{1-\beta}{\beta}\right]\left[\frac{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(g)}{1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)}\right]$.

Secondly, notice that $\frac{1}{\beta}\left[1-\left[1-\beta I_{RDEU}^{\beta,\varepsilon,\phi}(f)\right]BC\right]\geq I_{RDEU}^{\beta,\varepsilon,\phi}(f)$, since $0<B\leq 1$ and $0<C\leq 1$. Since, moreover, it follows from Lemma 2 that whenever ϕ is (strictly) convex, expression (14) is at least as great as (is strictly greater than) expression (15), (a) follows. Notice that whenever $\mu(f)\neq\mu(g)$ and $I_{RDEU}^{\beta}(f)=$

$I^{\beta,\varepsilon,\phi}_{RDEU}(g)$, it holds that $B = 1$ but $C < 1$, so that $\frac{1}{\beta}\left[1-\left[1-\beta I^{\beta,\varepsilon,\phi}_{RDEU}(f)\right]BC\right] > I^{\beta}_{RDEU}(f)$, and whenever $\mu(f) \neq \mu(g)$ and $I^{\beta,\varepsilon,\phi}_{RDEU}(f) < I^{\beta,\varepsilon,\phi}_{RDEU}(g)$, $B < 1$, so that, again, $\frac{1}{\beta}\left[1-\left[1-\beta I^{\beta,\varepsilon,\phi}_{RDEU}(f)\right]BC\right] > I^{\beta,\varepsilon,\phi}_{RDEU}(f)$. Together with the fact that convexity of ϕ implies that expression (14) is at least as great as expression (15), (b) follows.

We now consider the second case where $\preceq$ is representable by (13). Using W as in (11) and u as in (9), we have

$$I^{0,\gamma,\phi}_{RDEU}(\alpha fg) = \mu(\alpha fg) + \frac{1}{\gamma}\ln(-W(\alpha fg)). \tag{18}$$

The following has to be shown: (c) expression (18) is at least as great as (strictly greater than) $I^{0,\gamma,\phi}_{RDEU}(f)$ whenever ϕ is (strictly) convex, and (d) expression (18) is strictly greater than $I^{0,\gamma,\phi}_{RDEU}(f)$ whenever $\mu(f) \neq \mu(g)$.

Consider

$$\mu(\alpha fg) + \frac{1}{\gamma}\ln(-[\alpha W(f) + (1-\alpha)W(g)]), \tag{19}$$

and

$$\alpha\mu(f) + (1-\alpha)\mu(g) + \frac{1}{\gamma}\alpha\ln(-W(f)) + (1-\alpha)\ln(-W(g)) \tag{20}$$

$$= \alpha I^{0,\gamma,\phi}_{RDEU}(f) + (1-\alpha)I^{0,\gamma,\phi}_{RDEU}(g). \tag{21}$$

It follows from Lemma 2 that whenever ϕ is (strictly) convex, expression (18) is at least as great as (is strictly greater than) expression (19). Since, moreover, expression (19) is at least as great as expression (20) by concavity of the ln function, we have (c). In the case where $\mu(f) \neq \mu(g)$ and $I^{0,\gamma,\phi}_{RDEU}(f) = I^{0,\gamma,\phi}_{RDEU}(g)$, it holds that $W(f) \neq W(g)$ and, hence, expression (19) is strictly greater than expression (20) by strict concavity of the ln function. If $\mu(f) \neq \mu(g)$ and $I^{0,\gamma,\phi}_{RDEU}(f) \neq I^{0,\gamma,\phi}_{RDEU}(g)$, then expression (21) is strictly greater than $I^{0,\gamma,\phi}_{RDEU}(f)$. Hence, (d) follows. ■

Proof of Proposition 3

Consider an income growth process with as first and final stage income distributions respectively any $g \in \mathscr{F}$ and $f \in \mathscr{F}$ such that $\mu(f) > \mu(g)$.

Consider the following two subpatterns, both of which describe how inequality evolves as α increases over some subinterval $(\underline{\alpha},\overline{\alpha}) \subseteq (0,1)$:

(a) A constant pattern over $(\underline{\alpha},\overline{\alpha})$, i.e., for any $\alpha,\alpha' \in (\underline{\alpha},\overline{\alpha})$, $\alpha fg \sim \alpha' fg$.

(b) A U pattern over $(\underline{\alpha},\overline{\alpha})$, i.e., there exists an $\alpha^* \in (\underline{\alpha},\overline{\alpha})$ such that, for any $\alpha,\alpha' \in (\underline{\alpha},\alpha^*]$, if $\alpha > \alpha'$ then $\alpha fg \prec \alpha' fg$, and, for any $\alpha,\alpha' \in [\alpha^*,\overline{\alpha})$, if $\alpha > \alpha'$ then $\alpha' fg \prec \alpha fg$.

We first show by contradiction that neither subpattern can be the case for any $(\underline{\alpha},\overline{\alpha})$. Suppose that (a) or (b) holds over some subinterval $(\underline{\alpha},\overline{\alpha}) \subseteq (0,1)$. Both subpatterns imply that there exist some $\alpha,\alpha',\alpha'' \in (\underline{\alpha},\overline{\alpha})$ where $\alpha > \alpha' > \alpha''$

such that $\alpha' fg \preceq \alpha fg$ and $\alpha' fg \preceq \alpha'' fg$. This is obvious in the case of (a). In the case of (b), choose α' equal to the α^* from the definition of (b). Note now that, for $\alpha''' = \frac{\alpha'-\alpha''}{\alpha-\alpha''}$, it holds that $\alpha'''(\alpha fg)(\alpha'' fg) = \alpha' fg$. Consequently, it holds that $\alpha'''(\alpha fg)(\alpha'' fg) \preceq \alpha fg$ and $\alpha'''(\alpha fg)(\alpha'' fg) \preceq \alpha'' fg$. Since, moreover, $\alpha''' \in (0,1)$ and $\mu(\alpha fg) \neq \mu(\alpha'' fg)$, we have a violation of (CSQC).

Now, any pattern for which subpattern (a) is not the case for some $(\underline{\alpha}, \overline{\alpha})$, is either pattern (i), (ii), (iii), or a pattern for which pattern (b) is the case for some $(\underline{\alpha}, \overline{\alpha})$. However, as we have seen, the latter is impossible. ■

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