A consistent multidimensional Pigou-Dalton transfer principle[*]

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December 12, 2006

Abstract. The Pigou-Dalton principle demands that a regressive transfer decreases social welfare. In the unidimensional setting this principle is consistent, because regressivity in terms of attribute amounts and regressivity in terms of individual well-being coincide in the case of a single attribute. In the multidimensional setting, however, the relationship between the various attributes and well-being is complex. To formulate a multidimensional Pigou-Dalton transfer principle, a concept of well-being must therefore first be defined. We propose a version of the Pigou-Dalton principle that defines regressivity in terms of the individual well-being ranking that underlies the social ranking on which the principle is imposed. This well-being ranking (of attribute bundles) is induced from the social ranking over distributions in which all individuals have the same attribute bundle. It is shown that this new principle—the consistent Pigou-Dalton principle—imposes a quasi-linear structure on the well-being ranking. We discuss the implications of this result within the literature on multidimensional inequality measurement and within the literature on needs.

Keywords. Pigou-Dalton principle · Multidimensional inequality measurement · Majorization · Budget dominance · Needs · Weak equity axiom

JEL classification. D63

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1 Introduction

We consider the problem of ranking social states in a multidimensional setting. In this setting, a social state is a distribution of attribute bundles over the individuals in society, one bundle for each individual. It is assumed that these attribute bundles incorporate all information relevant to the problem and, hence, that the individuals can be treated as identical except for their attribute bundles. In addition, all attributes are considered to be good for individuals.

Our focus is on the formulation and examination of a multidimensional Pigou-Dalton principle, which expresses the social ranking's basic concern for inequality between individuals. In the unidimensional setting, the Pigou-Dalton principle demands that a regressive transfer in the single attribute decreases social welfare. Since there is a one-to-one correspondence between the amount an individual has of the single attribute and her level of well-being, transfers that are regressive in terms of the attribute are also regressive in terms of well-being. The latter is what ultimately matters: the Pigou-Dalton principle basically demands that if a transfer from one individual to another increases the well-being of the better-off individual and decreases that of the worse-off, then it decreases social welfare.

Two difficulties arise when one attempts to carry over the idea of the unidimensional Pigou-Dalton principle to the multidimensional setting. *First*, it is not necessarily the case that each of the attributes can be considered as transferable. More precisely, the idea of a transfer that preserves the total amount of an attribute in society is not necessarily meaningful and desirable for all attributes. However, this problem can be straightforwardly solved by defining the transfer principle only in terms of transferable attributes. The *second* problem is less easily tackled: in the multidimensional setting there is in general no one-to-one correspondence between the level of any one attribute and the level of well-being (there may even be no explicit concept of individual well-being at all). To illustrate the difficulty, let us consider an example with two attributes, the first transferable, the second not. Individual i has the attribute bundle $(70, 50)$ while individual j has the attribute bundle $(120, 30)$. Suppose now that an amount of the first (i.e., the transferable) attribute is transferred from individual i to individual j. Is this transfer regressive in terms of individual well-being or not? It is clear that in order to answer this question, we need to determine whether or not the bundle $(70, 50)$ yields a lower level of well-being than $(120, 30)$. In other words, to define an appropriate multidimensional Pigou-Dalton principle, we need a ranking of attribute bundles on the basis of individual well-being.

Such an individual well-being ranking can be induced from any social ranking that is complete: it is sufficient to consider how the social ranking evaluates distributions in which all individuals have the same attribute bundle. In

terms of the example above, the bundle $(70, 50)$ yields a lower (higher) level of well-being than the bundle $(120, 30)$ if the social ranking considers the distribution $((70, 50), (70, 50), \ldots, (70, 50))$ to be worse (better) than the distribution $((120, 30), (120, 30), \ldots, (120, 30))$. The well-being ranking is common to all individuals since they are identical except for their attribute bundles. We propose to consider a version of the Pigou-Dalton principle that defines regressivity in terms of the well-being concept that underlies the social ranking. Because this version of the Pigou-Dalton principle is in this sense *consistent* with the social ranking on which it is imposed, we refer to the new principle as the "consistent Pigou-Dalton principle." The principle is defined in Section 3.

We investigate the effect of imposing the consistent Pigou-Dalton principle in the standard framework of social rankings satisfying anonymity, monotonicity, and additive representability. Social rankings that satisfy the latter three properties can be represented by a social welfare function which is a sum of utilities: the utility function is the same for each individual and represents the common well-being ranking alluded to above. The imposition of the consistent Pigou-Dalton principle in this framework has the strong implication that the common utility function is quasi-linear in the transferable attributes. This result, which is our main result, is presented in Section 4.

We discuss the main result in the settings of the multidimensional inequality measurement literature and the needs literature, respectively. We start in Section 5 with the multidimensional inequality measurement literature, which usually assumes all attributes to be transferable. The consistent Pigou-Dalton principle implies that social welfare has to be measured as the sum of utilities of individual budgets, the latter defined as a weighted sum of all attributes of an individual. As such, the main result can be rephrased to provide a normative justification of the concept of budget dominance (Kolm, 1977). Furthermore, we show that the resulting social ranking can never reverse the ranking obtained using three majorization principles that have been proposed in the normative approach to multidimensional inequality measurement, to wit, the uniform majorization, the uniform Pigou-Dalton majorization, and the correlation increasing majorization principles (see Weymark, 2006, for an overview).

In Section 6, we consider the needs literature, which typically assumes only one attribute, usually income, to be transferable. In this setting, the consistent Pigou-Dalton principle implies that social welfare has to be measured as the sum of utilities of adjusted income, i.e., income adjusted for needs by subtracting a certain amount, which depends on the non-income attributes, from the nominal income. We show that this social ranking obeys Sen's (1973) weak equity axiom. Moreover, we discuss the necessity of using additive needs corrections (also known as absolute equivalence scales) in the light of a recent incompatibility result between welfarism and the so-called between type Pigou-

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Dalton transfer principle, a multidimensional Pigou-Dalton transfer principle based on multiplicative needs corrections using so-called relative equivalence scales (Ebert, 1997, Ebert and Moyes, 2003, Shorrocks, 2004). Finally, the social ranking also provides a normative justification for a dominance criterion, which turns out to coincide with Bourguignon's (1989) dominance criterion in case of a single non-transferable attribute, e.g., an ordinal needs index to classify household types.

2 Notation

We consider a finite set M of at least two individuals and a non-empty finite set A of attributes. We distinguish transferable from non-transferable attributes. Whether or not an attribute is transferable is ultimately a normative choice: an attribute is transferable if one believes that transferring attribute amounts from better to worse off individuals, while preserving the total amount of this attribute, is desirable. Income would be a typical example of a transferable attribute, whereas subjective health status would be a typical example of a possibly non-transferable attribute. The set T collects the transferable attributes and is non-empty; the set N collects the non-transferable attributes; $A = T \cup N$. Concerning the sets T and N, two extreme positions have received considerable attention. First, the multidimensional inequality literature puts $N = \varnothing$, i.e., only considers transferable attributes. Second, in the needs literature all attributes except for one are considered as non-transferable. In this framework, income is considered as the single transferable attribute.

The variable that measures the amount of attribute k runs over some closed interval $A_k \subset \mathbb{R}$. As such, the domains $\mathcal{A}_T$ and $\mathcal{A}_N$ of transferable and non-transferable attribute bundles are orthotopes, i.e., cartesian products of closed intervals. The domain of attributes is[1]

$$\mathcal{A} \; = \; \underbrace{\Pi_{k \in T} A_k}_{\mathcal{A}_T} \times \underbrace{\Pi_{k \in N} A_k}_{\mathcal{A}_N} \; \subset \; \underbrace{\Pi_{k \in T} \mathbb{R}}_{\mathbb{R}^T} \times \underbrace{\Pi_{k \in N} \mathbb{R}}_{\mathbb{R}^N} .$$

Each attribute bundle x in $\mathcal{A}$ can be decomposed into (x_T, x_N) with x_T in $\mathcal{A}_T$ the transferable part and x_N in $\mathcal{A}_N$ the non-transferable part. We extend this decomposition to arbitrary vectors in $\mathbb{R}^A$ and we write $\varepsilon = (\varepsilon_T, \varepsilon_N)$ with ε_T in $\mathbb{R}^T$ and ε_N in $\mathbb{R}^N$.

Each individual i in M is endowed with some attribute bundle $x^i = (x_k^i)_{k \in A}$ in $\mathcal{A}$. The number x_k^i in A_k measures the amount of attribute k individual i is endowed with. Superscripts refer to individuals and subscripts to attributes. A distribution of attributes over the set of individuals is an $|A| \times |M|$ matrix X with the attribute bundle x^i at the ith column. The set $\mathcal{D} = \mathcal{A}^M$ is said to be

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[1]The notation $\mathbb{R}^A$ follows the notation B^A for the collection of maps from a set A to a set B.

the domain of distributions. We assume that the attribute bundles completely capture the relevant differences between the individuals. In other words, the individuals are identical except for their attribute bundles.

Vector and matrix inequalities are denoted by $\geq$, $>$, and $\gg$: we write $X \geq Y$ if the inequality $x_k^i \geq y_k^i$ holds for each individual i and each attribute k, $X > Y$ if in addition at least one of the inequalities holds strictly, and $X \gg Y$ if all the inequalities hold strictly. We write 0 for zero vectors. For two vectors x and y in $\mathbb{R}^\ell$, we write $x \cdot y$ for the sum $x_1 y_1 + x_2 y_2 + \cdots + x_\ell y_\ell$.

A social ranking is a quasi-ordering $\succsim$ in $\mathcal{D}$.[2] The asymmetric and symmetric components of $\succsim$ are denoted by $\succ$ and $\sim$, respectively.[3] The social ranking $\succsim$ induces a quasi-ordering in $\mathcal{A}$. We denote this induced relation by $R_{\underset{\sim}{\succ}}$: for each x and y in $\mathcal{A}$,

$$x \, R_{\underset{\sim}{\succ}} \, y \quad \text{if and only if} \quad \begin{pmatrix} x & x & \cdots & x \end{pmatrix} \succsim \begin{pmatrix} y & y & \cdots & y \end{pmatrix}. \tag{1}$$

It is compelling to interpret the relation $R_{\underset{\sim}{\succ}}$ as the ranking of attribute bundles in terms of individual well-being that underlies the social ranking $\succsim$. Since individuals only differ with respect to attribute bundles, a choice between two distributions in which they all have the same attribute bundle boils down to a choice of the best attribute bundle at the individual level. In case the social ranking $\succsim$ in $\mathcal{D}$ is complete, then also the induced relation $R_{\underset{\sim}{\succ}}$ in $\mathcal{A}$ is complete. The asymmetric part of $R_{\underset{\sim}{\succ}}$ is denoted by $P_{\underset{\sim}{\succ}}$.

We now introduce three properties for a social ranking $\succsim$ of distributions. Monotonicity and anonymity are natural requirements.

Monotonicity. For each X and Y in $\mathcal{D}$, the matrix inequality $X > Y$ implies $X \succ Y$.

Anonymity. For each X in $\mathcal{D}$, we have indifference between X and all distributions that are equal to X up to a rearrangement of its columns (individuals).

Monotonicity makes sense in the multidimensional context if each attribute is a *good*—not a bad. A monotonic social ranking registers an increase in an attribute as an improvement. Anonymity imposes that the names of the individuals are not taken into account. This property presupposes that all relevant characteristics of individual i are incorporated in the attribute bundle x^i. In other words, the relevant differences between the individuals are captured by the attribute bundles. The third property incorporates completeness, continuity, and separability.

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[2]A transitive and reflexive binary relation is a quasi-ordering. A complete quasi-ordering is an ordering.

[3]That is, $X \succ Y$ if $X \succsim Y$ and not $Y \succsim X$, and $X \sim Y$ if $X \succsim Y$ and $Y \succsim X$.

Additive representability. There exist C^2-maps $u^i : \mathcal{A} \to \mathbb{R}$, one for each i in M, such that, for each X and Y in $\mathcal{D}$,

$$X \succsim Y \quad \text{if and only if} \quad \sum_{i \in M} u^i(x^i) \geq \sum_{i \in M} u^i(y^i). \tag{2}$$

Imposing this property forces the social ranking to be complete, continuous (hence, representable), and separable over individuals (in order to compare two distributions, only those individuals who experience a change in their attribute bundles are taken into account, individuals who experience a status quo have no impact). The other way around, if the social ranking $\succsim$ satisfies monotonicity, anonymity, continuity, and separability over individuals, then it is representable as in (2) with $u^i = u$ for each i in M (Blackorby, Donaldson, and Auersperg, 1981). This map u represents the ordering $R_{\succsim}$ in $\mathcal{A}$ induced by the social ranking $\succsim$ in $\mathcal{D}$. The technical condition that the representation involves C^2—i.e., twice continuously differentiable—functions is not always imposed upon the relation $\succsim$. Here, we follow Atkinson and Bourguignon (1982, 1987).

3 A consistent Pigou-Dalton principle

According to the unidimensional Pigou-Dalton principle, a small transfer from poor to rich results in a distribution that is socially worse than the initial distribution. We propose a multidimensional analogue as follows. First, we select a quasi-ordering R in $\mathcal{A}$. The relation R ranks attribute bundles in terms of individual well-being. The R-Pigou-Dalton principle requires that whenever an individual is—according to R—not worse off than another, then a mean-preserving transfer in one or more attributes between these individuals that is regressive in terms of well-being, decreases social welfare.

R-**Pigou-Dalton principle.** For each X and Y in $\mathcal{D}$, for each i and j in M with $x^i \, R \, x^j$, and for each $\varepsilon = (\varepsilon_T, \varepsilon_N)$ in $\mathbb{R}^A$ with $\varepsilon_T > 0$ and $\varepsilon_N = 0$, we have that if

$$Y = \begin{pmatrix} \cdots & x^i + \varepsilon & \cdots & x^j - \varepsilon & \cdots \end{pmatrix},$$

with X and Y coinciding except for individuals i and j, then $X \succ Y$.

In this definition, the move from distribution X to distribution Y involves a transfer from j to i. The restriction $\varepsilon_N = 0$ reflects that only transferable attributes are involved. According to the relation R, individual i is not worse off than j *before* the transfer and definitely better off than j *after* the transfer (this at the cost of individual j). The unidimensional $\geq$-Pigou-Dalton principle (with $\geq$ the natural ordering in $\mathbb{R}$) coincides with the unidimensional Pigou-Dalton principle. The normative contents of the R-Pigou-Dalton principle

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crucially depends on the *choice* of the quasi-ordering R in $\mathcal{A}$. It seems natural to choose R equal to the well-being concept underlying $\succsim$, i.e., to choose $R = R_{\underset{\sim}{\succ}}$. We refer to this version of the R-Pigou-Dalton principle as the consistent Pigou-Dalton principle.

Consistent Pigou-Dalton principle. The social ranking $\succsim$ in $\mathcal{D}$ satisfies the consistent Pigou-Dalton principle if it satisfies the $R_{\underset{\sim}{\succ}}$-Pigou-Dalton principle, where the quasi-ordering $R_{\underset{\sim}{\succ}}$ in $\mathcal{A}$ is induced by the social ranking $\succsim$ in $\mathcal{D}$ as in (1).

Note that a unidimensional social ranking $\succsim$ satisfies the consistent Pigou-Dalton principle if and only if the relation $R_{\underset{\sim}{\succ}}$ coincides with $\geq$, i.e., if and only if the social ranking $\succsim$ satisfies a weak form of monotonicity (if $X \gg Y$, then $X \succ Y$).

4 Main result

The next theorem investigates the effect of imposing monotonicity, anonymity, additive representability, and the consistent Pigou-Dalton principle upon a social ranking of distributions.

Theorem 1. *A social ranking $\succsim$ satisfies monotonicity, anonymity, additive representability, and the consistent Pigou-Dalton principle if and only if there exists*

- *a vector p_T in $\mathbb{R}^T$ with $p_T \gg 0$,*

- *a strictly increasing and strictly concave C^2-map $\varphi : \mathbb{R} \to \mathbb{R}$, and*

- *a C^2-map $\psi : \mathcal{A}_N \to \mathbb{R}$ which is strictly increasing in each variable,*

such that, for each X and Y in $\mathcal{D}$, we have

$$X \succsim Y \quad \text{if and only if} \quad \sum_{i \in M} \varphi\big(p_T \cdot x_T^i + \psi(x_N^i)\big) \geq \sum_{i \in M} \varphi\big(p_T \cdot y_T^i + \psi(y_N^i)\big).$$

Proof. The particular representation of the social ranking $\succsim$ satisfies the four conditions. We focus on the reverse implication. Therefore, let $\succsim$ be an ordering in $\mathcal{D}$ that satisfies the four conditions.

Since the social ranking $\succsim$ is anonymous, monotonic, and additive separable, it can be represented by a C^2-map on the domain of distributions:

$$SR : \mathcal{D} \longrightarrow \mathbb{R} : X \longmapsto \sum_{i \in M} u(x^i),$$

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with $u : \mathcal{A} \to \mathbb{R}$ a strictly increasing function that represents the induced ordering $R_{\succeq}$. The monotonicity of the relation $\succsim$ implies the monotonicity of $R_{\succeq}$ (if $x > y$, then $x P_{\succeq} y$).

Next, we study the indifference surfaces of the map u. The monotonicity of $R_{\succ}$ implies that the indifference surfaces are thin.

Let $t \in T$. Let x and y in $\mathcal{A}$ satisfy $u(x) \geq u(y)$. The consistent Pigou-Dalton principle implies that the distribution $\begin{pmatrix} x & y & z & \cdots & z \end{pmatrix}$ is socially preferred to the distribution $\begin{pmatrix} x+\varepsilon & y-\varepsilon & z & \cdots & z \end{pmatrix}$ with ε in $\mathbb{R}^A$, $\varepsilon_t > 0$, and $\varepsilon_k = 0$ for each $k \neq t$. Given additive representability, it follows that

$$u(y) - u(y - \varepsilon) \; > \; u(x + \varepsilon) - u(x) \; > \; 0.$$

Divide by ε_t, take the limits for ε_t to 0, and obtain $D_t u(y) \geq D_t u(x) \geq 0$. In sum,

for each t in T, if $u(x) \geq u(y)$, then $0 \; \leq \; D_t u(x) \; \leq \; D_t u(y)$.

Let x and y belong to the same indifference surface. Then, $u(x) \geq u(y)$, $u(y) \geq u(x)$, and the partial derivative with respect to a transferable attribute is a constant, $D_t u(x) = D_t u(y)$. The derivative $D_t u(x)$ does not depend upon the particular position of the vector x in $\mathcal{A}$. The utility level $u(x)$ completely determines the derivative $D_t u(x)$.

As a consequence, for each t in T and for each x in $\mathcal{A}$, it holds that $D_t u(x) \; = \; V_t(u(x))$, with $V : \mathbb{R} \to (\mathbb{R}^+)^T$ a vector valued map. Differentiate the identity $D_t u(x) = V_t(u(x))$ with respect to x_s (s in T) and obtain

$$D_s D_t u(x) \; = \; DV_t\big(u(x)\big) \times V_s\big(u(x)\big).$$

Because the map u is twice continuously differentiable, we have $D_s D_t u = D_t D_s u$. Hence, for each x in $\mathcal{A}$ and for each t and s in T, we have

$$DV_t\big(u(x)\big) \times V_s\big(u(x)\big) \; = \; DV_s\big(u(x)\big) \times V_t\big(u(x)\big).$$

Therefore, on the image set of u, it holds that $D[\ln \circ V_t] = D[\ln \circ V_s]$, or that $V_s = p_{st} V_t$ with $p_{st} > 0$. In conclusion,

$$\text{for each } x \text{ in } \mathcal{A}, \; \big(D_t u(x)\big)_{t \in T} \; = \; v\big(u(x)\big) \, p_T, \tag{3}$$

with p_T in $\mathbb{R}^T$, $p_T \gg 0$ ($p_{st} = p_s/p_t > 0$), and with v a decreasing map from $\mathbb{R}$ to $\mathbb{R}^+$.

Eliminate in (3) the term $v(u(x))$ and obtain $D_t u(x) = (p_t/p_s) D_s u(x)$ for each s and t in T. This is a system of $|T|-1$ linear first order partial differential equations. The solution is

$$u(x) = \tilde{\varphi}(p_T \cdot x_T, x_N),$$

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with the map $\tilde{\varphi}$ from $\mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}$ strictly increasing in each variable. The map $\tilde{\varphi}$ is strictly concave in its first argument. Indeed, if $D_1\tilde{\varphi}(z, x_N) = D_1\tilde{\varphi}(z', x_N)$ with $z' > z$, then a small progressive transfer might result in the same social welfare. This contradicts the Pigou-Dalton principle. Furthermore, the map v satisfies $D_1\tilde{\varphi} = v \circ \tilde{\varphi}$ and is strictly decreasing.

In the particular case where N is empty, we are done. The map $\varphi = \tilde{\varphi}$ has the properties stated in the theorem, and the social ranking is represented by

$$ SR : \mathcal{D} \longrightarrow \mathbb{R} : X \longmapsto \sum_{i \in M} \varphi(p_T \cdot x^i). $$

In the case where N is not empty, we return to equation (3) and plug in the map $\tilde{\varphi}$. We obtain

$$ D_1\tilde{\varphi}(z, x_N) \;=\; v\big(\tilde{\varphi}(z, x_N)\big), $$

for each z in $\mathbb{R}$ between the minimum and the maximum value of $p_T \cdot x_T$. The solution of this quasi-linear first-order partial differential equation reads

$$ \int \frac{d\tilde{\varphi}}{v(\tilde{\varphi})} \;=\; z + \psi(x_N), $$

with ψ an arbitrary map from $\mathbb{R}^N$ to $\mathbb{R}$.[4] It follows that $\tilde{\varphi}(z, x_N)$ depends upon $z + \psi(x_N)$ rather than upon z and x_N separately. In conclusion: $\tilde{\varphi}(z, x_N) = \varphi(z + \psi(x_N))$ and the social ranking is represented by

$$ SR : \mathcal{D} \longrightarrow \mathbb{R} : X \longmapsto \sum_{i \in M} \varphi\big(p_T \cdot x_T^i + \psi(x_N^i)\big), $$

with φ and ψ as stated in the theorem. □

As Theorem 1 shows, the imposition of the consistent Pigou-Dalton principle in an additively separable framework strongly limits the possibilities to rank distributions in $\mathcal{D}$: the induced well-being ranking in $\mathcal{A}$ obtains a quasi-linear structure. The next two sections apply this result to the framework of multidimensional inequality measurement and to the framework of needs, respectively.

5   Multidimensional inequality measurement

The literature on multidimensional social evaluation assumes that each attribute is transferable, $A = T$ and $N = \varnothing$. Imposing the four properties results in the following criterion: for each X and Y in $\mathcal{D}$,

$$ X \succsim Y \quad \text{if and only if} \quad \sum_{i \in M} \varphi\big(p \cdot x^i\big) \geq \sum_{i \in M} \varphi\big(p \cdot y^i\big), \tag{4} $$

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[4]See Polyanin, Zaitsev, and Moussiaux (2002).

with $\varphi : \mathbb{R} \to \mathbb{R}$ a strictly increasing and strictly concave C^2-map, and $p \gg 0$ in $\mathbb{R}^T$. From here, we distinguish the dominance approach (5.1) and the normative approach (5.2). For surveys of these two approaches, we refer to Trannoy (2006) and Weymark (2006), respectively.

5.1. In the dominance approach, the Lorenz dominance criterion plays a central role. We recall the generalized Lorenz criterion and the budget dominance relation.

Generalized Lorenz criterion. Let a and b be two n-tuples of real numbers. Denote the ordered coordinates of a by $a_{\langle 1 \rangle} \leq a_{\langle 2 \rangle} \leq \cdots \leq a_{\langle n \rangle}$ (similar for b). Then a is said to generalized Lorenz dominate b—denoted $a \succsim_{GL} b$—if one of the following equivalent conditions is fulfilled (Kolm, 1969, Marshall and Olkin, 1979, Shorrocks, 1983):

- $a_{\langle 1 \rangle} + a_{\langle 2 \rangle} + \cdots + a_{\langle k \rangle} \geq b_{\langle 1 \rangle} + b_{\langle 2 \rangle} + \cdots + b_{\langle k \rangle}$ for each $k = 1, 2, \ldots, n$,

- $g(a_1) + g(a_2) + \cdots + g(a_n) \geq g(b_1) + g(b_2) + \cdots + g(b_n)$ for each C^2-map $g : \mathbb{R} \to \mathbb{R}$ that is strictly increasing and strictly concave.

Budget dominance. Let X and Y be two distributions in $\mathcal{D}$. Then, X is said to budget dominate Y—denoted $X \succsim_B Y$—if

$$\left(p \cdot x^i \right)_{i \in M} \succsim_{GL} \left(p \cdot y^i \right)_{i \in M} \qquad \text{for each } p \gg 0.$$

Interpret p in $\mathbb{R}^T$ as a price vector and the inner product $b^i = p \cdot x^i$ as the budget of individual i. Then, distribution X budget dominates distribution Y if, for each price vector, the distribution of budgets induced by X generalized Lorenz dominates the distribution of budgets induced by Y.

Recall the relation described by (4) and compare it with the definitions of the generalized Lorenz criterion and budget dominance. Theorem 1 allows us to rephrase the concept of budget dominance in terms of the consistent Pigou-Dalton principle. We obtain the following normative underpinning for the concept of budget dominance.

Corollary 1. *Let each attribute be transferable. Let X and Y be two distributions in $\mathcal{D}$. Then, $X \succsim_B Y$ if and only if $X \succsim Y$ for each social ranking $\succsim$ that satisfies monotonicity, anonymity, additive representability, and the consistent Pigou-Dalton principle.*

5.2. Although the functional form (2) is frequently used as a starting point in the normative approach to multidimensional inequality measurement, the relation described in (4) hardly receives any attention.

We investigate the relationship between the consistent Pigou-Dalton principle and three other multidimensional principles, to wit, the uniform majorization principle, the uniform Pigou-Dalton majorization principle, and the correlation increasing majorization principle.

First, we introduce some additional notation. Let I denote the $|M| \times |M|$ identity matrix. A non-negative square matrix is said to be bistochastic if all of its row and column sums are equal to 1. A bistochastic matrix with only zeros and ones is a permutation matrix. The permutation matrix that interchanges the i and j coordinates is denoted by $I_{i,j}$. Furthermore, a strict t-transform is a linear transformation defined by an $|M| \times |M|$ matrix S of the form

$$S \ = \ \lambda I + (1 - \lambda) I_{i,j} \quad \text{with } \lambda \text{ in } (0, 1).$$

In this notation, anonymity postulates that post-multiplying a distribution by a permutation matrix results in an equally good distribution. The uniform majorization principle requires that post-multiplying a distribution by a non-permutation bistochastic matrix increases social welfare. The uniform Pigou-Dalton majorization principle demands that post-multiplying a distribution by a strict t-transform increases social welfare.

Uniform majorization principle. For each X in $\mathcal{D}$ and for each non-permutation bistochastic matrix B, we have $XB \succ X$.

Uniform Pigou-Dalton majorization principle. For each X in $\mathcal{D}$ and for each strict t-transform S, we have $XS \succ X$.

The uniform Pigou-Dalton majorization principle is weaker than the uniform majorization principle. Indeed, each strict t-transform is a non-permutation bistochastic matrix while the converse does not hold (Marshall and Olkin, 1979, p. 431).

It appears that social rankings of the form (4) do not satisfy these majorization principles. We illustrate this claim using a counterexample. Let there be two individuals and two transferable attributes. Consider the distributions

$$X \ = \ \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = XS \text{ with } S = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

Both uniform majorization principles rank Y strictly higher than X (the matrix S defines a strict t-transfer with $\lambda = 0.5$). On the other hand, each social ranking of the form (4) with weights $p_1 = p_2$ judges X and Y as equally good. This example easily extends to more individuals and more attributes. On the other hand, the consistent Pigou-Dalton principle is compatible with the weak versions (with $\succ$ replaced by $\succsim$) of the two majorization principles. The next proposition captures this phenomenon.

Proposition 1. *Let each attribute be transferable. Let $\succsim$ be a social ranking that satisfies monotonicity, anonymity, additive representability, and the consistent Pigou-Dalton principle. Then, for each X in $\mathcal{D}$ and for each non-permutation bistochastic matrix B, we have $XB \succsim X$.*

Proof. Let X in $\mathcal{D}$ be a distribution, b the corresponding M-tuple of budgets, and B an $|M| \times |M|$ non-permutation bistochastic matrix.

The properties of φ imply that the map $W : a \mapsto \sum_{i \in M} \varphi(a^i)$, which represents the social ranking, is strictly Schur-concave.[5] Indeed, for each i and j in M and for each M-tuple a, we have

$$a^i > a^j \quad \text{implies} \quad D_i W(a) < D_j W(a).$$

It follows that $W(bB) > W(b)$ except for the particular case in which bB and B are identical up to a permutation (implying that $W(bB) = W(b)$). □

The above proof reveals that the case $XB \sim X$ occurs if and only if there exists a permutation $\pi : M \to M$ such that each individual i in M is—according to the induced relation $R_{\underset{\sim}{\succsim}}$—indifferent between his attribute bundle x^i (the ith column in X) and the $\pi(i)$th column in XB.

We close this section by comparing the consistent Pigou-Dalton principle with the correlation increasing majorization principle of Tsui (1999). For each x and y in $\mathbb{R}^A$, let $x \wedge y = (\min\{x_k, y_k\})_{k \in A}$ and $x \vee y = (\max\{x_k, y_k\})_{k \in A}$.

Correlation increasing majorization. For each X and Y in $\mathcal{D}$, and for each i and j in M, we have that if

$$Y = \begin{pmatrix} \cdots & x^i \vee x^j & \cdots & x^i \wedge x^j & \cdots \end{pmatrix} \neq X,$$

with X and Y coinciding except for individuals i and j, then $X \succ Y$.

Note that in the above definition, individual i is better off than j in distribution Y. Under the assumption that each attribute is transferable, the above principle—in combination with anonymity—boils down to Tsui's (1999) dependence-sensitivity axiom.[6] The move from distribution Y to X involves the transfer $(x^j - x^i) \vee 0$ from individual i to j. Therefore, the consistent

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[5]A map $f : D \subset R^n \to \mathbb{R}$ is said to be strictly Schur-concave if $f(yB) > f(y)$ for each y in D and each $n \times n$ non-permutation bistochastic matrix B for which the n-tuple yB is not a rearrangement of y. Marshall and Olkin (1979) provide the equivalent conditions that we use.

[6]Tsui's definition permits transfers as described in the definition of correlation increasing majorization in combination with rearrangements of the individuals. Therefore, the anonymity principle is needed in order to arrive at Tsui's definition. In the presence of anonymity, the restriction $x^i R x^j$ in the definition of correlation increasing majorization is redundant.

Pigou-Dalton principle—in combination with anonymity—implies the correlation increasing majorization principle. The next proposition formulates this observation.

Proposition 2. *Let each attribute be transferable. Let $\succsim$ be a complete social ranking that satisfies anonymity and the consistent Pigou-Dalton principle. Then, $\succsim$ satisfies the correlation increasing majorization principle.*

6 The needs framework

The starting point in the needs literature is the assumption that mean-preserving transfers only make sense for incomes. Let the income level appear as the first coordinate in the attribute bundle. Imposing the four properties results in the following criterion: for each X and Y in $\mathcal{D}$,

$$X \succsim Y \quad \text{if and only if} \quad \sum_{i \in M} \varphi\big(x_1^i + \psi(x_N^i)\big) \geq \sum_{i \in M} \varphi\big(y_1^i + \psi(y_N^i)\big), \quad (5)$$

the map $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, strictly concave, and C^2, and the map $\psi : \mathcal{A}_N \to \mathbb{R}$ is strictly increasing and C^2. Theorem 1 extends and allows for different income-dimensions. Nevertheless, we focus on the criterion expressed in (5) with one single income variable. From here, we distinguish between the literature following Sen's (1973) weak equity axiom (6.1), the cardinal equivalence scale literature (6.2), and the ordinal equivalence scale literature (6.3).

6.1. In his book "On economic inequality," Sen (1973) defines a utilitarian welfare function as a map of distributions X into a number $\sum_{i \in M} U^i(x_1^i) = \sum_{i \in M} u(x_1^i, x_N^i)$. All the relevant non-income variables (x_N^i) are compressed in the superscript i of the utility function U^i. He argues that such a utilitarian approach is a blunt approach to rank multidimensional income distributions because it conflicts with a simple notion of equity, the so-called weak equity principle. This principle states that if some individual has higher needs compared to another—i.e., a lower utility level for all income levels—, then the former should receive more income than the latter when dividing a fixed amount of income.

The consistent Pigou-Dalton principle implies the weak equity principle. Suppose one individual is worse off compared to another for all income levels. Then, any distribution of income which would give strictly more to the latter cannot be optimal according to a social ranking obeying the consistent Pigou-Dalton principle, because such a distribution is strictly inferior to a distribution obtained by transferring a small amount of income from the latter to the former.

As a consequence, also the social ranking in (5) must obey the weak equity principle. The other way around, Theorem 1 tells us that utilitarianism can be equity-regarding in a consistent way only if the utility functions have a specific quasi-linear structure.

6.2. In contrast with Sen's ordinal notion of needs based on utility levels—i.e., higher needs correspond with a lower utility level for all income levels—, the cardinal equivalence scale literature tries to quantify the needs differences. For an attribute bundle x^i, the correction term $\psi(x_N^i)$ in (5) adjusts income for needs in an additive, rather than a multiplicative, way: higher values correspond with lower needs. In the equivalence scale literature, the additive correction is called an absolute equivalence scale, while the multiplicative correction is called a relative equivalence scale. Furthermore, the additive correction term $\psi(x_N^i)$ is exact, i.e., it does not depend on the income level x_1^i.

The use of exact and absolute equivalence scales as in (5) is the only way to reconcile utilitarianism with a notion of equity. This sheds some light on an incompatibility result between welfarism and the so-called between type Pigou-Dalton transfer principle. This equity principle is what we have called an R-Pigou-Dalton principle, where the exogenous well-being ranking R is based on relative, rather than absolute, equivalence scales. As a result, there are two conflicting rankings—one exogenous (R) and one induced ($R_{\succsim}$)—to assess individual well-being. As shown by Ebert (1997), Ebert and Moyes (2003), and Shorrocks (2004), this inner contradiction ultimately conflicts with the welfarist nature of utilitarianism, which requires a unique utility metric.[7]

6.3. As an alternative to Sen's ordinal notion of needs based on utility levels (see 6.1 above), the ordinal needs literature proposes a notion based on utility differences. More precisely, higher needs correspond with a higher marginal utility of income. We present the criteria due to Atkinson and Bourguignon (1987) and Bourguignon (1989); we follow Ebert's (2000) presentation.

The marginal distribution of needs is taken to be fixed. Hence, we compare distributions X and Y in $\mathcal{D}$ with the same non-transferable attributes, i.e., $x_N^i = y_N^i$ for each individual i in M. The set M of individuals is partitioned into different needs groups from least to most needy. Let κ denote the number of different classes of needs types. This ordinal needs ranking presupposes the existence of some strictly increasing function ψ to rank the non-transferable attributes such that

$$\underbrace{\psi(x_N^i) = \cdots = \psi(x_N^j)}_{M_1 \,:\, \text{lowest needs}} \;>\; \cdots \;>\; \underbrace{\psi(x_N^k) = \cdots = \psi(x_N^\ell)}_{M_\kappa \,:\, \text{highest needs}}.$$

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[7]See Capéau and Ooghe (2006) for a reconciliation of welfarism and the between type Pigou-Dalton transfer principle in a non-utilitarian setting.

The partitioning $M = M_1 \cup M_2 \cup \cdots \cup M_\kappa$ depends upon ψ. If there is only one non-transferable attribute, e.g., an ordinal index which is inversely related to needs, then there is no dispute about how to partition individuals in needs groups: each increasing function ψ induces the same ordering and the same partition. Next, individuals in the same needs group have the same utility function which is C^2. A profile is a κ-tuple $U = (U_1, U_2, \ldots, U_\kappa)$ of utility functions from $\mathbb{R}^+$ to $\mathbb{R}$, one for each needs group.

Atkinson and Bourguignon (1987) and Bourguignon (1989) define—for a given profile U of utility functions—the total welfare $W_U(X)$ of a distribution X in $\mathcal{D}$ as the sum

$$W_U(X) = \sum_{i \in M_1} U_1(x_1^i) + \sum_{i \in M_2} U_2(x_1^i) + \cdots + \sum_{i \in M_\kappa} U_\kappa(x_1^i).$$

Furthermore, distribution X is said to dominate distribution Y if, for each profile U of utility functions with $0 \leq U_1' \leq U_2' \leq \cdots \leq U_\kappa'$ and $U_k'' \leq 0$ for each k, we have $W_U(X) \geq W_U(Y)$. The conditions $U_k' \leq U_{k+1}'$ for each $k = 1, 2, \ldots, \kappa - 1$ ensure that higher needs correspond with higher marginal utilities of income. If X dominates Y, then we write $X \succsim_\psi Y$; the subscript ψ refers to the map behind the partitioning in different needs groups. The next proposition shows that the social ranking $\succsim_\psi$ can be interpreted as unanimity among utilitarian welfare functions based on a wide set of absolute equivalence scales; see Fleurbaey et al. (2003) for a characterization based on relative equivalence scales.

Proposition 3. *Let there be exactly one transferable attribute (attribute 1) and at least one non-transferable attribute. Let X and Y be two distributions in $\mathcal{D}$ with $x_N^i = y_N^i$ for each i in M. Let the map $\psi : \mathcal{A}_N \to \mathbb{R}$ be strictly increasing and C^2. Then, $X \succsim_\psi Y$ if and only if*

$$\sum_{i \in M} \varphi\left(x_1^i + \vartheta \circ \psi(x_N^i)\right) \geq \sum_{i \in M} \varphi\left(y_1^i + \vartheta \circ \psi(y_N^i)\right),$$

for each strictly increasing and strictly concave map φ and for each strictly increasing map ϑ.

Proof. The utility functions $t \mapsto \varphi(t + \vartheta \circ \psi(x_N^i))$ generate a profile that satisfies the imposed conditions. Hence, the "if-then" implication follows. The reverse implication is more involved.

The map ψ partitions the individuals in κ different needs groups. Let $F_{X,k}$ be the distribution in X of incomes for the kth needs group. The distribution $F_{X,k}$ has a finite support. Let p_k be the marginal distribution of the needs types. We rewrite the welfare $W_{\varphi,\vartheta}(X)$ of X as measured by the maps φ and ϑ:

$$W_{\varphi,\vartheta}(X) \quad = \quad \frac{1}{|M|} \sum_{i \in M} \varphi\big(x_1^i + \vartheta \circ \psi(x_N^i)\big)$$

$$= \quad \sum_{k=1}^{\kappa} p_k \int_{-\infty}^{+\infty} \varphi\big(t + \vartheta \circ \psi(x_N^i)\big) \, dF_{X,k}(t).$$

The inequality $W_{\varphi,\vartheta}(X) \geq W_{\varphi,\vartheta}(Y)$ holds for each suitable φ and ϑ if and only if

$$\sum_{k=1}^{\kappa} p_k \int_{-\infty}^{+\infty} \varphi(t + m_k) \, d\big(F_{X,k}(t) - F_{Y,k}(t)\big) \; \geq \; 0,$$

for each strictly increasing and strictly concave C^2-map φ and for each κ-tuple $(m_1, m_2, \ldots, m_\kappa)$ of real numbers with $m_1 > m_2 > \cdots > m_\kappa$. In each integral we shift the variable t by m^k and we rewrite the previous inequality as

$$\sum_{k=1}^{\kappa} p_k \int_{-\infty}^{+\infty} \varphi(t) \, d\big(F_{X,k}(t - m_k) - F_{Y,k}(t - m_k)\big) \; \geq \; 0.$$

Lambert (2001, p. 54, Lemma 3.1) shows that this inequality holds for each strictly increasing and strictly concave map φ and for each κ-tuple m of real numbers with $m_1 > m_2 > \cdots > m_\kappa$ if and only if

$$\sum_{k=1}^{\kappa} p_k \int_{-\infty}^{t_k = t - m_k} \big(F_{X,k}(z) - F_{Y,k}(z)\big) \, dz \; \leq \; 0,$$

for each κ-tuple m of real numbers with $m_1 > m_2 > \cdots > m_\kappa$ and for each real number t; or—equivalently—for each κ-tuple $(t_1, t_2, \ldots, t_\kappa)$ of real numbers with $t_1 < t_2 < \cdots < t_\kappa$. In this final condition, we recognize Bourguignon's (1989) criterion,[8] which he shows to be equivalent to $W_U(X) \geq W_U(Y)$ for each profile $U = (U_1, U_2, \ldots, U_\kappa)$ of utility functions that satisfy $0 \leq U_1' \leq U_2' \leq \cdots \leq U_\kappa'$ and $U_k'' \leq 0$ for each $k = 1, 2, \ldots, \kappa$. $\qquad \square$

The next corollary provides a normative justification for a generalization of Bourguignon's criterion.

Corollary 2. *Let there be exactly one transferable attribute (attribute 1) and at least one non-transferable attribute. Let X and Y be two distributions in $\mathcal{D}$ such that $x_N^i = y_N^i$ for each i in M. Then, $X \succsim_\psi Y$ for each strictly increasing C^2-map $\psi : \mathcal{A}_N \to \mathbb{R}$ if and only if $X \succsim Y$ for each social ranking $\succsim$ that satisfies monotonicity, anonymity, additive representability, and the consistent Pigou-Dalton principle.*

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[8]The Bourguignon criterion is $\sum_{k=1}^{\kappa} p_k \int_{-\infty}^{t_k} (F_{X,k}(z) - F_{Y,k}(z)) \, dz \leq 0$ for each κ-tuple $(t_1, t_2, \ldots, t_\kappa)$ of real numbers with $t_1 \leq t_2 \leq \cdots \leq t_\kappa$. As each map $t \mapsto \int_{-\infty}^{t} (F_{X,k}(z) - F_{Y,k}(z)) \, dz$ is continuous, the strict inequalities can be replaced by weak inequalities.

If there is only one non-transferable attribute, e.g., an ordinal index inversely related to needs, then Bourguignon's criterion turns out to be equivalent with unanimity among all social rankings satisfying the above mentioned properties. If there are two or more non-transferable attributes, then one should check Bourguignon's criterion for each ordinal classification of individuals in needs groups that guarantees the implication "if $x_N^i < x_N^j$, then i has higher needs than j."

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