Lorenz comparisons of nine rules for the adjudication of conflicting claims*

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January 17, 2007

Abstract

Consider the following nine rules for adjudicating conflicting claims: the proportional, constrained equal awards, constrained equal losses, Talmud, Piniles', constrained egalitarian, adjusted proportional, random arrival, and minimal overlap rules. For each pair of rules in this list, we examine whether or not the two rules are Lorenz comparable. We allow the comparison to depend upon whether the amount to divide is larger or smaller than the half-sum of claims. In addition, we provide Lorenz-based characterizations of the constrained equal awards, constrained equal losses, Talmud, Piniles', constrained egalitarian, and minimal overlap rules.

Keywords. Claims problem · Bankruptcy · Taxation · Lorenz dominance · Progressivity · Proportional rule · Constrained equal awards rule · Constrained equal losses rule · Talmud rule · Piniles' rule · Constrained egalitarian rule · Adjusted proportional rule · Random arrival rule · Minimal overlap rule

JEL classification. D63

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*We thank Bart Capéau, Frank Cowell, Erik Schokkaert, Frans Spinnewyn, William Thomson, and Bertil Tungodden for helpful remarks and comments. All remaining shortcoming are ours. Kristof Bosmans gratefully acknowledges financial support from the Fund for Scientific Research - Flanders (grant G.0005.04) and the Interuniversity Attraction Poles network funded by the Federal Public Planning Service, Belgian Science Policy (grant P5/21-A).

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1 Introduction

How should an amount of money be divided among a group of individuals if the amount available falls short of the sum of the individuals' claims? Several distribution problems take the form of this "claims problem"—two typical examples are the problems of bankruptcy and taxation. In the case of bankruptcy, the amount to divide is the liquidation value of the firm that goes bankrupt, and the claims are the entitlements of the creditors; in the case of taxation, the amount to divide is the difference between the total pre-tax income and the tax revenue, and the claims are the pre-tax incomes. The literature on the claims problem is largely devoted to the axiomatic study of rules that associate with each possible claims problem a division among the individuals.[1]

We examine how the divisions selected by nine well known rules compare in terms of inequality (or, in the terminology of the taxation, how the rules compare in terms of progressivity). As a criterion for making inequality comparisons, we use the Lorenz dominance relation which constitutes the cornerstone of the literature on inequality measurement.[2] In order to illustrate our analysis, let us consider a claims problem involving three individuals with claims equal to 500, 2000, and 3500, and an amount to divide equal to 1500. Table 1 presents the divisions proposed for this claims problem by the nine rules that we consider in our examination: the proportional (P), constrained equal awards (CEA), constrained equal losses (CEL), Talmud (T), Piniles' (Pin), constrained egalitarian (CE), adjusted proportional (A), random arrival (RA), and minimal overlap (MO) rules. The final row of the table presents totals, with the total claim equal to 6000 and the amount to divide equal to 1500.

Table 1. Example with an amount to divide of 1500

Claims	CEA	CE, Pin, T	A	RA, MO	P	CEL
500	500	250	214	167	125	0
2000	500	625	643	667	500	0
3500	500	625	643	667	875	1500
6000	1500	1500	1500	1500	1500	1500

All divisions in Table 1 can be compared with respect to inequality. As we move from left to right in the table, the award allocated to the smallest claim decreases, while the award allocated to the largest claim increases. In more general terms, moving from left to right, divisions become more unequal according to the Lorenz dominance relation. So, for instance, the adjusted proportional rule selects a less unequal division than the minimal overlap rule for the given claims problem. The key question is whether this type of conclusion holds in general or depends upon the particular characteristics of the example.

The next theorem—which is proven in Section 3—summarizes a first set of results. A rule R is said to Lorenz dominate a rule R' if, for each claims problem, the division proposed by R Lorenz dominates the division proposed by R'.

[1] See Moulin (2002) and Thomson (2003) for surveys.

[2] See, e.g., the survey of this literature by Sen and Foster (1997).

Theorem 1. *The Lorenz dominance relation ranks the nine rules as follows.*

$$
\begin{tikzcd}
 & & & RA \arrow[dr] & & \\
 & & & A \arrow[r] & MO \arrow[dr] & \\
CEA \arrow[r] & CE \arrow[r] & Pin \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & & CEL. \\
 & & & T \arrow[uur] & & \\
 & & & P \arrow[uurr] & &
\end{tikzcd}
$$

An arrow (or a sequence of arrows) from R to R' indicates that R Lorenz dominates R'. The absence of an arrow (or of a sequence of arrows) indicates the absence of a Lorenz relationship.

Related results have been established in the literature. Thomson (2002) provides a Lorenz ranking of the members of the family of increasing-constant-increasing rules which includes the constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules. Moreno-Ternero and Villar (2006) rank the family of TAL-rules which includes the constrained equal awards, constrained equal losses, and Talmud rules.

Theorem 1 shows that some rules are Lorenz incomparable. This is the case, for instance, for the proportional rule and the Talmud rule. To illustrate this point, we consider a claims problem with claims as in Table 1, but with an amount to divide of 4500 instead of 1500. The divisions proposed by the different rules are given in Table 2.

Table 2. Example with an amount to divide of 4500

Claims	*CEA*, *CE*	*Pin*	*P*	*RA*	*A*	*T*	*MO*	*CEL*
500	500	500	375	333	286	250	167	0
2000	2000	1625	1500	1333	1357	1375	1417	1500
3500	2000	2375	2625	2833	2857	2875	2917	3000
6000	4500	4500	4500	4500	4500	4500	4500	4500

Whereas, for the claims problem of Table 1, the Talmud rule proposes a less unequal division than the proportional rule, the converse is true for the claims problem of Table 2. As some rules—viz., the Talmud, Piniles', and constrained egalitarian rules—explicitly treat claims problems differently according to whether the amount to divide is smaller or larger than the half-sum of claims, it seems natural to consider restrictions of the Lorenz dominance relation on those two subsets of claims problems. It turns out that significantly more rules are Lorenz comparable on these restricted domains: for instance, for each claims problem with an amount to divide smaller than the half-sum of claims, the division selected by the Talmud rule Lorenz dominates that selected by the proportional rule, whereas the converse relation holds for each claims problem with an amount available larger than the half-sum of claims. The results for the restricted domains are summarized in Theorem 2, which is stated and proven in Section 4.

In addition to providing the Lorenz relationships between the nine rules, we characterize six of them as maximal or minimal with respect to the Lorenz dominance relation: the constrained equal awards, constrained equal losses, Talmud, Piniles', constrained egalitarian, and minimal overlap rules. For instance, we show that the minimal overlap rule is Lorenz-minimal in the set of rules satisfying order preservation of awards, order preservation of losses, reasonable lower bounds on awards, limited consistency, and super-modularity in claims (a new property that is analogous to the traditional super-modularity property). The characterizations of the constrained equal awards and constrained egalitarian rules are closely related to results by Schummer and Thomson (1997) and Chun, Schummer, and Thomson (2001), respectively. The propositions in Sections 3 and 4 formulate these characterization results and Table 4 in Section 4 provides a summary.

2 Nine rules and ten properties[3]

An amount E in $\mathbb{R}_+$ has to be divided among a set $N = \{1, 2, \ldots, n\}$ of at least two individuals with claims adding up to more than E. Let c_i in $\mathbb{R}_+$ be individual i's claim. The n-tuple $c = (c_1, c_2, \ldots, c_n)$ is said to be the vector of claims. The total claim $c_1+c_2+\cdots+c_n$ is assumed to be positive and is denoted by C. A claims problem is an ordered pair (c, E) with c the claims vector and E the amount to divide. The set $\mathcal{C}$ collects all claims problems that involve n individuals. A rule is a map from the set $\mathcal{C}$ to the set $\mathbb{R}^n_+$ of nonnegative n-tuples, i.e.

$$R : \mathcal{C} \longrightarrow \mathbb{R}^n_+ : (c, E) \longmapsto R(c, E),$$

that satisfies the conditions $R_1(c, E)+R_2(c, E)+\cdots+R_n(c, E) = E$ and $c \geq R(c, E) \geq 0$.[4] The division $R(c, E)$ is said to be an awards vector for (c, E). Sometimes, we use R_i as shorthand for $R_i(c, E)$. The difference $c_i - R_i(c, E)$ is said to be the loss for claimant i. We only consider anonymous rules, i.e. rules for which the identity of the claimants does not matter. Accordingly, we limit our attention to claims vectors c with $0 \leq c_1 \leq c_2 \leq \cdots \leq c_n$.

2.1 Nine rules

We list nine rules, starting with the four classical ones.[5] The most commonly used rule in practice makes awards proportional to claims.

Proportional rule, P**.** For each (c, E) in $\mathcal{C}$, we have $P(c, E) = \frac{E}{C}c$.

The next two rules both implement the idea of equality, albeit in different ways. The constrained equal awards rule equalizes awards under the constraint that no individual's award exceeds her claim.

Constrained equal awards rule, *CEA***.** For each (c, E) in $\mathcal{C}$ and each i in N, we have $CEA_i(c, E) = \min\{c_i, \lambda\}$, where λ is chosen so that $\sum_{i=1}^n \min\{c_i, \lambda\} = E$.

The constrained equal losses rule equalizes losses under the constraint that no award is negative.

Constrained equal losses rule, *CEL***.** For each (c, E) in $\mathcal{C}$ and each i in N, we have $CEL_i(c, E) = \max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum_{i=1}^n \max\{0, c_i - \lambda\} = E$.

The fourth classical rule, known as the Talmud rule, specifies two regimes depending upon whether or not the amount to divide exceeds the half-sum of the claims. If the amount available is less than the half-sum, then the Talmud rule coincides with the constrained equal awards rule applied to the vector of half-claims. If the amount available is larger than the half-sum, then the Talmud rule gives each claimant her half-claim and applies

[3]This section is strongly based on Thomson's (2003) survey.
[4]Vector inequalities: $x \geq y$, $x > y$, and $x \gg y$.
[5]Herrero and Villar (2001) provide a comparative examination of these four rules.

the constrained equal losses rule to divide the remainder (where both the claims and the amount to divide are truncated).

Talmud rule, T**.** For each (c, E) in $\mathcal{C}$, we have that

(i) if $\frac{1}{2}C \geq E$, then $T(c, E) = CEA(\frac{1}{2}c, E)$,

(ii) if $\frac{1}{2}C \leq E$, then $T(c, E) = \frac{1}{2}c \ + \ CEL(\frac{1}{2}c, E - \frac{1}{2}C)$.

In the case where the amount to divide is equal to the half-sum of claims, the awards vector in both regimes coincides with the vector of half-claims. If $\frac{1}{2}C < E$, then a typical awards vector for the Talmud rule looks like $(\frac{1}{2}c_1\,, \frac{1}{2}c_2\,, \ldots,\ \frac{1}{2}c_k\,,\ c_{k+1}{-}\lambda,\ c_{k+2}{-}\lambda, \ldots,\ c_n{-}\lambda)$.

We continue with two rules that coincide with the Talmud rule whenever the amount available is less than the half-sum of the claims. To illuminate the difference with the Talmud rule, we present a typical awards vector for the case where the amount available exceeds the half-sum of claims.

Piniles' rule, Pin**.** For each (c, E) in $\mathcal{C}$, we have that

(i) if $\frac{1}{2}C \geq E$, then $Pin(c, E) = T(c, E) = CEA(\frac{1}{2}c, E)$,

(ii) if $\frac{1}{2}C \leq E$, then $Pin(c, E) = \frac{1}{2}c \ + \ CEA(\frac{1}{2}c, E - \frac{1}{2}C)$.

If $\frac{1}{2}C < E$, then $(c_1, c_2, \ldots, c_k, \frac{1}{2}c_{k+1} + \lambda, \frac{1}{2}c_{k+2} + \lambda, \ldots, \frac{1}{2}c_n + \lambda)$ is a typical awards vector for Piniles' rule.

Constrained egalitarian rule, CE**.** For each (c, E) in $\mathcal{C}$, we have that

(i) if $\frac{1}{2}C \geq E$, then $CE(c, E) = T(c, E) = CEA(\frac{1}{2}c, E)$,

(ii) if $\frac{1}{2}C \leq E$, then, for each i in N, we have $CE_i(c, E) = \max\{\frac{1}{2}c_i\,, \min\{c_i, \lambda\}\}$, where λ is chosen so that $\sum_{i=1}^{n} \max\{\frac{1}{2}c_i\,, \min\{c_i, \lambda\}\} = E$.

If $\frac{1}{2}C < E$, then $(c_1, c_2, \ldots, c_k, \lambda, \lambda, \ldots, \lambda, \frac{1}{2}c_\ell, \frac{1}{2}c_{\ell+1}, \ldots, \frac{1}{2}c_n)$ is a typical awards vector for the constrained egalitarian rule.

We close this list of rules with three rules that coincide with the Talmud rule in the case of two claimants. The adjusted proportional rule first allocates to each claimant her minimal right, i.e. the part of the amount to divide that is left after each other claimant is fully compensated (such a minimal right might be zero). Next, the claims and the amount to divide are revised and the resulting problem is solved using the proportional rule. Formally, for the claims problem (c, E), claimant i's minimal right $m_i(c, E)$ is defined as $\max\{0\,,\ E - C + c_i\}$ and $m(c, E) = (m_1(c, E), m_2(c, E), \ldots, m_n(c, E))$. Furthermore, the adjusted amount to divide $E - \sum_{i=1}^{n} m_i(c, E)$ is denoted by E_A.

Adjusted proportional rule, A**.** For each (c, E) in $\mathcal{C}$, we have

$$A(c, E) \ = \ m(c, E) + P\Big((\min\{c_i - m_i(c, E)\,,\ E_A\})_{i \in N}\ ,\ E_A\Big).$$

To define the next rule, suppose the individuals arrive one at a time and are fully compensated until the money runs out. By averaging the awards vectors obtained in this way over all possible orders of arrival, we get the division proposed by the random arrival rule. For a formal definition, let $\mathcal{P}$ collect the $n!$ different orderings in the set N. For each ordering π in $\mathcal{P}$ and for each individual i in N, the set $\pi[i]$ collects the predecessors of i with respect to the ordering π.[6]

Random arrival rule, RA**.** For each (c, E) in $\mathcal{C}$ and for each i in N, we have

$$RA_i(c,E) \;=\; \frac{1}{n!}\sum_{\pi\,\in\,\mathcal{P}} \min\Big\{c_i\,,\; \max\{0, E - \sum_{j\,\in\,\pi[i]} c_j\}\Big\}.$$

Finally, the minimal overlap regards each individual i as claiming the part $[0, c_i]$ of the interval $[0, E]$. The rule distinguishes two cases. (i) In the case where there exists a claim at least as great as the amount to divide, all claims are first truncated by the amount available. Next, each part of $[0, E]$ is divided equally among all individuals claiming it. For instance, the interval $[0, c_1]$ is claimed by everyone, and so everyone gets c_1/n. The interval $(c_1, c_2]$ is claimed by everyone except individual 1, and so each member of $N - \{1\}$ receives in addition $(c_2 - c_1)/(n-1)$. This process continues until the entire interval $[0, E]$ is covered. (ii) In the case where all claims are smaller than the amount to divide, one lets $c_0 = 0$ and looks for the largest k^* in $\{0, 1, 2, \ldots, n-2\}$ for which there exists a t in $\mathbb{R}_+$ that satisfies

$$c_{k^*} < t \le c_{k^*+1} \qquad \text{and} \qquad (c_{k^*+1} - t) + (c_{k^*+2} - t) + \cdots + (c_n - t) = E - t.^7 \tag{1}$$

Each individual i in the set $\{k^* + 1, k^* + 2, \ldots, n\}$ obtains a first share equal to $c_i - t$, i.e. the part of the interval $(t, E]$ that i alone claims. The remaining part $[0, t]$ is divided as in case (i) (with t as the amount to divide). For the definition, we follow Chun and Thomson (2005, p. 138).

Minimal overlap rule, MO**.** Let $c_0 = 0$. For each (c, E) in $\mathcal{C}$, we have the following.

(i) Let $c_{k^*} < E \le c_{k^*+1} \le c_n$ with k^* in $\{0, 1, 2, \ldots, n-1\}$. Then,

$$\begin{aligned} MO_i &= \tfrac{c_1}{n} + \tfrac{c_2 - c_1}{n-1} + \tfrac{c_3 - c_2}{n-2} + \cdots + \tfrac{c_i - c_{i-1}}{n-i+1} && \text{for each } i = 1, 2, \ldots, k^*, \\ MO_j &= MO_{k^*} + \tfrac{E - c_{k^*}}{n-k^*} && \text{for each } j = k^*+1, k^*+2, \ldots, n. \end{aligned}$$

(ii) Let $c_n < E$. Let $c_{k^*} < t \le c_{k^*+1}$ with k^* in $\{0, 1, 2, \ldots, n-2\}$ and t as in (1). Then,

$$\begin{aligned} MO_i &= \tfrac{c_1}{n} + \tfrac{c_2 - c_1}{n-1} + \tfrac{c_3 - c_2}{n-2} + \cdots + \tfrac{c_i - c_{i-1}}{n-i+1} && \text{for each } i = 1, 2, \ldots, k^*, \\ MO_j &= (c_j - t) + MO_{k^*} + \tfrac{t - c_{k^*}}{n-k^*} && \text{for each } j = k^*+1, k^*+2, \ldots, n. \end{aligned}$$

Individual k^* is said to be pivotal. Observe the particular position of claims problems (c, E) with pivotal individual k^* equal to 0: if (i) applies, then the minimal overlap proposes equal division, whereas if (ii) applies, then the minimal overlap rule selects the constrained equal losses division (each claimant loses $\frac{C-E}{n} \le \frac{n-1}{n} c_1$).

[6] For example, let $N = \{1, 2, 3\}$ and consider the ordering $\pi = (1, 3, 2)$ in which individual 1 queues first, followed by 3, and finally 2. In this case, $\pi[3] = \{1\}$.

[7] If the claims happen to be feasible, i.e. $c_1 + c_2 + \cdots + c_n = E$, then $k^* = 0$ and we allow $t = c_0 = 0$.

2.2 *Ten properties*

We consider ten properties. Table 3 indicates which of these properties are satisfied by each of the nine rules defined in the previous subsection.

Order preservation of awards requires that awards are ordered as claims are.

Order preservation of awards. For each (c, E) in $\mathcal{C}$, we have that if $c_i \leq c_j$, then $R_i(c, E) \leq R_j(c, E)$.

Order preservation of losses demands that losses are ordered as claims are.

Order preservation of losses. For each (c, E) in $\mathcal{C}$, we have that if $c_i \leq c_j$, then $c_i - R_i(c, E) \leq c_j - R_j(c, E)$.

Resource monotonicity holds that if the amount to divide increases, then each individual should receive at least as much as she did initially.

Resource monotonicity. For each pair (c, E) and (c, E') in $\mathcal{C}$, we have that if $E \leq E' \leq C$, then $R(c, E) \leq R(c, E')$.

The next two properties describe responses of the awards vector to changes in the amount available and in the claims vector, respectively. The first requires that if the amount to divide increases, of two individuals, the one with the greater claim benefits more than the other. This property is commonly known as super-modularity,[8] but we refer to it as "super-modularity in amount available" in order to distinguish it from an analogous property. This property—which we refer to as "super-modularity in claims"—requires that if the claim of individual k decreases, of two individuals in $N - \{k\}$, the one with the greater claim benefits more than the other.[9]

Super-modularity in amount available. For each pair (c, E) and (c, E') in $\mathcal{C}$ with $E \leq E' \leq C$, and for each pair i and j in N with $c_i \leq c_j$, we have $R_i(c, E') - R_i(c, E) \leq R_j(c, E') - R_j(c, E)$.

We write (c'_k, c_{-k}) for the claims vector obtained from c by replacing c_k with c'_k.

Super-modularity in claims. For each k in N, for each pair (c, E) and (c', E) in $\mathcal{C}$ with $c' = (c'_k, c_{-k})$ and $c'_k < c_k$, and for each pair i and j in $N - \{k\}$ with $c_i \leq c_j$, we have $R_i(c', E) - R_i(c, E) \leq R_j(c', E) - R_j(c, E)$.

We postpone the discussion of which rules satisfy super-modularity in claims to the end of this subsection.

The next property requires that truncating the claims at the level of the amount to divide has no impact on awards.

Invariance under claims truncation. For each claims problem (c, E) in $\mathcal{C}$, we have $R(c, E) = R((\min\{c_i, E\})_{i \in N}, E)$.

A self-dual rule treats the problem of dividing the amount available and the problem of dividing the shortfall (i.e. the difference between the total claim and the amount to divide) in a symmetrical way.

Self-duality. For each (c, E) in $\mathcal{C}$, we have $R(c, E) = c - R(c, C - E)$.

[8] See Thomson (2003, p. 270).

[9] The property appears in Thomson (2006, p. 106) with the label "order preservation under claims variations."

The adjusted proportional rule inherits self-duality from the proportional rule (Thomson and Yeh, 2006, Corollary 1).

The midpoint property requires the awards vector to coincide with the vector of half-claims whenever the amount to divide coincides with the half-sum of claims.

Midpoint property. For each (c, E) in $\mathcal{C}$ with $E = \frac{1}{2}C$, we have $R(c, E) = \frac{1}{2}c$.

Self-duality implies the midpoint property: indeed, if $R(c, E) + R(c, C - E) = c$, then, for a claims problem (c, E) with $C = 2E$, we have $2R(c, E) = c$.

Limited consistency states that adding an individual with a zero claim does not change the awards of the claimants initially present. We abuse notation and use R to denote both the n-claimants and the $(n+1)$-claimants version of a rule. Obviously, if $(c_1, c_2, \ldots, c_n, E)$ is a claims problem involving n claimants, then $(0, c_1, c_2, \ldots, c_n, E)$ is a claims problem with $n + 1$ claimants.

Limited consistency. For each $(c, E) = (c_1, c_2, \ldots, c_n, E)$ in $\mathcal{C}$ involving n individuals, we have $R(0, c_1, c_2, \ldots, c_n, E) = (0, R(c_1, c_2, \ldots, c_n, E))$.

Finally, reasonable lower bounds on awards ensures that each claimant receives at least the minimum of (i) her claim divided by the number of claimants and (ii) the amount available divided by the number of claimants (see Moreno-Ternero and Villar, 2004, and Dominguez and Thomson, 2006).

Reasonable lower bounds on awards. For each (c, E) in $\mathcal{C}$ and for each i in N, we have $R_i(c, E) \geq \frac{1}{n^*} \min\{c_i, E\}$, where n^* denotes the number of individuals with a positive claim.

We now turn to the question of which rules satisfy super-modularity in claims. First, it can easily be established that each consistent[10] rule satisfies super-modularity in claims if and only if it satisfies super-modularity in amount available. The proportional, constrained equal awards, constrained equal losses, Talmud, and Piniles' rules all satisfy consistency and super-modularity in amount available, and therefore satisfy super-modularity in claims. The constrained egalitarian rule, on the other hand, is consistent but does not satisfy super-modularity in amount available. Hence, it does not satisfy super-modularity in claims.

Now consider the adjusted proportional rule. The larger the claim is, the larger the minimal right. If one of the claims decreases, then the minimal rights of the other claimants increase in an order preserving manner (the minimal right either stays at the zero level, or becomes positive, or increases with the amount the particular claim decreases). The proportional rule is applied to the adjusted claims problem, and its proposal is added to the minimal rights vector. Since, moreover, the proportional rule satisfies super-modularity in claims, the adjusted proportional rule also satisfies the property.

Next, we check whether the random arrival rule satisfies super-modularity in claims. Let (c, E) be a claims problem, let c_k decrease to c'_k, and let i and j be two claimants different from k such that $c_i \leq c_j$. Consider an order of arrival π in which the decrease in c_k generates an increase in claimant i's award (i.e. $k \in \pi[i]$ and $0 < E - \sum_{\ell \in \pi[i]} c_\ell < c_i$). Switch the positions of i and j in π and obtain the order π'. Then, $k \in \pi'[j]$ and $0 < E - \sum_{\ell \in \pi'[j]} c_\ell < c_j$. With respect to the order π', claimant j experiences at least the same increase as i does with respect to π. By consequence, the random arrival rule satisfies super-modularity in claims.

[10]See Thomson (2003, p. 279) for a definition of the consistency property.

Finally, the minimal overlap rule also satisfies super-modularity in claims, as will become clear in the proof of Proposition 5 in Section 3.

Table 3. Rules and properties (for at least three individuals)

	P	CEA	CEL	T	Pin	CE	A	RA	MO
Order preservation of awards	yes	yes	yes	yes	yes	yes	yes	yes	yes
Order preservation of losses	yes	yes	yes	yes	yes	yes	yes	yes	yes
Resource monotonicity	yes	yes	yes	yes	yes	yes	yes	yes	yes
Super-modularity in amount available	yes	yes	yes	yes	yes	*NO*	yes	yes	yes
Super-modularity in claims	yes	yes	yes	yes	yes	*NO*	yes	yes	yes
Invariance under claims truncation	*NO*	yes	*NO*	yes	yes	yes	yes	yes	yes
Self-duality	yes	*NO*	*NO*	yes	*NO*	*NO*	yes	yes	*NO*
Midpoint property	yes	*NO*	*NO*	yes	yes	yes	yes	yes	*NO*
Limited consistency	yes	yes	yes	yes	yes	yes	yes	yes	yes
Reasonable lower bounds on awards	*NO*	yes	*NO*	yes	yes	yes	yes	yes	yes

3 Lorenz comparisons on the full domain

In this section we prove Theorem 1. First, we define the Lorenz dominance relation. Let $\mathbb{R}^n_{\leq}$ be the set of nonnegative n-tuples $x = (x_1, x_2, \ldots, x_n)$ ordered from small to large, i.e. $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. Let x and y be in $\mathbb{R}^n_{\leq}$. We say that x Lorenz dominates y if we have

$$x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k \qquad \text{for each } k = 1, 2, \ldots, n-1,$$

and $x_1+x_2+\cdots+x_n = y_1+y_2+\cdots+y_n$. If x Lorenz dominates y and $x \neq y$, then at least one of these $n-1$ inequalities is a strict inequality. We consider only Lorenz comparisons of vectors that are ordered from small to large as this seems natural in the given context.

Definition. Let R and R' be two rules that satisfy order preservation of awards and let $\mathcal{D} \subseteq \mathcal{C}$ be a set of claims problems. Then, R Lorenz dominates R' on the domain $\mathcal{D}$ if $R(c, E)$ Lorenz dominates $R'(c, E)$ for each (c, E) in $\mathcal{D}$.

We shorten "Lorenz dominance on the domain $\mathcal{C}$" to "Lorenz dominance." The transitivity and reflexivity of the Lorenz dominance relation in the set $\mathbb{R}^n_{\leq}$ implies the transitivity and reflexivity of the Lorenz dominance relation in the set of rules.

According to the next lemma, the duality operator reverses the Lorenz dominance relation. The dual rule R^d of R treats what is available for division in the same way as R treats what is missing. Formally, for each (c, E) in $\mathcal{C}$, we have $R^d(c, E) = c - R(c, C - E)$. The claims problems $(c, C - E)$ and (c, E) are said to be dual.

Lemma 1. *Let R and S be two rules that satisfy order preservation of awards and let $\mathcal{D} \subseteq \mathcal{C}$ be a set of claims problems. Then, R Lorenz dominates S on the domain $\mathcal{D}$ if and only if S^d Lorenz dominates R^d on the domain $\mathcal{D}^d$ of dual claims problems.*

Proof. Let (c, E) be a claims problem in $\mathcal{D}$. Then, $(c, C - E)$ belongs to $\mathcal{D}^d$. Duality implies

$$R(c, E) + R^d(c, C - E) = c \qquad \text{and} \qquad S(c, E) + S^d(c, C - E) = c.$$

Conclude that $R(c,E)$ Lorenz dominates $S(c,E)$ if and only if $S^d(c,C-E)$ Lorenz dominates $R^d(c,C-E)$. $\square$

The rest of this section proves the Lorenz relationships indicated by the arrows in Theorem 1 (see Section 1): we proceed from left to right and from top to bottom. Incomparabilities are postponed until Section 4. A Lorenz-based characterization of a rule is stated as a proposition.

3.a. *CEA* $\rightarrow$ *CE***.** This Lorenz comparison follows from the fact that the constrained equal awards rule is Lorenz-maximal in the set of rules that preserve the order of awards.

Proposition 1.[11] *Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards. The constrained equal awards rule is the only rule in $\mathcal{R}$ that Lorenz dominates each rule in $\mathcal{R}$.*

Proof. It suffices to show that *CEA* Lorenz dominates each rule in $\mathcal{R}$. This is done by contradiction. Let R in $\mathcal{R}$ and (c,E) in $\mathcal{C}$ be such that $x = CEA(c,E)$ does not Lorenz dominate $y = R(c,E)$. Let k in N be the smallest number such that $x_1 + x_2 + \cdots + x_k < y_1 + y_2 + \cdots + y_k$.

Hence, $x_k < y_k \leq c_k$. Therefore, $x_k = \lambda$ (according to *CEA* the individuals that receive less than λ are fully compensated). As $\lambda < y_k$ and $y_k \leq y_{k+1} \leq \cdots \leq y_n$ (the rule R preserves the order of awards), the allocation y is not feasible. $\square$

3.b. *CE* $\rightarrow$ *Pin***.** This is a consequence of the next proposition. Recall that the constrained egalitarian and Piniles' rules satisfy resource monotonicity and the midpoint property.

Proposition 2.[12] *Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards, the midpoint property, and resource monotonicity. The constrained egalitarian rule is the only rule in $\mathcal{R}$ that Lorenz dominates each rule in $\mathcal{R}$.*

Proof. It suffices to show that *CE* Lorenz dominates each rule in $\mathcal{R}$. This is done by contradiction. Let R in $\mathcal{R}$ and (c,E) in $\mathcal{C}$ be such that $x = CE(c,E)$ does not Lorenz dominate $y = R(c,E)$. Let k in N be the smallest number such that $x_1 + x_2 + \cdots + x_k < y_1 + y_2 + \cdots + y_k$.

Hence, $x_k < y_k$ and $x_\ell > y_\ell$ for some $\ell > k$. The inequalities $x_k < y_k \leq c_k$ imply that either $x_k = \frac{1}{2}c_k$ or $x_k = \lambda < c_k$. In addition, $\ell > k$ implies $x_\ell < c_\ell$ (*CE* fully compensates only—if any—the smaller claims). We distinguish $\frac{1}{2}C > E$ from $\frac{1}{2}C < E$. The midpoint property tackles the case $\frac{1}{2}C = E$.

Case 1, $\frac{1}{2}C > E$ and $x_k = \frac{1}{2}c_k$. Then, $y_k > x_k$ cannot hold because of resource monotonicity and the midpoint property. Indeed, if E increases towards $\frac{1}{2}C$, then y_k should increase towards $\frac{1}{2}c_k$, and a contradiction follows.

Case 2, $\frac{1}{2}C > E$ and $x_k = \lambda$. Then, the inequalities $\lambda < y_k \leq y_{k+1} \leq \cdots \leq y_n$ (R preserves the order of awards) make the vector y infeasible.

Case 3, $\frac{1}{2}C < E$ and $y_\ell < x_\ell = \frac{1}{2}c_\ell$. Again, a contradiction follows: if E decreases towards $\frac{1}{2}C$, then y_ℓ should decrease towards $\frac{1}{2}c_\ell$.

Case 4, $\frac{1}{2}C < E$ and $y_\ell < x_\ell = \lambda$. We obtain the configuration $x_k = x_{k+1} = \cdots = x_\ell = \lambda$. Then, the rule R does not preserve the order of awards: $y_k > \lambda$, while $y_\ell < \lambda$. $\square$

[11] See Schummer and Thomson (1997, Propositions 3 and 4) for related results.

[12] See Chun, Schummer, and Thomson (2001, Theorem 3) for a different proof.

3.c. $Pin \to \{RA, A, T, P\}$**.** In the case where the half-sum of claims is larger than the amount to divide, Piniles' rule coincides with the constrained egalitarian and Talmud rules. Proposition 2 implies that *CE* Lorenz dominates *Pin*, *RA*, *A*, *T*, and *P*. Hence, on the domain $\mathcal{C}_1$ of claims problems (c, E) with $\frac{1}{2}C \geq E$, the rule $Pin = T = CE$ Lorenz dominates *RA*, *A*, and *P*.

Now, we focus on problems with a half-sum of claims less than the amount to divide. Let $\mathcal{C}_0$ collect the problems (c, E) for which $\frac{1}{2}C \leq E$. Since the rules *Pin*, *RA*, *A*, *T*, and *P* all satisfy the properties required, the next lemma simultaneously tackles the relationships on the restricted domain $\mathcal{C}_0$.

Lemma 2. *Restrict the domain to $\mathcal{C}_0$. Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards, the midpoint property, and super-modularity in amount available. Piniles' rule ($Pin|_{\mathcal{C}_0}$) is the only rule in $\mathcal{R}$ that Lorenz dominates each rule in $\mathcal{R}$ on the domain $\mathcal{C}_0$.*

Proof. It suffices to show that *Pin* Lorenz dominates each rule in $\mathcal{R}$. This is done by contradiction. Let R in $\mathcal{R}$ and (c, E) in $\mathcal{C}_0$ be such that $x = Pin(c, E)$ does not Lorenz dominate $y = R(c, E)$. Let k in N be the smallest number such that $x_1 + x_2 + \cdots + x_k < y_1 + y_2 + \cdots + y_k$.

Hence, $x_k < y_k \leq c_k$ and $y_\ell < x_\ell \leq c_\ell$ for some $\ell > k$. The definition of *Pin* implies that $x_k = \frac{c_k}{2} + \lambda$ and $x_\ell = \frac{c_\ell}{2} + \lambda$. As the rule R satisfies the midpoint property, we have $R_k(c, E) - R_k(c, C/2) = y_k - \frac{c_k}{2} > \lambda$, while $R_\ell(c, E) - R_\ell(c, C/2) = y_\ell - \frac{c_k}{2} < \lambda$. This contradicts the fact that R satisfies super-modularity in amount available (recall that $c_k \leq c_\ell$). $\square$

The combination of Lemma 2 and Proposition 2 entails a Lorenz-based characterization of Piniles' rule.

Proposition 3. *Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards, resource monotonicity, the midpoint property, and super-modularity in amount available. Piniles' rule is the only rule in $\mathcal{R}$ that Lorenz dominates each rule in $\mathcal{R}$.*

3.d. $\{RA, A, T\} \to MO$**.** If there are only two claimants, then the random arrival, adjusted proportional, Talmud, and minimal overlap rules coincide (with the concede-and-divide rule; see Thomson, 2003). The next proposition provides a Lorenz-based characterization of the minimal overlap rule.

Proposition 4. *Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards, order preservation of losses, super-modularity in claims, limited consistency, and reasonable lower bounds on awards. The minimal overlap rule is the only rule in $\mathcal{R}$ that is Lorenz dominated by each rule in $\mathcal{R}$.*

Proof. By induction on the number of claimants.

Proof for $n = 2$. Let (c, E) be a claims problem with two individuals. If $k^* = 0$, then either (i) $E \leq c_1$ and $MO(c, E) = (E/n, E/n)$, or (ii) $MO(c, E) = CEL(c, E)$. Since each rule R in $\mathcal{R}$ satisfies (i) reasonable lower bounds, and (ii) order preservation of losses, the awards vector $R(c, E)$ Lorenz dominates $MO(c, E)$.

If $k^* = 1$, i.e. $c_1 < E \leq c_2$, then the minimal overlap rule proposes $x = (c_1/2, E - c_1/2)$. Each rule that satisfies reasonable lower bounds on awards proposes a division either equal to or Lorenz dominating x.

Inductive step. Suppose that the proposition holds for claims problems with at most $n-1$ claimants. We have to show that the proposition holds for claims problems with n claimants. Consider a rule R—defined for each number of claimants—in $\mathcal{R}$.

Let (c, E) be a claims problem with n individuals and let k^* be its pivotal claimant. If $k^* = 0$, then either (i) $E \leq c_1$ and $MO(c, E) = (E/n, E/n, \ldots, E/n)$, or (ii) $MO(c, E) = CEL(c, E)$. Since each rule R in $\mathcal{R}$ satisfies (i) reasonable lower bounds, and (ii) order preservation of losses, the awards vector $R(c, E)$ Lorenz dominates $MO(c, E)$ (for (ii), see Proposition 5 below).

If $k^* > 0$, then $c_2 + c_3 + \cdots + c_n > E$ and $(0, c_2, c_3, \ldots, c_n, E)$ is a claims problem. In addition, the pivotal claimant for the problem $(0, c_2, c_3, \ldots, c_n, E)$ coincides with k^*.

The inductive hypothesis implies that the $(n-1)$-tuple $R(c_2, c_3, \ldots, c_n, E)$ Lorenz dominates $MO(c_2, c_3, \ldots, c_n, E)$. Since R and MO satisfy limited consistency, we have

$$\begin{array}{ccc} (0, R(c_2, c_3, \ldots, c_n, E)) & \text{Lorenz dominates} & (0, MO(c_2, c_3, \ldots, c_n, E)), \\ \| & & \| \\ R(0, c_2, c_3, \ldots, c_n, E) & \text{Lorenz dominates} & MO(0, c_2, c_3, \ldots, c_n, E). \end{array}$$

Start from $(0, c_2, c_3, \ldots, c_n, E)$ and let the claim of individual 1 move up from 0 to c_1. The minimal overlap rule transfers an amount $c_1/[n(n-1)]$ from each claimant $i = 2, 3, \ldots, n$ towards claimant 1, who obtains an award equal to c_1/n. On the other hand, the rule R allocates at least c_1/n to claimant 1. Furthermore, the rule R satisfies super-modularity in claims. Hence, when the claim of individual 1 moves from 0 to c_1, the decrease in the award R_i of individual $i \neq 1$ is increasing in i. Conclude that $R(c_1, c_2, \ldots, c_n, E)$ Lorenz dominates $MO(c_1, c_2, \ldots, c_n, E)$. □

3.e. $\{MO, P\} \rightarrow$ ***CEL*.** The constrained equal losses rule is the dual of the constrained equal awards rule (see, e.g., Herrero and Villar, 2001). Whereas the constrained equal awards rule is Lorenz-maximal, the constrained equal losses rule is Lorenz-minimal.

Proposition 5. *Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards and order preservation of losses. The constrained equal losses rule is the only rule in $\mathcal{R}$ that is Lorenz dominated by each rule in $\mathcal{R}$.*

Proof. Note that if a rule satisfies order preservation of awards, then its dual satisfies order preservation of losses. Take the dual of Proposition 1 and apply Lemma 1. □

This completes the proof of Theorem 1, except for the incomparabilities.

4 Lorenz comparisons on restricted domains

Theorem 1 presents an incomplete ranking: the set $\{RA, A, T, P\}$ and the pair $\{P, MO\}$ are not ranked. In this section, we focus on both these sets of rules and study their relationships on the restricted domains $\mathcal{C}_{0.5}$ $(\frac{1}{2}C{=}E)$, $\mathcal{C}_0$ $(\frac{1}{2}C{\leq}E)$, and $\mathcal{C}_1$ $(\frac{1}{2}C{\geq}E)$. As already mentioned, the position of the amount to divide against the half-sum of claims is crucial. We formulate our second theorem.

Theorem 2. *On the restricted domains $\mathcal{C}_{0.5}$, $\mathcal{C}_0$, and $\mathcal{C}_1$, the Lorenz dominance relation ranks the nine rules as follows.*

$$\mathcal{C}_{0.5}\ (\tfrac{1}{2}C{=}E): \quad \mathit{CEA} \longrightarrow \mathit{CE} = \mathit{Pin} = \mathit{RA} = A = T = P \longrightarrow \mathit{MO} \longrightarrow \mathit{CEL},$$

$$\mathcal{C}_0\ (\tfrac{1}{2}C{\le}E): \quad \begin{tikzcd} & & & P \arrow[r] & A \arrow[dr] & & & \\ \mathit{CEA} \arrow[r] & \mathit{CE} \arrow[r] & \mathit{Pin} \arrow[ur] \arrow[dr] & & & T \arrow[r] & \mathit{MO} \arrow[r] & \mathit{CEL}, \\ & & & \mathit{RA} \arrow[urr] & & & & \end{tikzcd}$$

$$\mathcal{C}_1\ (\tfrac{1}{2}C{\ge}E): \quad \begin{tikzcd} & & A \arrow[r] \arrow[dr] & P \arrow[dr] & \\ \mathit{CEA} \arrow[r] & \mathit{CE} = \mathit{Pin} = T \arrow[ur] \arrow[dr] & & & \mathit{CEL}. \\ & & \mathit{RA} \arrow[r] & \mathit{MO} \arrow[ur] & \end{tikzcd}$$

An arrow (or a sequence of arrows) from rule R to R' indicates that R Lorenz dominates R' on the relevant restricted domain. The absence of an arrow (or of a sequence of arrows) indicates the absence of a Lorenz relationship.

We subsequently discuss the domains $\mathcal{C}_{0.5}$, $\mathcal{C}_0$, and $\mathcal{C}_1$, and prove the relationships not covered by Theorem 1. We distinguish the Lorenz relations by denoting the domain in stack position, e.g. $P \xrightarrow{\mathcal{C}_0} A$.

4.a. *The domain $\mathcal{C}_{0.5}$***.** The arrows involving the constrained equal awards, minimal overlap, and constrained equal losses rules are implied by Theorem 1. The equalities $\mathit{CE} = \mathit{Pin} = \mathit{RA} = A = T = P$ follow from the midpoint property. $\square$

We now consider the domains $\mathcal{C}_0$ $(\frac{1}{2}C{\le}E)$ and $\mathcal{C}_1$ $(\frac{1}{2}C{\ge}E)$. Observe that several rankings reverse over these domains: we have $P \xrightarrow{\mathcal{C}_0} A \xrightarrow{\mathcal{C}_0} T$ and $T \xrightarrow{\mathcal{C}_1} A \xrightarrow{\mathcal{C}_1} P$, as well as $\mathit{RA} \xrightarrow{\mathcal{C}_0} T$ and $T \xrightarrow{\mathcal{C}_1} \mathit{RA}$.[13] Since the domains $\mathcal{C}_0$ and $\mathcal{C}_1$ are dual and the rules P, A, T, and RA are self-dual, these reversals are a consequence of Lemma 1.

4.b. *$P \xrightarrow{\mathcal{C}_0} A$ and $A \xrightarrow{\mathcal{C}_1} P$***.** By Lemma 1 and self-duality of P and A, we need only consider the second statement. Let (c, E) be a claims problem with $\frac{1}{2}C \ge E$. If the minimal right m_i of individual i is positive, then either $E - C + c_i > 0$, or $E + c_i > C \ge 2E$, or $c_i > E$. By consequence, at most one individual has a positive minimal right. We distinguish two cases.

[13]The switch from $P \xrightarrow{\mathcal{C}_0} A$ to $A \xrightarrow{\mathcal{C}_1} P$ is especially noteworthy because the proportional and adjusted proportional rules—in contrast to, for instance, the Talmud rule—do not have different recipes for claims problems according to whether the half-sum of claims is larger or smaller than the amount available.

Case 1, $m_n(c,E) = 0$. Then, each individual has a minimal right equal to 0. The definition of A implies

$$A(c,E) = A(\bar{c},E) = P(\bar{c},E) \qquad \text{with } \bar{c} = (\min\{c_i, E\})_{i \in N}.$$

Only the larger claims are truncated. Therefore, the vector $P(\bar{c},E)$ Lorenz dominates $P(c,E)$.

Case 2, $m_n(c,E) > 0$. The minimal rights vector reads $m = (0,0,\ldots,0,E-C+c_n)$. The adjusted amount to divide E_A is equal to $E - m_n = C - c_n$. Also, $c_n - m_n > E - m_n$. Next, we determine $P(c_1, c_2, \ldots, c_{n-1}, C-c_n\,,\, C-c_n)$. As the claims add up to $2(C-c_n)$, we obtain $P_i(\bar{c},E) = \frac{1}{2}c_i$ for each i in N. By consequence,

$$A(c,E) = \Big(c_1/2,\, c_2/2,\, \ldots,\, c_{n-1}/2,\, E - (C - c_n)/2\Big).$$

On the other hand, $\frac{E}{C} \leq \frac{1}{2}$ implies $P(c,E) \leq \frac{1}{2}c$. The Lorenz dominance result follows. □

4.c. $\{A, RA\} \xrightarrow{\mathcal{C}_0} T$ *and* $T \xrightarrow{\mathcal{C}_1} \{A, RA\}$**.** The second statement already appeared in the previous section (see 3.c, first paragraph). Self-duality transforms the second statement into the first by Lemma 1. In addition, we provide the next Lorenz-based characterization of the Talmud rule.

Proposition 6. *Restrict the domain to $\mathcal{C}_0$. Let $\mathcal{R}$ be the set of rules that satisfy order preservation of awards, order preservation of losses, resource monotonicity, and the midpoint property. The Talmud rule ($T|_{\mathcal{C}_0}$) is the only rule in $\mathcal{R}$ that is Lorenz dominated by each rule in $\mathcal{R}$ on the domain $\mathcal{C}_0$.*

Proof. The midpoint property and resource monotonicity are self-dual properties: if a rule R satisfies these properties, then so does R^d (Thomson, 2003). Also, the domains $\mathcal{C}_0$ and $\mathcal{C}_1$ are dual. Restrict Proposition 2 to $\mathcal{C}_1$, take its dual, and apply Lemma 1. □

Table 4 presents an overview of the propositions.

Table 4. A summary of the Lorenz-based characterizations

Proposition	1	2	3	4	5	6
Lorenz-maximal	*CEA*	*CE* ($T\|_{\mathcal{C}_1}$)	*Pin*			
Order preservation of awards	•	•	•	•	•	•
Midpoint property		•	•			•
Resource monotonicity		•	•			•
Super-modularity in amount available			•			
Super-modularity in claims				•		
Reasonable lower bounds on awards				•		
Order preservation of losses				•	•	•
Lorenz-minimal				*MO*	*CEL*	$T\|_{\mathcal{C}_0}$

We continue by illustrating the incomparabilities. Each example involves a small number of claimants: to increase this number, just add individuals with zero claims and use limited consistency.

4.d. *RA and P are incomparable in $\mathcal{C}_0$ and in $\mathcal{C}_1$.* Consider $(c, E) = (20, 30,\ 10)$ and $(c', E') = (10, 30, 30,\ 30)$ in $\mathcal{C}_1$. We have that $RA(c, E) = (5, 5)$ Lorenz dominates $P(c, E) = (4, 6)$, while $P(c', E') = (4.3, 12.9, 12.9)$ Lorenz dominates $RA(c', E') = (3.3, 13.3, 13.3)$. The dual problems $(c, E)^* = (20, 30,\ 40)$ and $(c', E')^* = (10, 30, 30,\ 40)$ tackle the incomparability on the domain $\mathcal{C}_1$. □

4.e. *RA and A are incomparable in $\mathcal{C}_0$ and in $\mathcal{C}_1$.* Consider $(c, E) = (10, 10, 10, 20,\ 20)$ and $(c', E') = (10, 20, 30,\ 20)$ in $\mathcal{C}_1$. We have that $RA(c, E) = (4.2, 4.2, 4.2, 7.5)$ Lorenz dominates $A(c, E) = (4, 4, 4, 8)$, while $A(c', E') = (4, 8, 8)$ Lorenz dominates $RA(c', E') = (3.3, 8.3, 8.3)$. The dual problems $(c, E)^* = (10, 10, 10, 20,\ 30)$ and $(c', E')^* = (10, 20, 30,\ 40)$ tackle the incomparability on the domain $\mathcal{C}_0$. □

4.f. *MO and P are incomparable in $\mathcal{C}_1$.* Consider the problems $(c, E) = (20, 30,\ 10)$ and $(c', E') = (10, 10, 10, 40,\ 20)$ in $\mathcal{C}_1$. We have that $MO(c, E) = (5, 5)$ Lorenz dominates $P(c, E) = (4, 6)$, while $P(c', E') = (2.9, 2.9, 2.9, 11.4)$ Lorenz dominates $MO(c', E') = (2.5, 2.5, 2.5, 12.5)$. □

This completes the proofs of Theorem 2 and of Theorem 1. The next corollary closes the paper. The Lorenz ranking for each of the pairs $\{T, A\}$ and $\{T, RA\}$ reverses in moving from $\mathcal{C}_1$ to $\mathcal{C}_0$. Furthermore, the Talmud, adjusted proportional, and random arrival rules all satisfy invariance under claims truncation. By consequence, for claims problems that shift from $\mathcal{C}_1$ to $\mathcal{C}_0$ if claims are truncated, the divisions proposed by these three rules should coincide. A claims problem (c, E) with $\frac{1}{2}C \geq E$ generates a truncated claims problem $(\bar{c}, E)$ with $\frac{1}{2}\bar{C} \leq E$ if and only if $c_1 + c_2 + \cdots + c_{n-1} \leq E \leq c_n$.

Corollary. *Let (c, E) be a claims problem with $C - c_n \leq E \leq c_n$. The Talmud, adjusted proportional, and random arrival rules propose $(\frac{1}{2}c_1,\ \frac{1}{2}c_2,\ \ldots,\ \frac{1}{2}c_{n-1},\ E - \frac{1}{2}(C - c_n))$.*

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