Comparing degrees of inequality aversion

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Comparing Degrees of Inequality Aversion*

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Abstract

We propose a straightforward dominance procedure for comparing social welfare orderings (SWOs) with respect to the degree of inequality aversion they express. We consider three versions of the procedure: (i) a criterion based on the Lorenz quasi-ordering which we argue to be the ideal version, (ii) a criterion based on a minimalist concept of inequality, and (iii) a criterion based on the relative differentials quasi-ordering. It turns out that the traditional Arrow-Pratt approach is equivalent to the latter two criteria for important classes of SWOs, but that it is profoundly inconsistent with the Lorenz-based criterion. With respect to the problem of combining extreme inequality aversion and monotonicity, criteria (ii) and (iii) identify as extremely inequality averse a set of SWOs that includes leximin as a special case, whereas the Lorenz-based criterion concludes that extreme inequality aversion and monotonicity are incompatible.

Keywords: Inequality Aversion, Lorenz, Leximin, Maximin, Risk Aversion

JEL Classification Numbers: D63, D81

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1 Introduction

How should we compare different social preference relations over income distributions with respect to the degree of inequality aversion, i.e., the degree of dislike towards inequality, they express? We propose a procedure for comparing degrees of inequality aversion that can be loosely formulated as follows:

Procedure ($\star$): An SWO $\succeq_W$ is *at least as inequality averse as* an SWO $\succeq'_W$ if and only if, for all income distributions $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}$ is *less unequal than* $\mathbf{y}$ according to a pre-specified inequality quasi-ordering, (i) $\succeq_W$ strictly prefers $\mathbf{x}$ to $\mathbf{y}$ ($\mathbf{x} \succ_W \mathbf{y}$) whenever $\succeq'_W$ strictly prefers $\mathbf{x}$ to $\mathbf{y}$ ($\mathbf{x} \succ'_W \mathbf{y}$), and (ii) $\succeq_W$ weakly prefers $\mathbf{x}$ to $\mathbf{y}$ ($\mathbf{x} \sim_W \mathbf{y}$ or $\mathbf{x} \succ_W \mathbf{y}$) whenever $\succeq'_W$ is indifferent between $\mathbf{x}$ and $\mathbf{y}$ ($\mathbf{x} \sim'_W \mathbf{y}$).

Note that in order to make this procedure operational, an inequality quasi-ordering must first be chosen. This feature of Procedure ($\star$) makes explicit the fact that, underlying any criterion for comparing degrees of inequality aversion, there necessarily has to be a concept for making comparisons according to inequality—obviously, to be able to check whether an SWO expresses more or less *dislike towards inequality* than another SWO, it must be clear what is meant by inequality in the first place. Once an inequality quasi-ordering is chosen, Procedure ($\star$) turns into a fully operational criterion which entails a straightforward check for dominance: an SWO is referred to as at least as inequality averse as another if it implies, in all relevant choice situations (i.e., those pairs of income distributions that are strictly ranked using the chosen inequality quasi-ordering), an at least as inequality averse choice as the other (as defined in (i) and (ii) of Procedure ($\star$)). Procedure ($\star$) can furthermore be shown to be consistent with the common approach of measuring the degree of inequality aversion by the amount of mean income an SWO is prepared to forego in exchange for a given decrease in inequality (see Section 3).

Interestingly, the traditional Arrow-Pratt criterion for comparing degrees of inequality aversion[1] is a special case of Procedure ($\star$). Roughly speaking, the Arrow-Pratt criterion is obtained in the case where the chosen inequality quasi-ordering is the extremely simplistic one which allows only (strict) inequality comparisons between, on the one hand, unequal income distributions and, on the other hand, perfectly equal ones (see Section 4). In this paper, we take the point of view that while Procedure ($\star$) is the appropriate way to approach the problem of comparing degrees of inequality aversion, the Arrow-Pratt version of the procedure is unattractive because it is based on an unduly restrictive inequality quasi-ordering. Taking into consideration its central place in the literature on inequality measure-

[1] The Arrow-Pratt approach is discussed thoroughly in Lambert (2001).

ment, the Lorenz inequality quasi-ordering seems a much more suitable candidate for this role. This critique of the Arrow-Pratt criterion echoes that of Ross (1981) in the context of decision under risk. Ross argues that for a comparison of risk aversion between two expected utility maximizers, it is not sufficient to compare the premia they are maximally prepared to pay for an insurance against all risks, as the Arrow-Pratt criterion prescribes, but it is also necessary to consider premia for insurances that decrease risk to a lower, but still risky, level. Our proposal to consider the criterion based on Procedure ($\star$) with the Lorenz inequality quasi-ordering is similar to that proposed by Ross since his concept of decreases in risk is close to the Lorenz concept.

Throughout the paper, we will often be concerned with comparing results yielded by, on the one hand, the version of Procedure ($\star$) that is equivalent to the Arrow-Pratt criterion and, on the other hand, the favoured version of Procedure ($\star$) using the Lorenz inequality quasi-ordering (henceforth referred to as the "Lorenz-based criterion"). It is interesting, however, to consider also a third criterion which is intermediate between the Arrow-Pratt criterion and the Lorenz-based criterion—this third criterion is based on the relative differentials quasi-ordering, a concept that is stronger than the minimalist inequality concept that underlies the Arrow-Pratt criterion and weaker than the Lorenz quasi-ordering (see Moyes, 1994).

In this paper, we first compare the three criteria for the class of continuous and monotonic SWOs, the broadest class of SWOs to which the conventional Arrow-Pratt criterion is commonly applied. We show that the relative differentials-based criterion yields the same results as the Arrow-Pratt criterion if SWOs are in addition separable, but not necessarily otherwise. Unfortunately, such consistency turns out not to hold between the Lorenz-based criterion and the Arrow-Pratt criterion, not even with respect to the important class of constant elasticity of substitution (CES) SWOs (a subclass of the class of continuous, monotonic and separable SWOs). Usually, a CES SWO with a higher value of the single parameter, ε, is considered more inequality averse than one with a lower value of ε. This role of ε as a measure for the degree of inequality aversion, is justified in the framework of the Arrow-Pratt criterion for comparing degrees of inequality aversion. However, as straightforward examples show, it is *not* justified if the Lorenz-based criterion is adopted: given two income distributions such that one Lorenz dominates the other, it is quite possible that a CES SWOs with ε strictly prefers the Lorenz dominating income distribution, while a CES SWOs with $\varepsilon' > \varepsilon$ strictly prefers the other one. Moreover, using a result by Ross (1981) we show that such examples can be found for any two CES SWOs. In other words, if the Lorenz-based criterion is adopted, no two CES SWOs can be compared with respect to degree of inequality aversion.

Second, we study the concept of "extreme inequality aversion" for the three different criteria for comparing degrees of inequality aversion. We call an SWO extremely inequality averse in a class of SWOs S if and only if it is at least as inequality averse as all SWOs in S (and, moreover, is itself a member of S). In the literature, leximin is often seen as a typical example of an SWO that combines extreme inequality aversion with monotonicity. We show that, in the class of monotonic SWOs, both the Arrow-Pratt criterion and the relative differentials-based criterion identify the entire class of weakly maximin SWOs as the extremely inequality averse ones—an SWO is said to be *weakly maximin* if and only if it implies a strict preference for a given income distribution over another if the worst off is strictly better off in the given income distribution. The class includes leximin and, by consequence, the Arrow-Pratt criterion and the relative differentials based criterion can be said to support the conventional view (see also Tungodden and Vallentyne, 2004). However, if the Lorenz-based criterion is adopted, this view has to be abandoned: we show that in this case the set of extremely inequality averse monotonic SWOs is empty. Finally, we provide evidence that the incompatibility between extreme inequality aversion and monotonicity is robust with respect to certain reasonable changes in the definition of the concept of extreme inequality aversion.

The paper is structured as follows. Section 2 deals with preliminaries. In Section 3 we formally introduce and discuss the three criteria for comparing degrees of inequality aversion that constitute the topic of the paper. The questions of how the three criteria compare with respect to the class of continuous and monotonic SWOs, and with respect to the concept of extreme inequality aversion, are dealt with in Sections 4 and 5, respectively. Some concluding remarks are given in Section 6. All proofs are contained in the Appendix.

2 Preliminaries

We begin by introducing basic definitions and notation. An *income distribution* is a vector $(x_1,\ldots,x_n) \in \mathbb{R}^n_{++}$ where $n \geq 3$ is the number of individuals in society and x_i is the income of individual i. For convenience we assume a fixed population size throughout the paper. The set of individuals is N and the set of income distributions is X. We assume that, for any income distribution $\mathbf{x} \in X$, individuals are indexed such that it holds that $x_1 \leq x_2 \leq \cdots \leq x_n$. In accordance with this assumption, we suppose that all considered concepts for welfare and inequality comparisons satisfy *anonymity*—that is, any given income distribution is treated equivalently as each of its permutations. The arithmetic mean of any income distribution $\mathbf{x} \in X$ is written as $\mu(\mathbf{x})$. We use the symbol $\mathbf{1}_n$ to denote an n-dimensional vector of which all components are equal to 1. For any pair of income distributions, $\mathbf{x},\mathbf{y} \in X$, we write $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ for all $i \in N$ with at least one strict inequality, and we write $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for all $i \in N$.

Social preferences are represented by a *social welfare ordering* (SWO) $\succeq_W$ ("is at least as good as") on X.[2] The asymmetric and symmetric parts of $\succeq_W$ are denoted with $\succ_W$ ("is better than") and $\sim_W$ ("is equally good as"), respectively. A *social welfare function* is a function $W: X \to \mathbb{R}$ which represents some SWO.

We shall require certain axioms in our analysis. Roughly speaking, continuity ensures that small changes in an income distribution cause only small changes in its social welfare ranking with respect to other income distributions.

Axiom 1 (Continuity). For all $\mathbf{x} \in X$, $\{\mathbf{y} \mid \mathbf{y} \in X, \mathbf{y} \succeq_W \mathbf{x}\}$ and $\{\mathbf{y} \mid \mathbf{y} \in X, \mathbf{x} \succeq_W \mathbf{y}\}$ are closed in X.

Monotonicity says that it is an improvement if some individuals get better off without any individuals getting worse off.

Axiom 2 (Monotonicity). For all $\mathbf{x}, \mathbf{y} \in X$, if $\mathbf{x} > \mathbf{y}$ then $\mathbf{x} \succ_W \mathbf{y}$.

Separability requires that the social welfare ranking of any pair of income distributions is not influenced by the incomes that are the same in both income distributions.

Axiom 3 (Separability). For all $\hat{N} \subset N$ and for all $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in X$, if $x_i = y_i$ and $x'_i = y'_i$ for all $i \in \hat{N}$, and $x_i = x'_i$ and $y_i = y'_i$ for all $i \in N \setminus \hat{N}$, then $\mathbf{x} \succeq_W \mathbf{y} \Leftrightarrow \mathbf{x}' \succeq_W \mathbf{y}'$.

Any SWO that satisfies continuity, monotonicity and separability can be represented by a social welfare function of the following form:

$$W(\mathbf{x}) = \sum_{i=1}^{n} u(x_i) \quad \text{for all } \mathbf{x} \in X, \tag{1}$$

where $u: \mathbb{R}_{++} \to \mathbb{R}$ is a continuous and strictly increasing function, referred to as a *utility function*. We shall pay special attention in our analysis to the constant elasticity of substitution (CES) class of SWOs, an important subclass of the class of continuous, monotonic and separable SWOs. An SWO $\succeq_W^\varepsilon$ is a member of the CES class if and only if there exists a nonnegative scalar ε such that $\succeq_W^\varepsilon$ can be represented by (1) with utility function $u(x) = \frac{1}{1-\varepsilon} x^{1-\varepsilon}$ for all $x \in \mathbb{R}_{++}$.

Since comparisons of income distributions *with respect to inequality* are conceptually prior to comparisons of SWOs *with respect to degree of inequality aversion*, we require the concept of an *inequality quasi-ordering* (IQO) $\preceq_I$ ("is not more unequal than") on X.[3] The asymmetric and symmetric parts of $\preceq_I$ are denoted by $\prec_I$ ("is less unequal than") and $\sim_I$ ("is equally unequal as"), respectively.

[2] An ordering is a reflexive, transitive and complete binary relation.

[3] A quasi-ordering is a reflexive and transitive binary relation.

An *inequality measure* is a function $I: X \to \mathbb{R}$ which represents some complete IQO. The strongest IQO to receive broad acceptance amongst economists is the Lorenz IQO. The *Lorenz IQO*, written as $\preceq_I^L$, is defined as follows: for all $\mathbf{x}, \mathbf{y} \in X$,

$$\mathbf{x} \preceq_I^L \mathbf{y} \Leftrightarrow \frac{1}{n}\sum_{i=1}^{k}\frac{x_i}{\mu(\mathbf{x})} \geq \frac{1}{n}\sum_{i=1}^{k}\frac{y_i}{\mu(\mathbf{y})} \quad \text{for all } k = 1, \ldots, n.$$

An IQO $\preceq_I$ will be referred to as *Lorenz consistent* if it agrees with all comparisons made by the Lorenz IQO, i.e., if $\prec_I^L \subset \prec_I$ and $\sim_I^L \subset \sim_I$. We shall refer to an SWO as Lorenz consistent if it follows the asymmetric part of the Lorenz IQO for comparisons between income distributions with the same mean incomes.[4]

Axiom 4 (Lorenz Consistency). For all $\mathbf{x}, \mathbf{y} \in X$, if $\mu(\mathbf{x}) = \mu(\mathbf{y})$ and $\mathbf{x} \prec_I^L \mathbf{y}$ then $\mathbf{x} \succ_W \mathbf{y}$.

In the literature, social welfare functions are often assumed to depend on mean income and inequality only, i.e., it is assumed that there exists an inequality measure I and a function $f: (\mathbb{R}_{++} \times \mathbb{R}) \to \mathbb{R}$, increasing in the first argument and decreasing in the second, such that $W = f(\mu, I)$. Given that perspective, it is clear that Lorenz consistency is a weak requirement for SWOs—it is sufficient that the underlying inequality measure is Lorenz consistent.[5] We note that all CES SWOs are Lorenz consistent and can be written as a function of mean income and inequality. Specifically, it can be shown that any CES SWO $\succeq_W^\varepsilon$ can be represented by a social welfare function of the form $W(\mathbf{x}) = \mu(\mathbf{x})[1 - I^\varepsilon(\mathbf{x})]$ for all $\mathbf{x} \in X$, where I^ε is a Lorenz consistent inequality measure.[6]

In this paper we study a general approach for comparing SWOs with respect to the degree of inequality aversion they express. This approach will be described in the next section. It is instructive, however, to first consider the standard so-called Arrow-Pratt approach, which in Section 4 will be compared with the approach suggested in this paper. The analysis of Pratt (1964) concerning risk aversion has provided several equivalent criteria that can be applied to the problem of comparing degrees of inequality aversion (see also Lambert, 2001, pp. 94-97). Some of these criteria can only be used to compare SWOs that can be written in the expected utility form, i.e., SWOs that satisfy continuity, monotonicity and separability. This class is important and we shall pay special attention to it. However,

[4]So we use the same term for two different concepts of Lorenz consistency. However, confusion is avoided because it will always be clear from the context whether the Lorenz consistency concept for IQOs or that for SWOs is meant.

[5]Note that, for any continuous, monotonic and separable SWO $\succeq_W$, Lorenz consistency is satisfied if the following weaker criterion is satisfied: $\mu(\mathbf{x})\mathbf{1} \succ_W \mathbf{x}$ for all $\mathbf{x} \in X$ such that $\mathbf{x}$ is not perfectly equal.

[6]See Atkinson (1970).

since we wish to initially consider the entire class of continuous and monotonic SWOs, we focus on the strongest of Pratt's criteria that is applicable also to non-separable SWOs, viz., the criterion based on the *equally distributed equivalent income* (EDEI). The EDEI, $\xi(\succeq_W; \mathbf{x})$, for any income distribution $\mathbf{x}$ and any SWO $\succeq_W$, is the income that, when equally distributed, yields the same level of welfare according to $\succeq_W$ as the income distribution $\mathbf{x}$.[7] Formally, for any SWO $\succeq_W$ and any $\mathbf{x} \in X$, $\xi(\succeq_W; \mathbf{x}) = e$ if and only if $e\mathbf{1}_n \sim_W \mathbf{x}$. The criterion for comparing degrees of inequality aversion based on the EDEI concept, referred to as the "EDEI-based criterion" (*EDEI*-BC), is defined as follows.

Definition 1 (*EDEI*-BC). Let $\succeq_W$ and $\succeq'_W$ be any two continuous and monotonic SWOs. Then, $\succeq_W$ is *at least as inequality averse as* $\succeq'_W$ if and only if, for all $\mathbf{x} \in X$, $\xi(\succeq_W; \mathbf{x}) \leq \xi(\succeq'_W; \mathbf{x})$.

As is conventional, we say that $\succeq_W$ is *more inequality averse than* $\succeq'_W$ if $\succeq_W$ is at least as inequality averse as $\succeq'_W$ while $\succeq'_W$ is not at least as inequality averse as $\succeq_W$, and we say that $\succeq_W$ is *equally inequality averse as* $\succeq'_W$ if $\succeq_W$ is at least as inequality averse as $\succeq'_W$ and $\succeq'_W$ is at least as inequality averse as $\succeq_W$.

The idea of the *EDEI*-BC is to compare, for all income distributions, how much sacrifice of mean income SWOs maximally allow in order to move from a given income distribution to a perfectly equal one—for an SWO $\succeq_W$ and an income distribution $\mathbf{x}$, this sacrifice equals $[\mu(\mathbf{x}) - \xi(\succeq_W; \mathbf{x})]$. According to the *EDEI*-BC, an SWO in the CES class is more inequality averse as the value of its corresponding ε is greater.[8] For this reason, ε is traditionally interpreted as being a parameter of inequality aversion.

We close the section by discussing the concept of extreme inequality aversion which will be the focus of Section 5. Conventionally, maximin and leximin, both of which give absolute priority to the worst off, are seen as typical examples of extremely inequality averse SWOs. Maximin implies indifference in all cases in which the worst off is equally well off, i.e., an SWO $\succeq_W$ is *maximin* if and only if, for all $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \succeq_W \mathbf{y} \Leftrightarrow x_1 \geq y_1$. Leximin, on the other hand, gives priority to the second worst off in the cases where the worst off is equally well off in both alternatives, and so on, i.e., an SWO $\succeq_W$ is *leximin* if and only if for all $\mathbf{x}, \mathbf{y} \in X$, it holds that

$$\mathbf{x} \succeq_W \mathbf{y} \Leftrightarrow \mathbf{x} = \mathbf{y}, \text{ or, there is an integer } k \text{ such that}$$
$$x_i = y_i \text{ for all } i < k \text{ and } x_k > y_k.$$

Both maximin and leximin are members of the class of weakly maximin SWOs, which is the class of SWOs that all have in common the asymmetric part of maximin, i.e., an SWO $\succeq_W$ is *weakly maximin* if and only if, for all $\mathbf{x}, \mathbf{y} \in X$, if $x_1 > y_1$

[7] See Atkinson (1970) and Kolm (1969).

[8] In fact, ε is the value of the relative Arrow-Pratt measure of risk/inequality aversion.

then $\mathbf{x} \succ_W \mathbf{y}$. It can straightforwardly be shown that maximin is the only continuous member of the class of weakly maximin SWOs and that leximin is the only separable member of the class. We will focus on leximin rather than on maximin in this paper because leximin satisfies monotonicity, whereas maximin does not. The role of leximin as being extremely inequality averse can be defended on the basis of the *EDEI*-BC. For instance, Hammond (1975) has demonstrated that leximin can be interpreted as the limit case, $\varepsilon \to \infty$, of the CES class of SWOs. This point can be generalized with respect to the entire class of continuous, monotonic and separable SWOs (see Lambert, 2001, p. 101, Theorem 4.4).

3 Three Criteria for Comparing Degrees of Inequality Aversion

As mentioned in the introduction, we wish to consider comparisons of SWOs with respect to degree of inequality aversion using criteria based on Procedure ($\star$). We shall now give a formal outline of this procedure. First, we determine a set which contains exactly all pairs of income distributions such that one income distribution is strictly more unequal than the other according to some "reference" IQO (clearly, this set is simply the asymmetric part of the reference IQO on X). These are exactly all pairs for which each SWO either implies an inequality averse choice (the less unequal income distribution is chosen), a neutral choice (indifference), or an inequality prone choice (the more unequal one is chosen)—three choices which can of course be unambiguously ranked from most inequality averse to least inequality averse. Second, two SWOs are compared with respect to the choices implied for each of the pairs of income distributions in the asymmetric part of the reference IQO: one SWO is referred to as at least as inequality averse as the other if and only if it implies an at least as inequality averse choice for all pairs belonging to the reference set.

In principle, any IQO can be chosen to determine the reference set in the first stage of the outlined procedure. However, since different people may have different reasonable views with respect to inequality comparisons, it seems preferable to consider the common part of all these views. Now, this is exactly the role that is often attributed to the Lorenz criterion in the literature. We argue, therefore, that it is most appropriate to use as the set of pairs of income distributions for which two SWOs are compared, the set $\prec_I^L$. We refer to the criterion for comparing degrees of inequality aversion based on the Lorenz IQO as the "*L*-based criterion" (*L*-BC) and define it as follows.

Definition 2 (*L*-BC). Let $\succeq_W$ and $\succeq'_W$ be any two SWOs. Then, $\succeq_W$ is *at least as inequality averse as* $\succeq'_W$ if and only if, for all $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \prec_I^L \mathbf{y}$, it holds that, (i) if $\mathbf{x} \succ'_W \mathbf{y}$ then $\mathbf{x} \succ_W \mathbf{y}$, and, (ii) if $\mathbf{x} \sim'_W \mathbf{y}$ then $\mathbf{x} \succeq_W \mathbf{y}$.

The "more inequality averse than" and "equally inequality averse as" relations corresponding to the L-BC are defined in the same way as with the *EDEI*-BC.

The L-BC is closely related to the concept of "strong risk aversion" studied by Ross (1981). Ross' concept is obtained if the L-BC is restricted to SWOs of the expected utility form, i.e., SWOs satisfying continuity, monotonicity and separability, and if the absolute version of the Lorenz IQO is used instead of the regular (relative) version.[9]

Given the broad acceptance of the Lorenz IQO we consider the L-BC to be the ideal criterion for comparing degrees of inequality aversion, but to allow for a stronger link with the existing literature on the topic, we shall also consider two different criteria that will appear to be closer to the conventional Arrow-Pratt framework, i.e., to the *EDEI*-BC. The idea behind these criteria is the same as that behind the L-BC, the only difference being that (weaker) alternatives for the Lorenz IQO are used—that is, in both cases, in comparing two SWOs a set is considered which is a proper subset of $\prec_I^L$. The first alternative IQO we consider is the *minimalist* IQO, written as $\preceq_I^M$: for all $\mathbf{x}, \mathbf{y} \in X$,

$$\mathbf{x} \preceq_I^M \mathbf{y} \Leftrightarrow \mathbf{x} = e\mathbf{1}_n \text{ for some scalar } e.$$

The minimalist IQO only allows inequality comparisons between pairs of income distribution of which at least one is perfectly equal. The second alternative is the *relative differentials* IQO, written as $\preceq_I^{RD}$: for all $\mathbf{x}, \mathbf{y} \in X$,

$$\mathbf{x} \preceq_I^{RD} \mathbf{y} \Leftrightarrow \frac{x_i}{y_i} \geq \frac{x_{i+1}}{y_{i+1}} \quad \text{for all } i = 1, \ldots, (n-1).$$

The relative differentials IQO, which was introduced into the literature on income distribution by Moyes (1994), says that any progressive redistribution decreases inequality. The criteria for comparing degrees of inequality aversion based on the minimalist IQO (M-BC) and the relative differentials IQO (RD-BC) are respectively defined as follows.

Definition 3 (*M*-BC). Let $\succeq_W$ and $\succeq'_W$ be any two SWOs. Then, $\succeq_W$ is *at least as inequality averse as* $\succeq'_W$ if and only if, for all $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \prec_I^M \mathbf{y}$, it holds that, (i) if $\mathbf{x} \succ'_W \mathbf{y}$ then $\mathbf{x} \succ_W \mathbf{y}$, and, (ii) if $\mathbf{x} \sim'_W \mathbf{y}$ then $\mathbf{x} \succeq_W \mathbf{y}$.

Definition 4 (*RD*-BC). Let $\succeq_W$ and $\succeq'_W$ be any two SWOs. Then, $\succeq_W$ is *at least as inequality averse as* $\succeq'_W$ if and only if, for all $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \prec_I^{RD} \mathbf{y}$, it holds that, (i) if $\mathbf{x} \succ'_W \mathbf{y}$ then $\mathbf{x} \succ_W \mathbf{y}$, and, (ii) if $\mathbf{x} \sim'_W \mathbf{y}$ then $\mathbf{x} \succeq_W \mathbf{y}$.

[9] Definition 2 moreover has a different phrasing than the concept of Ross (1981). Statement (ii) of Proposition 1 below and condition (3) in the proof of Lemma 2 below, are closer to the formulation used by Ross.

The "more inequality averse than" and "equally inequality averse as" relations corresponding to the *M*-BC and *RD*-BC are defined as before.

The *M*-BC is sometimes considered in the literature on risk aversion, but in a restricted version that makes the criterion applicable only to SWOs of the expected utility form. It is an established result in this context that, for SWOs of the expected utility form, the *M*-BC and the *EDEI*-BC are equivalent.[10] A more general result will be shown to hold in Section 4.

Since the three criteria rely on comparisons of choices over pairs of income distributions which are members of some set which represents a view on inequality, $\prec_I^M$, $\prec_I^{RD}$ and $\prec_I^L$, respectively, and given the fact that $\prec_I^M \subset \prec_I^{RD} \subset \prec_I^L$, the following remark is straightforwardly established.

Remark 1. Let $\succeq_W$ and $\succeq'_W$ be any two SWOs. Then, of the following three statements, (i) implies (ii) but (ii) does not imply (i), and (ii) implies (iii) but (iii) does not imply (ii):

(i) $\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the *L*-BC.

(ii) $\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the *RD*-BC.

(iii) $\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the *M*-BC.

It is easily verified that the relationships described in Remark 1 also hold for the relation "is equally inequality averse as," but not for the relation "is more inequality averse than."

Remark 1 shows that the *RD*-BC is more demanding than the *M*-BC and, in turn, the *L*-BC is more demanding than the *RD*-BC. A consequence is that if, for instance, the *M*-BC and the *L*-BC yield a different conclusion, then this disagreement will typically be of the type where the *M*-BC ranks two SWOs whereas the *L*-BC does not. The converse case, as well as cases in which the *M*-BC and the *L*-BC rank two SWOs in opposite ways, are excluded by Remark 1. In this respect it is important to note that if two SWOs, say $\succeq_W$ and $\succeq'_W$, are incomparable according to one of the three criteria, this does not simply mean that there is not sufficient evidence to refer to one SWO as at least as inequality averse as the other, but, more strongly, it means that the evidence is pointing in different directions: for some pair(s) of income distributions $\succeq_W$ is locally more inequality averse than $\succeq'_W$, while for (an)other pair(s) $\succeq'_W$ is locally more inequality averse than $\succeq_W$.

We saw in the previous section that the *EDEI*-BC can be interpreted as a criterion for comparing the willingness of SWOs to sacrifice mean income in return for a given decrease in inequality. Since this view of inequality aversion as essentially describing a trade-off between mean income and equality is popular, we wish to

[10] See, e.g., Mas-Colell et al. (1995, p. 191, Proposition 6.C.2)—the restricted version of the *M*-BC is close to their statement (v).

demonstrate that the L-BC, the M-BC and the RD-BC are consistent with it—i.e., that these three criteria can be rephrased in terms of the mean income-equality trade-off. The following propositions show that according to each of the three criteria, for any continuous and monotonic SWOs $\succeq_W$ and $\succeq'_W$, $\succeq_W$ is at least as inequality averse as $\succeq'_W$ if and only if, starting from any income distribution, $\succeq_W$ accepts a move to a given lower level of inequality at a loss of at least as much income as $\succeq'_W$ does.

Proposition 1. *Let $\succeq_W$ and $\succeq'_W$ be any two continuous and monotonic SWOs. Then, the following two statements are equivalent:*

(i) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the L-BC.*
(ii) *For all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ such that $\mathbf{x} \sim_I^L \mathbf{x}'$, $\mathbf{x} \prec_I^L \mathbf{y}$, $\mathbf{x} \sim_W \mathbf{y}$ and $\mathbf{x}' \sim'_W \mathbf{y}$, it holds that $\mu(\mathbf{x}) \leq \mu(\mathbf{x}')$.*

Proposition 2. *Let $\succeq_W$ and $\succeq'_W$ be any two continuous and monotonic SWOs. Then, the following two statements are equivalent:*

(i) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the M-BC.*
(ii) *For all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ such that $\mathbf{x} \sim_I^M \mathbf{x}'$, $\mathbf{x} \prec_I^M \mathbf{y}$, $\mathbf{x} \sim_W \mathbf{y}$ and $\mathbf{x}' \sim'_W \mathbf{y}$, it holds that $\mu(\mathbf{x}) \leq \mu(\mathbf{x}')$.*

Proposition 3. *Let $\succeq_W$ and $\succeq'_W$ be any two continuous and monotonic SWOs. Then, the following two statements are equivalent:*

(i) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the RD-BC.*
(ii) *For all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ such that $\mathbf{x} \sim_I^{RD} \mathbf{x}'$, $\mathbf{x} \prec_I^{RD} \mathbf{y}$, $\mathbf{x} \sim_W \mathbf{y}$ and $\mathbf{x}' \sim'_W \mathbf{y}$, it holds that $\mu(\mathbf{x}) \leq \mu(\mathbf{x}')$.*

The proofs of Propositions 2 and 3 are very similar to that of Proposition 1 and are therefore omitted.

To conclude the section, we mention two reasons for preferring the simple formulation used in Definitions 2 to 4—i.e., the formulation in terms only of preferences over pairs of income distributions—to the more traditional formulation in terms of the mean income-equality trade-off. First, the formulation in Definitions 2 to 4 has the advantage that it allows application of the criteria to *all* SWOs—also for instance to non-continuous SWOs, which will be useful in the discussion of extreme inequality aversion in Section 5. Second, a deeper concern is that an explicit reference to a mean income-equality trade-off may in certain cases misrepresent what comparisons of inequality aversion are really about. In general, there is no reason why equality should be traded off *only* with mean income. SWOs may express interest for other concerns, such as poverty alleviation for instance—then, the trade-off with mean income is just one of several trade-offs that are relevant for the concept of inequality aversion.[11] As the neutral formulation used in Definitions 2 to 4 does not refer to any particular trade-off, it seems to better capture the general essence of the concept of inequality aversion.

4 The Three Criteria Versus the Arrow-Pratt Framework

The objective of this section is to compare the Arrow-Pratt criterion, i.e., the *EDEI*-BC, with the three criteria presented in the previous section. We will focus on the conclusions yielded by these four criteria with respect to the class of continuous and monotonic SWOs since that is the broadest class to which the *EDEI*-BC is usually applied. If it is the case that the *M*-BC, *RD*-BC or *L*-BC, respectively, is not equivalent to the *EDEI*-BC for this class of SWOs, then we will examine whether this is at least the case for the important CES class.

Although we are most interested in the *L*-BC for the reason specified in Section 3, it is convenient for expositional purposes to start with the comparison of the *EDEI*-BC with the *M*-BC and *RD*-BC. These criteria will appear to be closer to the *EDEI*-BC than the *L*-BC is. We mentioned already in the previous section that the *M*-BC and the *EDEI*-BC are equivalent for continuous, monotonic and separable SWOs. As the following proposition shows, this equivalence also holds if separability is not demanded.

Proposition 4. *Let $\succeq_W$ and $\succeq'_W$ be any two continuous and monotonic SWOs. Then, the following two statements are equivalent:*

(i) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the EDEI-BC.*
(ii) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the M-BC.*

The result in Proposition 4 is not very surprising, given the fact that the definitions of both the *M*-BC and the *EDEI*-BC refer to preferences over pairs of income distributions of which one is perfectly equal. The following result shows that, when we take the step from the minimalist IQO to the relative differentials IQO as the underlying inequality concept for the criterion, we move away from convention.

Proposition 5. *Let $\succeq_W$ and $\succeq'_W$ be any two continuous and monotonic SWOs. Then, of the following two statements,* (ii) *implies* (i) *but* (i) *does not imply* (ii)*:*

(i) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the EDEI-BC.*
(ii) *$\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the RD-BC.*

[11] Note that, in principle, it may be the case that mean income is not even a concern at all (and that the SWO is not monotonic).

Proposition 5 reveals that the *RD*-BC and the *EDEI*-BC are inconsistent for the considered class of continuous and monotonic SWOs. The inconsistency of the two criteria consists of there being SWOs such that the *EDEI*-BC ranks them while the *RD*-BC does not. As we said earlier, we wish to pay special attention to the class of CES SWOs as they play such an important role in the literature. Do the *RD*-BC and the *EDEI*-BC at least agree on how to rank the CES SWOs? Consider first the following useful lemma.

Lemma 1. *Let* $\succeq_W$ *and* $\succeq'_W$ *be any two continuous, monotonic and separable SWOs. Consider, moreover, any pair* $\mathbf{x}, \mathbf{y} \in X$ *such that*

$$\text{there is an integer } k \text{ such that } x_i \geq y_i \text{ for all } i < k, \text{ and } x_i \leq y_i \text{ for all } i \geq k. \quad (2)$$

Then, of the following two statements, (i) *implies* (ii)*:*

(i) $\succeq_W$ *is at least as inequality averse as* $\succeq'_W$ *according to the EDEI-BC.*
(ii) *If* $\mathbf{x} \sim'_W \mathbf{y}$ *then* $\mathbf{x} \succeq_W \mathbf{y}$*, and, if* $\mathbf{x} \succ'_W \mathbf{y}$ *then* $\mathbf{x} \succ_W \mathbf{y}$*.*

The following result shows that not only do the *RD*-BC and the *EDEI*-BC agree on how to rank the members of the CES class of SWOs, but also on how to rank any pair of SWOs of the expected utility form.

Proposition 6. *Let* $\succeq_W$ *and* $\succeq'_W$ *be any two continuous, monotonic and separable SWOs. Then, the following three statements are equivalent:*

(i) $\succeq_W$ *is at least as inequality averse as* $\succeq'_W$ *according to the EDEI-BC.*
(ii) $\succeq_W$ *is at least as inequality averse as* $\succeq'_W$ *according to the M-BC.*
(iii) $\succeq_W$ *is at least as inequality averse as* $\succeq'_W$ *according to the RD-BC.*

With respect to the *M*-BC and the *RD*-BC we may conclude that the former, and to a lesser extent the latter, support the claims made traditionally in the literature on the basis of the Arrow-Pratt criterion. An important question we now turn to is whether the favoured *L*-BC is consistent with these claims. We already know, by Remark 1 and Proposition 5, that the *L*-BC and the *EDEI*-BC cannot be equivalent for the entire class of monotonic and continuous SWOs, so the question becomes whether this equivalence at least holds for the popular CES class as with the *RD*-BC. This appears *not* to be the case. There are several pairs of CES SWOs $\succeq_W^{\varepsilon}$ and $\succeq_W^{\varepsilon'}$ such that $\varepsilon > \varepsilon'$, and several pairs of income distributions $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \prec_I^L \mathbf{y}$, for which it holds that $\mathbf{y} \succ_W^{\varepsilon} \mathbf{x}$ while $\mathbf{x} \succ_W^{\varepsilon'} \mathbf{y}$. This is illustrated in the following example.

Example 1. The example is for the case $n = 3$. Take the income distributions $\mathbf{x} = (19, 57, 76)$ and $\mathbf{y} = (20, 20, 130)$. It can be shown that $\mathbf{x} \prec_I^L \mathbf{y}$. However, for all CES SWOs for which ε is such that $0.403 < \varepsilon < 14.513$ it holds that $\mathbf{x} \succ_W^{\varepsilon} \mathbf{y}$, while for all CES SWOs for which $\varepsilon > 14.514$ it holds that $\mathbf{y} \succ_W^{\varepsilon} \mathbf{x}$.

Using a result by Ross (1981) it is possible to draw even stronger conclusions with respect to the CES class. Ross' critique of the Arrow-Pratt framework can be interpreted as a confrontation of the L-BC and the M-BC in the framework of expected utility theory. The following lemma is based on one of his results. We use our own notation and terminology.

Lemma 2. *Let $\succeq_W^u$ and $\succeq_W^v$ be any two continuous, monotonic and separable SWOs such that the respective corresponding utility functions in* (1)*, u and v, are twice differentiable. Then, the following two statements are equivalent:*

(i) *$\succeq_W^u$ is at least as inequality averse as $\succeq_W^v$ according to the L-BC.*

(ii) *There exist a decreasing and concave function $f : \mathbb{R}_{++} \to \mathbb{R}$ and a scalar $\lambda > 0$ such that, for all $x \in \mathbb{R}_{++}$, $u(x) = \lambda v(x) + f(x)$.*

It can be shown now that in the entire class of CES SWOs there are no two SWOs that can be compared according to the L-BC.

Proposition 7. *Let $\succeq_W^\varepsilon$ and $\succeq_W^{\varepsilon'}$ be any CES SWOs such that $\varepsilon \neq \varepsilon'$. Then, $\succeq_W^\varepsilon$ and $\succeq_W^{\varepsilon'}$ are incomparable according to the L-BC, i.e., $\succeq_W^\varepsilon$ is not at least as inequality averse as $\succeq_W^{\varepsilon'}$ according to the L-BC, and $\succeq_W^{\varepsilon'}$ is not at least as inequality averse as $\succeq_W^\varepsilon$ according to the L-BC.*

The CES class of SWOs is often considered to be very useful in practice because, according to the conventional Arrow-Pratt approach, it encompasses a continuum of positions with respect to inequality aversion from the completely non-egalitarian mean income rule ($\varepsilon = 0$) to leximin ($\varepsilon \to \infty$). It owes its popularity furthermore to the fact that it has attractive properties from the theoretical perspective: it satisfies the basic axioms continuity, monotonicity and separability, and allows a natural decomposition into mean income and a Lorenz consistent inequality measure as explained in Section 2. However, the deep inconsistency between, on the one hand, the conventional interpretation of the parameter ε and, on the other hand, the L-BC may be seen as somewhat damaging for the CES class to operate as a canonical class of SWOs. The problem is aggravated by the fact that all members of the CES class ascribe importance to the Lorenz IQO—and thus the L-BC—because they are all Lorenz consistent. Is it possible to find another class of SWOs which both has attractive properties and encompasses a continuum of degrees of inequality aversion according to the L-BC? Although we shall not attempt to answer this question here, we wish to note that a sacrifice will have to be made irrespective of the direction in which an answer is sought. For instance, the analysis of Ross (1981) can be used to construct a class of SWOs to play a role similar to that of the CES class, in which case continuity, monotonicity and separability will still be satisfied. However, the drawback is that in that case the natural link between welfare and an underlying concept of (Lorenz consistent) inequality will be lost. Alternatively, such a natural link can be taken as a starting point to construct an alternative to the CES class, but at the cost of separability.

5 The Three Criteria and Extreme Inequality Aversion

In this section we characterize the classes of SWOs that reconcile monotonicity with an extreme form of inequality aversion according to each of the three criteria proposed in Section 3.[12] It will be of particular interest to see what role leximin plays in our analysis, since this is the only popular SWO that is usually viewed as combining extreme inequality aversion with monotonicity. Since our analysis brings us at certain points close to the work of Tungodden and Vallentyne (2004), we will at those points carefully explain the relation between their results and ours.

We start by defining the term "extreme inequality aversion" formally.

Definition 5. An SWO $\succeq_W$ is *extremely inequality averse* in the class S if and only if $\succeq_W$ is a member of S and $\succeq_W$ is at least as inequality averse as any member of S.

This definition assures that an extremely inequality averse SWO in S never implies a choice over a pair of income distributions that is less inequality averse than that implied by any other member of S. Note also that all extremely inequality averse SWOs are equally inequality averse.

In what follows, we shall identify the members of the class of monotonic SWOs that are extremely inequality averse according to the M-BC, the RD-BC and the L-BC. Since we do not require continuity, the standard $EDEI$-BC cannot be applied in this context—however, it is natural to interpret the M-BC as being the evident extension of the $EDEI$-BC capable of such comparisons. Again, it is convenient to begin the analysis by considering the M-BC and the RD-BC.

Proposition 8. *Let $\succeq_W$ be any monotonic SWO. Then, the following five statements are equivalent:*

(i) $\succeq_W$ *is extremely inequality averse in the class of monotonic SWOs according to the M-BC.*

(ii) $\succeq_W$ *is extremely inequality averse in the class of monotonic SWOs according to the RD-BC.*

[12] How the ideals of extreme inequality aversion and monotonicity can be combined is an important question in egalitarian social ethics. See Tungodden (2003, pp. 10-23) for an overview of the economic and philosophical literature concerning this topic.

(iii) *For all* $\mathbf{x}, \mathbf{y} \in X$ *such that not* $\mathbf{x} < \mathbf{y}$*, it holds that, if* $\mathbf{x} \prec_I^M \mathbf{y}$ *then* $\mathbf{x} \succ_W \mathbf{y}$.

(iv) *For all* $\mathbf{x}, \mathbf{y} \in X$ *such that not* $\mathbf{x} < \mathbf{y}$*, it holds that, if* $\mathbf{x} \prec_I^{RD} \mathbf{y}$ *then* $\mathbf{x} \succ_W \mathbf{y}$.

(v) $\succeq_W$ *is weakly maximin.*

The equivalence of (i) and (v) in Proposition 8 says that, according to the *M*-BC, the case of extreme inequality aversion in the class of monotonic SWOs is covered by the monotonic weakly maximin SWOs.[13] To a certain extent, this result supports the conventional view that leximin constitutes the case of extreme inequality aversion. The reason is that the literature focuses virtually exclusively on separable SWOs when studying extreme inequality aversion,[14] combined with the fact that leximin is the only separable weakly maximin SWO. The finding that (i) and (v) are equivalent is important for two reasons. Firstly, given Remark 1, it follows from this result that the classes of extremely inequality averse SWOs that are implied by the *RD*-BC and the *L*-BC must be subsets of the class of monotonic weakly maximin SWOs. Secondly, it presents another way of seeing why the *M*-BC is unattractive. As an illustration of this point, consider the following SWO $\succeq_W$: for all $\mathbf{x}, \mathbf{y} \in X$, it holds that,

$$\begin{aligned}
&\text{if } x_1 > y_1 \text{ then } \mathbf{x} \succ_W \mathbf{y}, \text{ and} \\
&\text{if } x_1 = y_1 \text{ then } [\mathbf{x} \succeq_W \mathbf{y} \Leftrightarrow \ \mu(\mathbf{x}) \geq \mu(\mathbf{y})].
\end{aligned}$$

Clearly, this SWO is both monotonic and weakly maximin. Now note that whenever two income distributions have the same lowest incomes, this SWO ranks them according to the completely non-egalitarian mean income rule.[15] Probably, many would hesitate to refer to such an SWO as extremely inequality averse, thus implicitly accepting that the *M*-BC is too undemanding as a criterion for comparing degrees of inequality aversion. However, as the equivalence of (ii) and (v) shows, moving on to the *RD*-BC does not solve anything: the class of monotonic weakly maximin SWOs is still identified as the extremely inequality averse subclass of the class of monotonic SWOs. Before we consider which monotonic weakly maximin SWOs survive the test of Definition 5 when we move to the *L*-BC, we consider the other statements of Proposition 8.

The conditions expressed in statements (iii) and (iv) of Proposition 8 constitute a natural way of giving meaning to extreme inequality aversion for SWOs that

[13] With an approach analogous to that of Hammond (1975) and Lambert (2001) (mentioned at the end of Section 2), the same conclusion can be reached using the standard *EDEI*-BC (see Bosmans, 2005).

[14] See Lambert (2001, pp. 99-101).

[15] Note that the comparison of such income distributions is probably even quite common in practice—think of a change in the tax system that leaves the existing minimally guaranteed income unaffected.

satisfy monotonicity—the conditions say that one should prefer, for any pair of income distributions, the one which is less unequal (according to the minimalist IQO and the relative differentials IQO in statements (iii) and (iv), respectively) unless the income distribution is worse for some and better for none. In a recent study on the possibility of combining extreme inequality aversion and monotonicity, Tungodden and Vallentyne (2004) have taken natural conditions as those expressed in statements (iii) and (iv) as a starting point (so, relying only implicitly on the concepts defined in our Definitions 2 to 5). They have considered a condition similar to that of statement (iii) and also show that statements (iii) and (v) are equivalent. Later, we draw a more interesting parallel between the present work and theirs.

Now, we come to the important question of what extremely inequality averse SWOs are identified by the L-BC. It quickly appears that none are.

Proposition 9. *There is no SWO that is extremely inequality averse in the class of monotonic SWOs according to the L-BC.*

Accordingly, the following proposition shows that it is impossible to use the idea of statements (iii) and (iv) from Proposition 8 in conjunction with the Lorenz IQO.[16]

Proposition 10. *Let* $\succeq_W$ *be any monotonic SWO. Then, the following condition is not satisfied: for all* $\mathbf{x}, \mathbf{y} \in X$ *such that not* $\mathbf{x} < \mathbf{y}$*, it holds that, if* $\mathbf{x} \prec_I^L \mathbf{y}$ *then* $\mathbf{x} \succ_W \mathbf{y}$.

So, while the M-BC and the RD-BC identify all weakly maximin SWOs as extremely inequality averse, according to the L-BC no member of this class is extremely inequality averse. Moreover, given the L-BC, the weakly maximin SWOs do not only fail the test of extreme inequality aversion described in Definition 5, they do so in a particularly bad way.

Proposition 11. *Let* $\succeq_W$ *be any continuous and monotonic SWO satisfying Lorenz consistency. Then, there are several pairs* $\mathbf{x}, \mathbf{y} \in X$ *such that* $\mathbf{x} \prec_I^L \mathbf{y}$*, so that* $\mathbf{x} \succ_W \mathbf{y}$ *while all weakly maximin SWOs strictly prefer* $\mathbf{y}$ *to* $\mathbf{x}$.

The proposition says that, for instance, it is possible to find pairs of income distributions such that a CES SWO with ε arbitrarily close to, but greater than, zero, and hence arbitrarily close to the completely non-egalitarian mean income rule, is locally more inequality averse than all weakly maximin SWOs for these pairs.

[16]Combined with the fact that the condition of Proposition 10 is equivalent to an SWO being extremely inequality averse in the class of monotonic SWOs according to the L-BC (for a proof, consider the proof of the equivalence of (i) and (iii) in Proposition 8), Proposition 10 constitutes an alternative proof of Proposition 9.

So, we conclude that if we accept the L-BC, then extreme inequality aversion is incompatible with monotonicity. In their work, Tungodden and Vallentyne (2004) reach a similar conclusion. However, they implicitly use a criterion that lies in between the M-BC and the RD-BC, and find an incompatibility.[17] This is possible because they use a slightly (but significantly) different framework than the one used here: their result is driven by the fact that they reject anonymity as a property of SWOs, but accept it for IQOs. The present study shows that without this assumption, there is no incompatibility between their version of extreme inequality aversion and monotonicity (this is implied by the equivalence of (ii) and (v) in Proposition 8), but that the incompatibility crops up again when the L-BC is accepted (Proposition 9).[18]

What should egalitarians who agree with the L-BC and want both monotonicity and extreme inequality aversion choose as an SWO? It might at first glance seem natural to regard leximin or other monotonic weakly maximin SWOs as being "close enough"—these SWOs satisfy a necessary condition for being extremely inequality averse (they are extremely inequality averse if one looks only at the pairs in $\prec_I^M$ or $\prec_I^{RD}$), and a sufficient condition cannot be satisfied (being extremely inequality averse for those in $\prec_I^L$ is impossible), hence why not content ourselves with these? Proposition 11 illustrates already how unattractive it is to settle for a conclusion based on the less demanding criteria M-BC and RD-BC if the L-BC is the one which is deemed ideal. There is also a deeper reason for extreme egalitarians not to (necessarily) focus on the class of weakly maximin SWOs. It is perfectly acceptable to consider the pairs ordered by the minimalist IQO (i.e., the set $\prec_I^M$) as not being more important than some alternative set of pairs ordered by the Lorenz IQO (i.e., a subset of $\prec_I^L$ which differs from $\prec_I^M$). If one accepts the Lorenz IQO, these former pairs of income distributions are not special in any way. If such an alternative set of pairs is used in a criterion for comparing degrees of inequality aversion, in accordance with the explanation at the beginning of Section 3, then the set of extremely inequality averse monotonic SWOs need not be empty nor contain any weakly maximin SWOs. For instance, if the income distributions from Example 1 are members of this alternative set, then none of the weakly maximin SWOs pass the test of extreme inequality aversion, while (depending on the other elements of the set) other SWOs may pass the test.

To conclude the section, we consider two alternative ways of giving meaning to the view that inequality reduction should always be preferred unless no one

[17] More precisely, they use a condition similar to that stated in statements (iii) and (iv) of Proposition 8, but with, in the place of the minimalist or relative differentials IQO, an IQO that is a proper subrelation of the relative differentials IQO and a proper superrelation of the minimalist IQO.

[18] In Tungodden (2000) it is also shown that, without rejecting anonymity, their extreme inequality aversion condition and monotonicity are compatible.

wins by it. However, as we shall see, neither alternative produces a convincing way out of the incompatibility.

The first alternative is to consider the SWOs for which no monotonic SWO is more inequality averse according to the L-BC, instead of the ones that are at least as inequality averse as all the other monotonic ones according to the L-BC. Consider the following definition of this alternative concept of "maximal inequality aversion."

Definition 6. An SWO $\succeq_W$ is *maximally inequality averse* in the class S if and only if $\succeq_W$ is a member of S and no member of S is more inequality averse than $\succeq_W$.

The subset of SWOs that are maximally inequality averse in the set of monotonic SWOs according to the L-BC is not empty: as the following result shows, at least leximin is a member.

Proposition 12. *Leximin is maximally inequality averse in the set of monotonic SWOs according to the L-BC.*

However, the concept of maximal inequality aversion seems too undemanding, because it is not excluded that there are SWOs, which are themselves unlikely candidates for being considered extremely inequality averse, that are more inequality averse for at least some pairs of income distributions—in the case of leximin, Proposition 11 should suffice to make this point.

A second alternative is to start from the view that SWOs are functions of an underlying inequality measure, which represents a complete IQO, a view not uncommon in the literature as we saw in Section 2. In that perspective, the following alternative to Definition 5 seems reasonable: suppose we can find for some SWO an IQO which, for any choice not directly implied by monotonicity, indicates lower inequality for the income distribution which is chosen by the SWO, then, at least according to this view of inequality, the SWO can be considered to be extremely inequality averse. The question is whether it is possible to find an SWO and a corresponding IQO that satisfy the required condition. First we need to consider some minimal criteria that a sensible IQO ought to satisfy. The first is that it should have the minimalist IQO as a subrelation. The second is that it satisfies some invariance criterion. An invariance criterion defines the transformation which when applied to all incomes leaves inequality invariant. For instance, the invariance criterion underlying the Lorenz IQO and the relative differentials IQO is scale invariance, which says: for all $\mathbf{x} \in X$ and all scalars $\lambda > 0$, $\mathbf{x} \sim_I \lambda\mathbf{x}$. However, we will demand only that a much weaker invariance criterion is satisfied. Minimal invariance says that for any given income distribution there must exist an income distribution in which everyone is better off and which is at least as unequal as the given income distribution.

Axiom 5 (Minimal Invariance). For all $\mathbf{x} \in X$, there is a $\mathbf{x}' \in X$ such that $\mathbf{x}' \gg \mathbf{x}$ and $\mathbf{x} \preceq_I \mathbf{x}'$.

The following proposition shows that no SWO and IQO with the described properties exist.

Proposition 13. *Let $\succeq_W$ be any monotonic SWO and let $\preceq_I$ be any IQO that satisfies minimal invariance and such that $\prec_I \supset \prec_I^M$. Then, the following condition is not satisfied: for all $\mathbf{x}, \mathbf{y} \in X$ such that not $\mathbf{x} < \mathbf{y}$, $\mathbf{x} \prec_I \mathbf{y} \Leftrightarrow \mathbf{x} \succ_W \mathbf{y}$.*

The proposition says that whatever the concept of inequality used (requiring only that it satisfies minimal conditions—far weaker than Lorenz consistency for instance), leximin at least for some pairs of income distributions will not choose the least unequal one according to this concept of inequality, even though this income distribution is not worse by monotonicity. Moreover, this does not only hold for leximin, but for all monotonic SWOs.

6 Concluding Remarks

In this paper we considered a straightforward dominance procedure for comparing SWOs with respect to degree of inequality aversion. We considered three versions of the procedure based on three inequality concepts: the *L*-BC which we argued to be the ideal version, the *M*-BC which is roughly equivalent to the traditional Arrow-Pratt approach, and the *RD*-BC which is intermediate in strength between the other two criteria.

It was shown that the *L*-BC is in general incompatible with the *M*-BC. In the case of the CES class of SWOs, the difference between the conclusions produced by the two criteria was especially pronounced: whereas the *M*-BC ranks all members of this class, the *L*-BC ranks none. As we have said already, it would be interesting to think about theoretically agreeable alternatives to the CES class of which the members can be ranked using the *L*-BC and which covers a wide spectrum of positions with respect to inequality aversion. Probably the most attractive solution is to give up separability and to consider classes of SWOs such as those given by $W(\mathbf{x}) = \mu(\mathbf{x})[1 - I(\mathbf{x})]^{\alpha}$ for all $\mathbf{x} \in X$, where I is a Lorenz consistent inequality measure and α is a parameter that measures inequality aversion in accordance with the *L*-BC. It may be interesting to see whether classes of SWOs in the spirit of this example can be constructed in a theoretically and philosophically sound way starting directly from the idea of the natural decomposition of welfare in mean income and inequality.

We furthermore showed that if we accept the *L*-BC, then monotonicity and extreme inequality aversion are incompatible. Hence, egalitarians committed to monotonicity have to content themselves with being less than extremely inequality averse: it is always possible to find pairs of income distributions for which a less inequality averse choice than possible must be made. Those who are attracted to both the ideals of monotonicity and extreme inequality aversion have to determine which of the two to weaken. We have discussed that if extreme inequality aversion is weakened, nothing forces one to opt for a weakly maximin SWO such as leximin. It is perfectly possible to choose a different set over which one wants to make inequality averse choices than the set that forces one to give full priority to the worst off. The other possibility, not yet discussed, is to weaken monotonicity. For instance, a possibility is to demand only *ray-monotonicity*: for all $\mathbf{x} \in X$ and all $\lambda > 1$, $\lambda \mathbf{x} \succ_W \mathbf{x}$. It can easily be shown that there exist extremely inequality averse SWOs according to the L-BC in the class of ray-monotonic SWOs.[19] Interestingly, not only does the weakening to ray-monotonicity make it possible to have extremely inequality averse SWOs, but none of them is weakly maximin (and this holds even if we use the M-BC instead of the L-BC). In other words, whichever of the two ideals egalitarians choose to weaken in order to deal with the incompatibility, they should not feel required to restrict their consideration to leximin or other weakly maximin SWOs.

[19]Consider the example of an SWO $\succeq_W$: for all $\mathbf{x}, \mathbf{y} \in X$, it holds that, if $\mathbf{x} \prec_I \mathbf{y}$ then $\mathbf{x} \succ_W \mathbf{y}$, and if $\mathbf{x} \sim_I \mathbf{y}$ then $[\mathbf{x} \succeq_W \mathbf{y} \Leftrightarrow \mu(\mathbf{x}) \geq \mu(\mathbf{y})]$, where $\preceq_I$ is a Lorenz consistent and complete IQO.

Appendix: Proofs

Proof of Proposition 1

First, we show that statement (i) implies statement (ii). Assume that (i) holds. Take any $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ such that $\mathbf{x} \sim_I^L \mathbf{x}'$, $\mathbf{x} \prec_I^L \mathbf{y}$, $\mathbf{x} \sim_W \mathbf{y}$ and $\mathbf{x}' \sim_W' \mathbf{y}$. We have to show that $\mu(\mathbf{x}) \leq \mu(\mathbf{x}')$. Note first that, for all $\mathbf{z}, \mathbf{w} \in X$, $\mathbf{z} \sim_I^L \mathbf{w}$ if and only if there exists a scalar $\lambda > 0$ such that $\lambda \mathbf{z} = \mathbf{w}$. Hence, there exists a $\lambda > 0$ such that $\lambda \mathbf{x}' = \mathbf{x}$. Now, we have $\mathbf{x}' \succeq_W \mathbf{y}$ by (i). So, it follows that $\lambda \leq 1$ by monotonicity. Hence, we have $\mu(\mathbf{x}) \leq \mu(\mathbf{x}')$.

Second, we show that statement (ii) implies statement (i). Assume that (ii) holds. Take any $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \prec_I^L \mathbf{y}$. By continuity and monotonicity, there exist $\lambda, \lambda' > 0$ such that $\lambda \mathbf{x} \sim_W \mathbf{y}$ and $\lambda' \mathbf{x} \sim_W' \mathbf{y}$. Suppose first that $\mathbf{x} \succ_W' \mathbf{y}$. We have to show that in this case $\mathbf{x} \succ_W \mathbf{y}$. Note first that $\mathbf{x} \succ_W' \mathbf{y}$ implies $\lambda' < 1$ by monotonicity. Since furthermore $\lambda \leq \lambda'$ by (ii), we have $\lambda < 1$. Hence, monotonicity implies $\mathbf{x} \succ_W \mathbf{y}$. By a similar reasoning, $\mathbf{x} \sim_W' \mathbf{y}$ can be shown to imply $\mathbf{x} \succeq_W \mathbf{y}$. ■

Proof of Proposition 4

Statement (i) of Proposition 4 is clearly equivalent to statement (ii) of Proposition 2. Hence, the result follows from Proposition 2. ■

Proof of Proposition 5

That (ii) implies (i) follows from Proposition 4 and Remark 1.

We give an example to show that (i) does not imply (ii). Consider an SWO $\succeq_W$ such that, for all $\mathbf{x}, \mathbf{y} \in X$,

$$\mathbf{x} \succeq_W \mathbf{y} \Leftrightarrow F(\mathbf{x}) \geq F(\mathbf{y}),$$

where

$$F(\mathbf{x}) = \begin{cases} \frac{2}{3}x_1 + \frac{1}{3}x_2 + \sum_{i=3}^n x_i & \text{if } x_1 \geq \frac{2}{5}x_2; \\ \frac{3}{13}(4x_1 + x_2) + \sum_{i=3}^n x_i & \text{if } x_1 \leq \frac{2}{5}x_2. \end{cases}$$

Consider also an SWO $\succeq_W'$ such that, for all $\mathbf{x}, \mathbf{y} \in X$,

$$\mathbf{x} \succeq_W' \mathbf{y} \Leftrightarrow G(\mathbf{x}) \geq G(\mathbf{y}),$$

where

$$G(\mathbf{x}) = \begin{cases} \frac{3}{5}x_1 + \frac{2}{5}x_2 + \sum_{i=3}^n x_i & \text{if } x_1 \geq \frac{1}{2}x_2; \\ \frac{7}{30}(4x_1 + x_2) + \sum_{i=3}^n x_i & \text{if } x_1 \leq \frac{1}{2}x_2. \end{cases}$$

Both SWOs are clearly continuous and monotonic.

First, we show that $\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the *EDEI*-BC. We consider, in turn, the three possible cases. (a) Case where $\mathbf{x} \in A = \left\{\mathbf{x} \in X \,|\, x_1 \geq \frac{1}{2}x_2\right\}$: Note that $\xi(\succeq_W;\mathbf{x}) = \frac{1}{n-1}\left(\frac{2}{3}x_1 + \frac{1}{3}x_2 + \sum_{i=3}^n x_i\right)$ and $\xi(\succeq'_W;\mathbf{x}) = \frac{1}{n-1}\left(\frac{3}{5}x_1 + \frac{2}{5}x_2 + \sum_{i=3}^n x_i\right)$ for all $\mathbf{x} \in A$. By consequence, $\xi(\succeq_W;\mathbf{x}) \leq \xi(\succeq'_W;\mathbf{x})$ for all $\mathbf{x} \in A$. (b) Case where $\mathbf{x} \in B = \left\{\mathbf{x} \in X \,|\, \frac{2}{5}x_2 \leq x_1 \leq \frac{1}{2}x_2\right\}$: Note that $\xi(\succeq_W;\mathbf{x}) = \frac{1}{n-1}\left(\frac{2}{3}x_1 + \frac{1}{3}x_2 + \sum_{i=3}^n x_i\right)$ for all $\mathbf{x} \in B$. To calculate $\xi(\succeq'_W;\mathbf{x})$ for any given $\mathbf{x} \in B$, we first find an $\mathbf{y} \in X$ such that $\mathbf{x} \sim'_W \mathbf{y}$ and $(y_1,y_2) = \left(\frac{1}{2}y,y\right)$ and then use that $\xi(\succeq'_W;\mathbf{x}) = \xi(\succeq'_W;\mathbf{y})$. Now, $\mathbf{y}$ is such that $4x_1 + x_2 = 4\frac{1}{2}y + y$, so that $y = \frac{4x_1+x_2}{3}$, and $y_i = x_i$ for all $i = 3,4,\ldots,n$. Since $\mathbf{y} \in A$, we can calculate $\xi(\succeq'_W;\mathbf{y})$ as in the previous case, so that $\xi(\succeq'_W;\mathbf{x}) = \xi(\succeq'_W;\mathbf{y}) = \frac{1}{n-1}\left(\frac{28}{30}x_1 + \frac{7}{30}x_2 + \sum_{i=3}^n x_i\right)$. For all $\mathbf{x} = (x_1,x_2) \in X$, $\xi(\succeq_W;\mathbf{x}) < \xi(\succeq'_W;\mathbf{x})$ if and only if the condition is met that $x_1 > \frac{3}{8}x_2$, a condition that holds for all $\mathbf{x} \in B$. (c) Case where $\mathbf{x} \in C = \left\{\mathbf{x} \in X \,|\, x_1 \leq \frac{2}{5}x_2\right\}$: To calculate $\xi(\succeq_W;\mathbf{x})$ and $\xi(\succeq'_W;\mathbf{x})$ for any given $\mathbf{x} \in C$, we use the same method as in the previous case. So, first we find $\mathbf{y},\mathbf{y}' \in X$ such that $\mathbf{x} \sim_W \mathbf{y}$, $\mathbf{x} \sim'_W \mathbf{y}'$, $(y_1,y_2) = \left(\frac{2}{5}y,y\right)$ and $y_i = x_i$ for all $i = 3,4,\ldots,n$, $(y'_1,y'_2) = \left(\frac{2}{5}y',y'\right)$ and $y'_i = x_i$ for all $i = 3,4,\ldots,n$, and then calculate $\xi(\succeq_W;\mathbf{y})$ and $\xi(\succeq'_W;\mathbf{y}')$, which are equal to $\xi(\succeq_W;\mathbf{x})$ and $\xi(\succeq'_W;\mathbf{x})$, respectively. Note, however, that $y = y'$, so that, since $y,y' \in B$, we have that $\xi(\succeq_W;\mathbf{y}) < \xi(\succeq'_W;\mathbf{y}')$, and hence $\xi(\succeq_W;\mathbf{x}) < \xi(\succeq'_W;\mathbf{x})$ for all $x \in C$. We conclude from (a), (b) and (c) that $\succeq_W$ is at least as inequality averse as $\succeq'_W$ according to the *EDEI*-BC.

We now show that $\succeq_W$ is not at least as inequality averse as $\succeq'_W$ according to the *RD*-BC. Consider $\mathbf{x}$ and $\mathbf{y}$ such that $(x_1,x_n) = (120,785)$, $(y_1,y_n) = (100,800)$ and $x_i = y_i = 240$ for all $i = 2,3,\ldots,(n-1)$. Clearly, $\mathbf{x} \prec_I^{RD} \mathbf{y}$, but $\mathbf{y} \succ_W \mathbf{x}$ while $\mathbf{x} \succ'_W \mathbf{y}$. ■

Proof of Lemma 1

Let $\succeq_W^u$ and $\succeq_W^v$ be any two continuous, monotonic and separable SWOs with u and v as utility functions in (1), respectively. Suppose that statement (i) holds with respect to $\succeq_W^u$ and $\succeq_W^v$, i.e., $\xi(\succeq_W^u;\mathbf{x}) \leq \xi\left(\succeq_W^v;\mathbf{x}\right)$ for all $\mathbf{x} \in X$. Denote the sets of winners and losers in going from any $\mathbf{y} \in X$ to any $\mathbf{x} \in X$ as $W(\mathbf{y},\mathbf{x}) = \{i \,|\, x_i > y_i\}$ and $L(\mathbf{y},\mathbf{x}) = \{i \,|\, x_i < y_i\}$, respectively. Throughout the proof we consider the pair $\mathbf{x},\mathbf{y} \in X$, which is any pair for which condition (2) of Lemma 1 holds. What has to be shown is that statement (ii) holds with respect to this pair, i.e., if $\mathbf{x} \sim_W^v \mathbf{y}$ then $\mathbf{x} \succeq_W^u \mathbf{y}$, and, if $\mathbf{x} \succ_W^v \mathbf{y}$ then $\mathbf{x} \succ_W^u \mathbf{y}$. Therefore, it is assumed that either $\mathbf{x} \sim_W^v \mathbf{y}$ or $\mathbf{x} \succ_W^v \mathbf{y}$ holds. Since statement (ii) holds trivially with respect to the pair $\mathbf{x},\mathbf{y}$ in the cases where $W(\mathbf{y},\mathbf{x}) = \varnothing$ or $L(\mathbf{y},\mathbf{x}) = \varnothing$, we only consider cases in which both sets are nonempty.

We require a tool to bring any income distribution $\mathbf{z} \in X$ closer to $\mathbf{x}$ by replacing at least one of the components of $\mathbf{z}$ by a component of $\mathbf{x}$, and this in such a way that for the resulting income distribution $\mathbf{z}'$ it holds that $\mathbf{z} \sim_W^v \mathbf{z}'$. Consider a function T_{ij} which transforms any $\mathbf{z} \in X$ into $\mathbf{z}'$ by replacing two and only two incomes, z_i and z_j. The function T_{ij} has $\mathbf{z}$ in its domain if and only if $z_i < x_i \leq x_j < z_j$, in other words, if and only if $i \in W(\mathbf{z},\mathbf{x})$ and $j \in L(\mathbf{z},\mathbf{x})$, and, furthermore, it holds that $i < j$. The values of z_i' and z_j' are determined as follows: (a) if $v(x_i)+v(x_j) = v(z_i)+v(z_j)$, then $(z_i',z_j') = (x_i,x_j)$, (b) if $v(x_i)+v(x_j) > v(z_i)+v(z_j)$, then $(z_i',z_j') = (s,x_j)$, where s is such that $z_i < s < x_i$ and $v(z_i)+v(z_j) = v(s)+v(x_j)$, (c) if $v(x_i)+v(x_j) < v(z_i)+v(z_j)$, then $(z_i',z_j') = (x_i,t)$, where t is such that $x_j < t < z_j$ and $v(z_i)+v(z_j) = v(x_i)+v(t)$. The s and t considered in cases (b) and (c), respectively, always exist by continuity and monotonicity. Note that indeed $\mathbf{z} \sim_W^v \mathbf{z}'$.

Now, using T_{ij}, we transform $\mathbf{y}$ step by step into $\mathbf{x}$. First, transform $\mathbf{y}$ into $\mathbf{y}'$ by applying T_{ij} for some $i \in W(\mathbf{y},\mathbf{x})$ and some $j \in L(\mathbf{y},\mathbf{x})$. If $W(\mathbf{y}',\mathbf{x})$ and/or $L(\mathbf{y}',\mathbf{x})$ is empty, stop. Otherwise, perform the transformation on $\mathbf{y}'$ by again applying T_{ij} for some $i \in W(\mathbf{y}',\mathbf{x})$ and some $j \in L(\mathbf{y}',\mathbf{x})$ such that $i < j$. Repeat this until the set of winners and/or the set of losers is empty after the transformation, then stop. Note that the transformation can always be performed if the sets are both nonempty due to the fact that $\mathbf{x},\mathbf{y} \in X$ satisfy condition (2) of Lemma 1, since the condition implies that, for all $i \in W(\mathbf{y},\mathbf{x})$ and all $j \in L(\mathbf{y},\mathbf{x})$, it holds that $i < j$, and this continues to hold after every step. So, the income distribution that results from the final step, say y'', has the property that $W(\mathbf{y}'',\mathbf{x})$ and/or $L(\mathbf{y}'',\mathbf{x})$ is empty and $\mathbf{y}'' \sim_W^v \mathbf{y}$. Furthermore, it is impossible that $W(\mathbf{y}'',\mathbf{x})$ is empty while $L(\mathbf{y}'',\mathbf{x})$ is not, since otherwise $\mathbf{x} < \mathbf{y}''$ and so $\mathbf{y}'' \succ_W^v \mathbf{x}$ by monotonicity, so that the fact that $\mathbf{y}'' \sim_W^v \mathbf{y}$ would imply that $\mathbf{y} \succ_W^v \mathbf{x}$, contrary to what we assumed. So, given that $L(\mathbf{y}'',\mathbf{x})$ must be empty, we have either $\mathbf{y}'' < \mathbf{x}$ or $\mathbf{y}'' = \mathbf{x}$ and, by consequence, $\mathbf{x}$ is weakly preferred to $\mathbf{y}''$ by any monotonic SWO.

We shall now show that $\mathbf{y}'' \sim_W^v \mathbf{y}$ implies $\mathbf{y}'' \succeq_W^u \mathbf{y}$. It is known from Pratt (1964) that $\xi(\succeq_W^u;\mathbf{x}) \leq \xi\left(\succeq_W^v;\mathbf{x}\right)$ for all $\mathbf{x} \in X$ implies that $u = f \circ v$ where the function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and concave. Now for any $\mathbf{z} \in X$, $\mathbf{z} \sim_W^v \mathbf{z}'$ is equivalent to $v(z_i') + v(z_j') = v(z_i) + v(z_j)$ or $v(z_i') - v(z_i) = v(z_j) - v(z_j')$. Since furthermore $v(z_j) > v(z_j') > v(z_i') > v(z_i)$, it holds that $u(z_i') - u(z_i) \geq u(z_j) - u(z_j')$ by strict increasingness and concavity of f. So, we have $\mathbf{z}' \succeq_W^u \mathbf{z}$. Hence, by transitivity, it holds that $\mathbf{y}'' \succeq_W^u \mathbf{y}$

We can now conclude the following. In the case where $\mathbf{x} \sim_W^v \mathbf{y}$, we have indeed $\mathbf{x} \succeq_W^u \mathbf{y}$, since $\mathbf{y}'' \succeq_W^u \mathbf{y}$ and since $\mathbf{x}$ is weakly preferred to $\mathbf{y}''$ by any monotonic SWO. In the case where $\mathbf{x} \succ_W^v \mathbf{y}$ we have indeed $\mathbf{x} \succ_W^u \mathbf{y}$, since $\mathbf{x} \succ_W^v \mathbf{y}$ and $\mathbf{y}'' \sim_W^v \mathbf{y}$ imply $\mathbf{x} \succ_W^v \mathbf{y}''$ and hence $\mathbf{x} > \mathbf{y}''$, so that, by $\mathbf{x} \succ_W^u \mathbf{y}''$ and $\mathbf{y}'' \succeq_W^u \mathbf{y}$, $\mathbf{x} \succ_W^u \mathbf{y}$. ■

Proof of Proposition 6

Equivalence of (i) and (ii): Immediate from Proposition 4.

Equivalence of (i) and (iii): That (iii) implies (i) follows from Proposition 5. That (i) implies (iii) follows from Lemma 1 and the fact that, for any pair $\mathbf{x}, \mathbf{y} \in X$, if $\mathbf{x} \prec_I^{RD} \mathbf{y}$ then condition (2) of Lemma 1 holds. ■

Proof of Lemma 2

Ross (1981) shows (ii) to be equivalent to the condition: for all $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \prec_I^L \mathbf{y}$, if $\mathbf{x} - \pi^u \mathbf{1} \sim_W^u \mathbf{y}$ and $\mathbf{x} - \pi^v \mathbf{1} \sim_W^v \mathbf{y}$ then $\pi^u \geq \pi^v$. We consider the following condition:

$$\text{for all } \mathbf{x}, \mathbf{y} \in X \text{ such that } \mathbf{x} \prec_I^L \mathbf{y}, \text{ if } \gamma^u \mathbf{x} \sim_W^u \mathbf{y} \text{ and } \gamma^v \mathbf{x} \sim_W^v \mathbf{y} \text{ then } \gamma^u \leq \gamma^v. \quad (3)$$

If this latter condition is fitted into the proof of Ross instead of the former, it is easily seen that they play the same role and are equivalent.

What remains to be shown is that the condition in (3) is equivalent to (i). First we show that (3) is equivalent to

$$\begin{array}{l} \text{for all } \mathbf{x}, \mathbf{x}', \mathbf{y} \in X \text{ such that } \mathbf{x} \sim_I^L \mathbf{x}', \mathbf{x} \prec_I^L \mathbf{y}, \mathbf{x} \sim_W^u \mathbf{y} \text{ and } \mathbf{x}' \sim_W^v \mathbf{y}, \\ \text{it holds that } \mu(\mathbf{x}) \leq \mu(\mathbf{x}'). \end{array} \quad (4)$$

It is immediate that (4) implies (3). That (3) implies (4) follows from the fact that if there exist $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ such that $\mathbf{x} \sim_I^L \mathbf{x}'$, $\mathbf{x} \prec_I^L \mathbf{y}$, $\mathbf{x} \sim_W^u \mathbf{y}$ and $\mathbf{x}' \sim_W^v \mathbf{y}$, then there exists a $\mathbf{z} \in X$ and scalars γ^u, γ^v such that $\mathbf{x} = \gamma^u \mathbf{z}$ and $\mathbf{x}' = \gamma^v \mathbf{z}$. Now, since (4) is equivalent to (i) by Proposition 1, the required result follows. ■

Proof of Proposition 7

Seeking a contradiction, suppose that, without loss of generality, $\varepsilon > \varepsilon'$ and that $\succeq_W^\varepsilon$ is at least as inequality averse as $\succeq_W^{\varepsilon'}$ according to the L-BC. Then, by Lemma 2, there exist a decreasing and concave function $f : \mathbb{R}_{++} \to \mathbb{R}$ and a scalar $\lambda > 0$ such that, for all $x \in \mathbb{R}_{++}$,

$$\frac{x^{1-\varepsilon}}{1-\varepsilon} = \lambda \frac{x^{1-\varepsilon'}}{1-\varepsilon'} + f(x).$$

Decreasingness and concavity of f imply

$$\frac{d f(x)}{dx} = x^{-\varepsilon} - \lambda x^{-\varepsilon'} \leq 0 \quad \text{for all } x \in \mathbb{R}_{++}, \quad (5)$$

and

$$\frac{d f^2(x)}{dx^2} = -\varepsilon x^{-(1+\varepsilon)} + \lambda \varepsilon' x^{-(1+\varepsilon')} \leq 0 \quad \text{for all } x \in \mathbb{R}_{++}. \tag{6}$$

From (5) and (6) it follows that

$$\lambda \geq x^{-(\varepsilon-\varepsilon')} \quad \text{for all } x \in \mathbb{R}_{++}, \tag{7}$$

and

$$\lambda \leq \frac{\varepsilon}{\varepsilon'} x^{-(\varepsilon-\varepsilon')} \quad \text{for all } x \in \mathbb{R}_{++}, \tag{8}$$

respectively. Since the functions $x \mapsto x^{-(\varepsilon-\varepsilon')}$ and $x \mapsto \frac{\varepsilon}{\varepsilon'} x^{-(\varepsilon-\varepsilon')}$ map $\mathbb{R}_{++}$ *onto* $\mathbb{R}_{++}$, there exist $x, y \in \mathbb{R}_{++}$ such that $x^{-(\varepsilon-\varepsilon')} > \frac{\varepsilon}{\varepsilon'} y^{-(\varepsilon-\varepsilon')}$. By consequence, λ cannot satisfy both (7) and (8) and we have a contradiction. ■

Proof of Proposition 8

Equivalence of (iii) and (v): That (v) implies (iii) is immediate. We prove using contraposition that (iii) implies (v). Suppose $\succeq_W$ is a monotonic SWO for which (v) does not hold, that is, $\succeq_W$ is not weakly maximin. Then, there is a pair $\mathbf{x}, \mathbf{y} \in X$, where $x_1 > y_1$, such that $\mathbf{y} \succeq_W \mathbf{x}$. Since $\mathbf{x} \succeq_W x_1 \mathbf{1}_n$ by reflexivity and monotonicity, it holds by transitivity that $\mathbf{y} \succeq_W x_1 \mathbf{1}_n$ while not $x_1 \mathbf{1}_n < \mathbf{y}$ and $x_1 \mathbf{1}_n \prec_I^M \mathbf{y}$. Hence, (iii) does not hold for $\succeq_W$.

Equivalence of (i) and (iii): That (iii) implies (i) is immediate. We prove using contraposition that (i) implies (iii). Suppose that $\succeq_W$ is a monotonic SWO for which (iii) does not hold, that is, there is a pair $\mathbf{x}, \mathbf{y} \in X$ such that not $\mathbf{x} < \mathbf{y}$ and $\mathbf{x} \prec_I^M \mathbf{y}$, while $\mathbf{y} \succeq_W \mathbf{x}$. Now, take any monotonic SWO $\succeq_W'$ such that $\mathbf{x} \succ_W' \mathbf{y}$. Clearly, $\succeq_W$ is not at least as inequality averse as $\succeq_W'$ according to the the M-BC. Hence, (i) does not hold for $\succeq_W$.

Equivalence of (iv) and (v): We first show that (v) implies (iv). Let $\mathbf{x}, \mathbf{y} \in X$ be any pair such that not $\mathbf{x} < \mathbf{y}$ and $\mathbf{x} \prec_I^{RD} \mathbf{y}$. Suppose first that $\mu(\mathbf{x}) > \mu(\mathbf{y})$. Then, there must be an $i \in N$ such that $\frac{x_i}{y_i} > 1$. Since also $\mathbf{x} \prec_I^{RD} \mathbf{y}$, we have $\frac{x_1}{y_1} > 1$. Hence, $\mathbf{x} \succ_W \mathbf{y}$ for any weakly maximin SWO $\succeq_W$. Suppose alternatively that $\mu(\mathbf{x}) \leq \mu(\mathbf{y})$. Then, because $\mathbf{x} \prec_I^{RD} \mathbf{y}$, $x_1 \leq y_1$ would imply that $\frac{x_i}{y_i} \leq 1$ for all $i \in N$ and $\frac{x_i}{y_i} < 1$ for at least one $i \in N$ and, hence, that $\mathbf{x} < \mathbf{y}$ which contradicts our premise. By consequence, $x_1 > y_1$ and $\mathbf{x} \succ_W \mathbf{y}$ for any weakly maximin SWO $\succeq_W$. Since (iv) implies (iii), because $\prec_I^M \subset \prec_I^{RD}$, and (iii) implies (v), as shown above, it follows furthermore that (iv) implies (v).

Equivalence of (ii) and (iv): The proof is very similar to that of the equivalence of (i) and (iii) and is therefore omitted. ■

Proof of Proposition 9

Suppose $\succeq_W$ is an SWO that is extremely inequality averse in the class of monotonic SWOs according to the L-BC. Then $\succeq_W$ is weakly maximin by Proposition 8 and Remark 1. Next, consider any $\mathbf{x},\mathbf{y} \in X$ where $x_1 \leq x_2 < x_3$ and $\mathbf{y} = \left(\lambda x_1, \lambda \frac{\sum_{i=2}^n x_i}{n-1}, \lambda \frac{\sum_{i=2}^n x_i}{n-1}, \ldots, \lambda \frac{\sum_{i=2}^n x_i}{n-1}\right)$ where λ is a scalar such that $x_1 > y_1$ and $x_2 < y_2$. For any allowed value of λ, it holds that $\mathbf{y} \prec_I^L \mathbf{x}$. Now, $\mathbf{x} \succ_W \mathbf{y}$, while there exists a monotonic SWO that implies a preference of $\mathbf{y}$ over $\mathbf{x}$ (since not $\mathbf{x} > \mathbf{y}$). Hence, we have a contradiction. ■

Proof of Proposition 10

Suppose $\succeq_W$ is a monotonic SWO that satisfies the condition stated in the proposition. Now, consider $\mathbf{x},\mathbf{y},\mathbf{z} \in X$ such that $\mathbf{x} = 2\mathbf{1}^n$, $y_1 = 1$ and $y_i = 3$ for all $i = 2,3,\ldots,n$, and $z_i = 2$ for all $i = 1,2,\ldots,(n-1)$ and $z_n = 2+4(n-1)$. It holds that $\mathbf{x} \prec_I^L \mathbf{y}$ and $\mathbf{y} \prec_I^L \mathbf{z}$. So, $\mathbf{x} \succ_W \mathbf{y}$ and $\mathbf{y} \succ_W \mathbf{z}$ by the condition. Moreover, $\mathbf{z} \succ_W \mathbf{x}$ by monotonicity. Hence, the SWO is intransitive, which is a contradiction. ■

Proof of Proposition 11

Take any $\mathbf{x},\mathbf{y} \in X$ where $x_1 \leq x_2 < x_3$ and $\mathbf{y} = \left(\lambda x_1, \lambda \frac{\sum_{i=2}^n x_i}{n-1}, \lambda \frac{\sum_{i=2}^n x_i}{n-1}, \ldots, \lambda \frac{\sum_{i=2}^n x_i}{n-1}\right)$ where λ is a positive scalar. For any allowed value of λ, it holds that $\mathbf{y} \prec_I^L \mathbf{x}$. Whenever $\lambda = 1$, then $\mathbf{y} \succ_W \mathbf{x}$ by Lorenz consistency. By continuity and monotonicity, there is an infinite number of λs such that $0 < \lambda < 1$ and $\mathbf{y} \succ_W \mathbf{x}$. Now, for any such λ it holds that $\mathbf{x}$ is strictly preferred to $\mathbf{y}$ by all weakly maximin SWOs. ■

Proof of Proposition 12

Suppose $\succeq_W$ is a monotonic SWO that is more inequality averse than leximin. Then, there is some pair $\mathbf{x},\mathbf{y} \in X$ such that $\mathbf{x} \prec_I^L \mathbf{y}$, $\mathbf{x} \succeq_W \mathbf{y}$ and leximin strictly prefers $\mathbf{y}$ to $\mathbf{x}$. By the latter condition, it holds that, either (a) $x_1 < y_1$, or (b) there is a $k > 1$ such that, for all $i = 1,\ldots,(k-1)$ it holds that $x_i = y_i$ while $x_k < y_k$. Now, consider a $\mathbf{z} \in X$ such that, in case (a), $x_1 < z_1 < y_1$ and $\mathbf{z} = z_1\mathbf{1}_n$ and, in case (b), $z_i = x_i = y_i$ for all $i = 1,\ldots,(k-1)$, $x_k < z_k < \min\left(\frac{\sum_{i=k}^n x_i}{n-k}, y_k\right)$ and $z_i = z_k$ for all $i = (k+1),\ldots,n$. Then, by monotonicity, $\mathbf{y} \succ_W \mathbf{z}$, and hence by transitivity $\mathbf{x} \succ_W \mathbf{z}$. Now, $\mathbf{z} \prec_I^L \mathbf{x}$ and leximin strictly prefers $\mathbf{z}$ to $\mathbf{x}$. By consequence, $\succeq_W$ is not more inequality averse than leximin and we have a contradiction. ■

Proof of Proposition 13

Note first that a monotonic SWO $\succeq_W$ can only satisfy the condition stated in the proposition if it is weakly maximin. Next, take an income distribution $\mathbf{x} \in X$ where $x_1 < x_2 < x_3$. By minimal invariance there must be some $\mathbf{x}' \in X$ such that $\mathbf{x}' \gg \mathbf{x}$ and $\mathbf{x} \preceq_I \mathbf{x}'$. Now consider an $\mathbf{y}$ such that $x_1 < y_1 < x_1'$, $y_2 < x_2 < x_2'$, and $x_3 < x_3' < y_3$. Clearly, for any weakly maximin SWO $\succeq_W$, it holds that $\mathbf{y} \succ_W \mathbf{x}$ and $\mathbf{x}' \succ_W \mathbf{y}$. Now, suppose $\preceq_I$ is an inequality quasi-ordering that satisfies the condition specified in the proposition. Then, $\mathbf{x} \preceq_I \mathbf{x}'$, $\mathbf{y} \prec_I \mathbf{x}$ since not $\mathbf{y} > \mathbf{x}$, and $\mathbf{x}' \prec_I \mathbf{y}$ since not $\mathbf{x}' > \mathbf{y}$. The IQO is intransitive, which is a contradiction. ■

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