Legislative Lobbying under Political Uncertainty*

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Abstract

In this paper we develop a duopolistic model of legislative lobbying. Two lobbies compete to influence the votes of a group of legislators who have a concern for both social welfare and campaign contributions. The type of a legislator is the relative weight he/she places on social welfare as compared to money. We study the equilibria of this lobbying game under political certainty and uncertainty and examine the circumstances under which the policy is socially efficient, and the amount of money that has been invested in the political process. Special attention is paid to three primitives of the environment: the intensity of the competition between the lobbies, the internal organisation of the legislature and the proportion of bad and good legislators in the political arena.

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1 Introduction

In all real polities *special interest groups* or *lobbies*[1] participate actively in the policy-making process. Researchers have developed an analytical apparatus aiming to provide a description of the channels through which the influence of these interest groups is exerted and a characterization of the main features of the equilibrium policies when this influence is accounted for. A common denominator of the research done on that topic in the last decade[2] has been to study structural models of the political process: economic and political actors behave rationally within well-specified economic and political institutions, where the policy-making process is formulated as an extensive form game. Methodologically, much progress has been made relatively to the traditional approaches which were often based on inconsistent or irrational political and economic behavior, relying on non-derived influence functions, political support functions, or vote functions. While this new literature does not point out a single canonical model that would impose itself against its competitors, it is fair to say that the description of the competitive process among special interest groups as a *common agency game* (Bernheim and Whinston (1986), Laussel and Le Breton (2001)) has become a contender[3]. In this formulation, the principals are the lobbyists and the common agent is an incumbent politician depicted as having the power to select unilaterally the economic policy. The lobbyists move first: they, simultaneously or sequentially, offer a menu of monetary payments conditional on the policy that will be ultimately selected. After contemplating the profile of offers, the politician decides which policy to select.

Empirical evidences are quite controversial, since documenting that money affects policy outcomes is not an easy task. Indeed, as formulated by Grossman and Helpman (2001) "After all, it is difficult to know what a bill would have looked like absent the net effect of contributions. Even if we focus on roll-call votes, as many researchers have done, the effort is confounded by the counterfactual: how would a legislator have voted absent the contributions? Perhaps a representative's vote on a bill was dictated by a concern for jobs in his district, which happens to be associated with the economic health of a contributor, such as a large corporation. Or simply, the legislator was following the directives of party leaders".

[1]It is not an easy task to define what is a special interest group (see Grossman and Helpman (2001) for discussion on the matter). Here we use interchangeably the terms special interest groups and lobbies meaning that we ignore all the potential difficulties a group may face to get some identity, and that gives rise to political organization/representation which is efficient. Not all groups are equal in that respect as suggested and investigated by Olson (1965). In this paper we skip this important aspect of the lobbying process known as the Olsonian program to focus on some other dimensions.

[2]See, for instance, Grossman and Helpman (2001) and Persson (1998).

[3]The common agency framework has been pionnered by Grossman and Helpman (1994, 2001) and followers to study trade policy, commodity taxation and other policies.

To control for these different effects it is necessary to introduce variables; for instance the legislator's ideological stance is reflected by his ratings with political organizations. Baldwin and Magee (2000) find that the probability of a vote in favor of trade liberalization on the NAFTA and GATT Uruguay Round bills increases with the amount of contributions that a legislator receives from business interests and falls with the amount collected from labor unions. Stratmann (2003) studies the congressional votes on financial services legislation and concludes that contributions change voting behavior. These papers are just two examples of an entire genre of research[4], and several other authors have reached different if not opposite conclusions. Analyzing Tullock's (1972) puzzle about the small amount of money invested in U.S. politics, Ansolobehere, De Figueiredo and Snyder (2003) conclude that there is no econometric evidence that contributions have substantial effects on votes and legislative decisions and suggest an alternative explanation.

We depart from this literature in abandoning the assumption that policies are set by a single individual or by a cohesive, well-disciplined political party. In reality most policy decisions are made not by one person but by a group of elected representatives acting as a legislative body. Even when the *legislature*[5] is controlled by a single party (as it is necessarily the case in a two-party system if the legislature consists of a unique chamber[6]), the delegation members do not always follow the instructions of their party leaders. In situations with multiple independent legislators, special interest groups face a subtle problem in deciding how to allocate their resources to influence policy choices. For instance, should the lobby seek to solidify support among those legislators who would be inclined to support its positions anyway, or should it seek to win over those who might otherwise be hostile to its views? The answer to this question depends on the rules of the legislative process, i.e. the optimal strategy for wielding influence will vary with the institutional setting.

Many formal models of the *legislative process* have been developed by social scientists. The extensive game form describes the sequence of decision/information nodes of the legislators where a decision node typically consists in either the proposal of an alternative (there, the legislator acts as an agenda setter) or expressing an opinion on a proposal (there, the legislator acts as a voter). Some policy is attached to each terminal node, and the model of

[4]Smith (1995) cites more than 35 studies published between 1980 and 1992 that attempted to explain roll-call votes in the U.S. Congress by campaign contributions from interested parties and by various indicators of a legislator's ideology.

[5]Like Diermeier and Myerson (1999), by legislators we mean here all individuals who have a constitutional role in the process of passing legislation. This may include individuals from what is usually refered to as being the executive branch, for instance the president or the vice-president.

[6]If instead, the legislature gives some power to actors from the "executive" branch then, this assertion does not necessarily holds true in case of divided government.

the legislative process is likely to depend upon the type of policy space under consideration. A classical model in that vein is the bargaining model of Baron and Ferejohn (1989) that describes the rules of the legislature to divide a fixed budget among the legislators. This legislative model has been paired with lobbying by Helpman and Persson (2001). Another very nice model of this kind, constructed by Grossman and Helpman (2001), applies to any finite set of policies with one policy playing the role of the status quo. In this model, one legislator decides unilaterally upon an alternative (bill, amendment, motion, reform,...) that will confront the status quo through a binary majority vote. Lobbies have an opportunity to influence legislators on two occasions: first, they will try to exert influence on the agenda setter and second, they will also try to buy votes. In this paper, we focus on the binary setting, i.e. we assume that the policy space consists of two alternatives: the status quo (alternative 0) versus the change or reform (alternative 1). While simplistic, we think that many policy issues fit that formulation, for instance to ratify or not a free-trade agreement, to forbid or not a free market for guns, to allow abortion or not. In such cases, there is no room for agenda setting and the unique role of the legislature is to select one of the two options through voting. A legislature is then described by a *simple game* $(N, \mathcal{W})$ where N is the set of legislators (or parties, if there is some strong party discipline) and $\mathcal{W}$ is the list of winning coalitions: the reform is adopted if and only if the coalition of legislators voting for the reform belongs to that list.

The preference of lobby 0 (respectively lobby 1) is defined by the amount of dollars W_0 (respectively W_1) that would be lost (respectively gained) by its members if the reform was adopted. Following Grosssman and Helpman (1994), we assume that each legislator seeks to maximize a weighted sum of social welfare and monetary contributions. Therefore, in this setting, each legislator i is simply described by a single parameter α_i denoting the weight that he puts on social welfare[7]. This will be referred hereafter as being the type of the legislator. The lower the value of α_i is, the cheaper legislator i is, and therefore there is a sense in which we can qualify politicians with low α as "bad" or corrupted as they are more willing to depart from social welfare when deciding upon which policy to implement[8].

[7]The idea that α could be an adverse selection parameter is suggested in Grossman and Helpman (1992) and is the main motivation of Le Breton and Salanié (2003).

[8]Some empirical estimates of this parameter have been provided in the common agency setting. Interestingly, Golberg and Maggi (1999) find that the 1983 U.S. pattern of protection is consistent with the model of Grossman and Helpman and estimate the value of the parameter α to be between 50 and 88, a surprisingly high range of values. Gawande and Bandyopadhyay (2000) also conclude that the model of Grossman and Helpman is consistent with the data but estimate the value of α to be between 3 and 8. Bradford (2001) proceeds to an empirical investigation of a variant of a model of Grossman and Helpman where politicians maximize votes and finds that politicians weight a dollar of campaign contributions about 15% more than a dollar of national income. This would lead to a value of α very close to 1.

The main purpose of the paper is to proceed to an equilibrium analysis of the lobbying game where first the two lobbies make offers to the legislators who then vote in favor or against the reform. Several variants of this game are examined in turn. In the first part we assume that the types of the legislators are common knowledge, an environment that we call *political certainty* as all the relevant variables are known with certainty by all the players. Within that informational setting, we investigate alternatively two cases, depending on whether the two lobbies move simultaneously or sequentially. In the second part we assume instead that the types of the legislators are private information. We refer to this environment as *political uncertainty* as the lobbies when buying votes and the legislators when voting do not know with certainty the consequences of their choices. The exogenous ingredients of our strategic environment are:

· The economic stakes W_0 and W_1 whose respective magnitudes will define the intensity of the competition. We assume here that the reform is the socially efficient policy, i.e. $W_1 \geq W_0$, and the ratio $\frac{W_1}{W_0} \geq 1$ is called the *efficiency threshold*.

· The simple game $(N, \mathcal{W})$ which describes the legislative process.

· A probability distribution F over the positive real line which describes the respective frequencies of bad and good legislators.

We aim to examine the impact of each of these key parameters on the final equilibrium outcome of the political mechanism described by this influence game. The outcome has two dimensions:

· The policy which is ultimately selected by the legislators.

· The ex ante monetary offers of the lobbies and their ex post implementation.

In the first part of the paper, we assume political certainty. When the lobbies move simultaneously we demonstrate that the equilibrium policy is efficient but existence is obtained only under very stringent conditions on F or $(N, \mathcal{W})$. Further, the influence game collapses since at equilibrium there are no monetary transfers as soon as the legislative process is not oligarchic. When the lobbies move sequentially, things get more intricate. There is no guarantee that the equilibrium policy will be efficient. We demonstrate that it will be efficient if the efficiency threshold is larger than some critical real number which is a summary statistic of the legislative process. We also explore in details the lobbing strategy of lobby 1, and provide a full characterization of the strategy when the legislative process is the majority simple game with no ties. It includes a complete description of the conditions under which lobby 1 will target some specific coalition which could be a minority, a minimal winning coalition or a supermajority. For an abstract legislature where legislators (or their parties) may have different powers (as defined for instance in the literature on power measures in assemblies), we examine how the offer made to a legislator is related to his power. We exhibit a surprising connection[9] with the nucleolus of a cooperative game with transferable utility generated by $(N, \mathcal{W})$. We conclude this part with an investigation of the Nash equilibrium in mixed strategies for the majority game with three legislators.

In the second part of the paper, we move to the case of political uncertainty and limit our attention to the majority legislative process. The lobbying game is much more intricate as solving the continuation voting subgames gives rise to reduced payoff functions for the lobbies which may display some irregularities. We first examine the case where only lobby 0 is active and explore its optimal lobbying strategy. Once again the efficiency threshold plays a critical role in explaining the features of this strategy and the probability of getting the efficient policy selected. One surprising feature of the optimal offer is that the larger the stake W_0 of lobby 0, the smaller will be the coalition of legislators who receive an offer. In the last part, we return to the game and offer some preliminary insights into the best responses when there are three legislators.

Related Literature

Many of the questions examined in this paper have been investigated by other authors. Some general positive models aiming to describe the lobbying process of a legislature have been proposed by Bennedsen and Feldmann (2002), Boylan (2002), Dekel, Jackson and Wolinsky (2006 a, b), Helpman and Persson (2001), Polborn (2002) and Snyder (1991) among many others. Recent papers by Dal Bo (2002) and Felgenhauer and Gruner (2004) study the impact of external influence on a legislature or committee from a mechanism design angle. In particular, they compare open and closed voting and reach interesting conclusions. In contrast to this paper they model the committee choice issue as a problem with common values as in Condorcet juries.

The most related papers[10] are Banks (2000), Diermeier and Myerson (1999), Groseclose and Snyder (1996), Young (1978 a, b, c) and Shubik and Young (1978). They also consider a binary setting, but focus exclusively on the sequential version of the lobbying game under political uncertainty. Banks and Groseclose and Snyder look at the majority game with a heterogeneous legislature and show under which conditions a supermajority is optimal. Diermeier and Myerson consider the general case but with a homogeneous legislature and concentrate on the architecture of the legislative process that would maximize monetary

[9] Another appearance of the nucleolus in a non cooperative setting is Montero (2006) in a bargaining framework a la Baron-Ferejohn.

[10] Prat and Rustichini (2000) present a general abstract model that extends the common agency framework as it has many principals (lobbies here) and many agents (legislators here). However, it does not cover our setting as they assume that each agent (legislator here) cares only about his own action.

offers. Young (1978 a, b, c) and Shubik and Young (1978) were the first to point out the relevance of the least core and the nucleolus to predict some dimensions of the equilibrium strategies of the lobbyists.

To the best of our knowledge, nobody has yet investigated the case of political uncertainty. This model follows the common agency model with adverse selection of Le Breton and Salanié (2003), except that in contrast to them we ignore the free-riding dimension of the lobbying process and its impact on efficiency.

2 The Model

In this section, we describe the main ingredients of the problem as well as the lobbying game which constitutes our model of vote-buying by lobbyists.

The external forces that seek to influence the legislature are represented by two players, whom we call lobby 0 and lobby 1. Lobby 1 wants the legislature to pass a bill (change, proposal, reform) that would change some area of law[11]. Lobby 0 is opposed to the bill and wants to maintain the status quo. Lobby 0 is willing to spend up to W_0 dollars to prevent passage of the bill while lobby 1 is willing to pay up to W_1 dollars to pass the bill. Sometimes, we will refer to these two policies in competition as being policies 0 and 1. We assume that $\Delta W \equiv W_1 - W_0 > 0$, i.e. that policy 1 is the socially efficient policy. The ratio $\frac{W_1}{W_0}$ which is (by assumption) larger than 1 will be called the *efficiency threshold*. It measures the intensity of the superiority of the reform as compared to the status quo and will be used repeatedly in the analysis.

The legislature is described by a *simple game*[12], i.e. a pair $(N, \mathcal{W})$ where $N = \{1, 2, ..., n\}$ is the set of legislators and $\mathcal{W}$ is the set of *winning* coalitions. The interpretation is as follows. A bill is adopted if and only if the subset of legislators who voted for the bill forms a winning coalition. From that perspective, the set of winning coalitions describes the rules operating in the legislature to make decisions. A coalition C is *blocking* if $N \backslash C$ is not winning: some legislators (at least one) are needed to form a winning coalition. We will denote by $\mathcal{B}$ the subset of blocking coalitions[13]; from the definition, the status quo is maintained as soon as the set of legislators who voted against the bill forms a blocking coalition. The simple

[11] The framework also covers the case of private bills as defined and analysed by Boylan (2002).

[12] In social sciences it is sometimes called a committee or a voting game. In computer science, it is called a quorum system (Holzman, Marcus and Peleg (1997)) while in mathematics, it is called a hypergraph (Berge (1989), Bollobas(1986)). An excellent reference is Taylor and Zwicker (1999).

[13] In game theory, $(N, \mathcal{W})$ is called the dual game.

game is called *strong* if $\mathcal{B} = \mathcal{W}$[14]. The set of minimal (with respect to inclusion) winning (blocking) coalitions will be denoted $\mathcal{W}_m(\mathcal{B}_m)$. A legislator is a *dummy* if he is not a part of any minimal winning coalition, while a legislator is a *vetoer* if he belongs to all blocking coalitions. A group of legislators forms an *oligarchy* if a coalition is winning if and only if it contains that group i.e. each member of the oligarchy is a vetoer and the oligarchy does not need any extra support to win[15] i.e. legislators outside the oligarchy are dummies. When the oligarchy consists of a single legislator, the game is called dictatorial.

In this paper, all legislators are assumed to act on behalf of social welfare, i.e. they will all vote for policy 1 against policy 0 if no other event interferes with the voting process. In contrast to Banks (2000) and Groseclose and Snyder (1996) we rule out the existence of a horizontal heterogeneity across legislators. However, legislators also value money and we introduce instead some form of vertical heterogeneity. More precisely, we assume that legislators differ among themselves according to their willingness to depart from social welfare. The type of legislator i, denoted by α_i, is the minimal amount of dollars that he needs to receive in order to sacrifice one dollar of social welfare. Therefore, if the policy adopted generates a level of social welfare equal to W, the payoff of legislator i if he receives a transfer t_i is:

$$t_i + \alpha_i W.$$

To promote passage of the bill, lobby 1 can promise to pay money to individual legislators conditional on their support of the bill. Similarly, lobby 0 can promise to pay money to individual legislators conditional on their support of status quo. We denote by $t_{i0} \geq 0$ and $t_{i1} \geq 0$ the (conditional) offers made to legislator i by lobbies 0 and 1 respectively. The corresponding n-dimensional vectors will be denoted respectively by t_0 and t_1.

The timing of actions and events that we consider to describe the lobbying game is as follows[16].

1. Nature draws the type of each legislator.
2. Both lobby 0 and lobby 1 make contingent monetary offers to individual legislators.
3. Legislators vote.
4. Payments (if any) are implemented.

This game has $n + 2$ players. A strategy for a lobby is a vector in $\Re^n_+$. Each legislator can chose among two (pure) strategies: to oppose or to support the bill.

[14] When the simple game is strong, the two competing alternatives are treated equally.

[15] The five countries of the security council of the United Nations are vetoers but still do not form an oligarchy as they need some extra support to make a decision.

[16] Specific details and assumptions will be provided in due time.

The game is not fully described as we have not yet precisely defined the information held by the players when they act. In this paper we consider two distinct settings concerning the move of player nature, but otherwise we assume that the votes of the legislators are observable, i.e. we assume open voting[17]. The first setting to which we refer as *political certainty* corresponds to the case where the vector of legislators's types is common knowledge. This informational specification has two implications: first, the lobbies know the types of the legislators when making their offers and second, each legislator knows the type of any other legislator when voting. The second setting to which we refer as *political uncertainty* corresponds instead to the case where the type of a legislator is private information. In such a case, not only the lobbies ignore the types of the legislators but each potential continuation voting subgame is a Bayesian game. This means that there is an adverse selection feature in the strategic relationship between lobbies and legislators and a Bayesian feature in the strategic interaction among legislators.

To conclude, the description it remains to specify the details of the decision nodes. We assume that the legislators know the offers when they are asked to vote. For the lobbies, we alternate between two specifications. We either assume that they move simultaneously or instead, that lobby 0 moves second knowing the offers made by lobby 1.

We examine the *subgame perfect Nash equilibria*[18] of this lobbying game. In section 3 we investigate the case of political certainty. Then, in section 4 we move to the case of political uncertainty.

3 Political Certainty

In this section, we consider the case where the vector $(\alpha_1, \alpha_2, ..., \alpha_n)$ of legislators's types is common knowledge and, without loss of generality, we assume that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$.. We first examine the set of subgame perfect Nash equilibria in pure strategies when the two lobbies act simultaneaously, that we call compactly (with a slight abuse of the terminology) Nash equilibria. We show that they are efficient but exist only under very stringent conditions. Then, we explore the set of subgame perfect Nash equilibria in the case where the two lobbies move in sequence, that we call Stackelberg equilibria and show the critical role played by the efficiency threshold. These results are derived without putting too

[17]The comparative analysis of closed(secret) versus open voting is the subject of several contributions among which Dal Bo (2002) and Felgenhauer and Grüner (2004) are relevant.

[18]In the case of political uncertainty, the ultimate subgame is truly a Bayesian game that we solve using Bayesian-Nash equilibria. We don't use the term Bayesian subgame perfect Nash equilibrium as there is no updating operation of beliefs in our game.

much structure on the simple game. In our final part, we look specifically at the case of the majority game with three legislators and calculate the Nash equilibrium in mixed strategies.

3.1 Nash Equilibria in Pure Strategies

In this section we assume that the two lobbies choose their offers simultaneously and let $(t_0, t_1) \in \Re^n_+ \times \Re^n_+$ be a profile of lobbying strategies. In the continuation voting subgame, each legislator's behavior strongly depends on whether he is pivotal or not. Consider a legislator who expects to be non-pivotal, i.e. who expects that the outcome does not change no matter which policy he votes for. Then, such a legislator votes in favor of the policy preferred by the lobby that offers the largest monetary payment. If legislator i believes that he is not decisive, he votes for policy 1 if and only if

$$t_{i1} \geq t_{i0} \tag{1}$$

and for policy 0 otherwise.

If instead, legislator i thinks that he is pivotal, he votes for 1 if and only if

$$\alpha_i \Delta W + t_{i1} \leq t_{i0} \tag{2}$$

and for policy 0 otherwise. Clearly, a legislator with no offers from lobby 0 votes for policy1.

The first result asserts that under complete information only the efficient policy is chosen at equilibrium[19].

Proposition 1 *All Nash equilibria in pure strategies are Pareto efficient.*

Proof. Suppose on the contrary that there is a Nash equilibrium (t_0^*, t_1^*) for which policy 0 is chosen. Let N_0 be the coalition of voters supporting policy 0. Then, $N_0 \in \mathcal{B}$.

Case 1: $N_0 \in \mathcal{B}_m$. Then, each agent in this set is pivotal. Therefore, for any $i \in N_0$ (2) are satisfied. Any agent $i \in N_1$ is not pivotal and (1) should be satisfied. The net payoff of lobby 0 is $W_0 - \sum_{i \in N_0} t_{i0}^*$ while the net payoff of lobby 1 is $-\sum_{i \in N_1} t_{i1}^*$. Lobby 1 could pay 0 to all legislators in N_1 and gets 0 instead of a negative payoff. Therefore in such an equilibrium, $t_{i1}^* = 0$ for all $i \in N_1$, and from (1) $t_{i0}^* = 0$ for all $i \in N_1$.

[19] Some readers may be surprised by the fact that we do not need to refine the set of Nash equilibria to reach that conclusion. In the common agency setting, Bernheim and Whinston (1986) use the truthful refinement to rule our inefficient Nash equilibria. But such a refinement is not needed here as we have only two possible decisions.

Therefore, both lobbies 0 and 1 make offers only to the legislators in N_0.

Next, if the inequalities (2) are strict, for each i, $t^*_{i0} > 0$ since the left-hand side is non-negative. Lobby 0 could reduce his transfer slightly without changing the outcome. Thus, the equalities must hold. Summing up these equalities for all $i \in N_0$ we get

$$\sum_{i\in N_0} t^*_{i1} = \sum_{i\in N_0} t^*_{i0} - \Delta W \sum_{i\in N_0} \alpha_i \leq W_0 - \Delta W \sum_{i\in N_0} \alpha_i < W_1.$$

Since $\sum_{i\in N_0} t^*_{i1} < W_1$ and $t^*_{i0} = 0$ for all $i \in N_1$, lobby 1 could slightly increase its offers to all $i \in N_0$ and change the outcome from 0 to 1 in contradiction with our assumption.

Case 2: $N_0 \notin \mathcal{B}_m$. Since none of the legislators from N_0 is pivotal, the following holds true:

$$t^*_{i0} \geq t^*_{i1} \text{ for all } i \in N_0.$$

Then, the arguments used in case 1 apply. ■

The next proposition exhibits several necessary conditions on such equilibria. While stringent, these conditions cover the traditional common agency setting.

Proposition 2 *Let* (t^*_0, t^*_1) *be a Nash equilibrium.*
(i) If $\Delta W \sum_{i\in S} \alpha_n \geq W^0$ *for all* $S \in \mathcal{B}_m$*, then* $t^*_1 = 0$.
(ii) If $\widehat{S}$ *is an oligarchy and* $\Delta W \alpha_i < W_0$ *for all* $i \in \widehat{S}$*, then* $t^*_1 = \#SW_0 - \Delta W \sum_{i\in\widehat{S}} \alpha_i$

Proof. The proof of (i) follows immediately from the observation that lobby 0 gross benefit is not enough to compensate a minimal blocking coalition of legislators. the proof of (ii) is also very simple. By proposition 1 the equilibrium is efficient. This implies that each veto player and therefore each member i of the oligarchy $\widehat{S}$ must receive at least $W_0 - \Delta W \alpha_i$. There is no need to make an offer to any other player as they are dummies or to pay more to the vetoers, as it does not add anything. ■

Unfortunately, these results are mitigated by the fact that the lobbying game typically does not possess Nash equilibria in pure strategies. In the case where $\mathcal{W}$ is the majority game, the lobbying game has the structure of an asymmetric Colonel Blotto game (Gross and Wagner (1950), Laslier and Picard (2002)) for which it is well known that Nash equilibria in pure strategies do not exist as soon as the asymmetry is too small. In this literature, the two competitors are constrained by their budgets while here there are no such financial constraints. Note however that as long as we consider pure strategies, none of the lobby will spend more than its gross benefit and will spend the totality of this gross benefit if this can prevent the other lobby from winning. This equivalence does not hold in the case of mixed strategies. While discontinuous, this game admits equilibria in mixed strategies; some features of these equilibria are described in subsection 3.3.

If the asymmetry between the lobbyists is large enough, the existence of an equilibrium in pure strategies follows. It can immediately be seen that if $\Delta W \sum_{i \in S} \alpha_n \geq W^0$, $(t_0^*, t_1^*) = (0, 0)$ is a Nash equilibrium. The second part of proposition 2 generalizes Le Breton and Salanié (2003) who consider the common agency framework i.e. the case of a dictatorial simple game. In that case, a Nash equilibrium always exists as we have assumed $W_1 > W_0$: the unique dictator receives W_0 from lobby 1. When there are least two vetoers in the oligarchy, existence is not guaranteed. Consider the case where $N = \widehat{S} = \{1, 2\}$, $\alpha_1 = \alpha_2 = 0$ and $W_1 < 2W_0$. From proposition 2 (ii), we deduce that $t_1^* = 2W_0$ which is not an equilibrium behavior as $W_1 < 2W_0$.

In fact, this logical argument against the existence of a Nash equilibrium in pure strategies applies to any simple game which is not oligarchic. Let (t_0^*, t_1^*) be a Nash equilibrium. From proposition 1, the reform is selected. This implies that a winning coalition S of legislators votes for the reform. Therefore lobby 0 does not implement any monetary offer. This means that $t_{i0}^* - t_{i1}^* < \alpha_i \Delta W$ for all $i \in N$. We deduce that there is at most a minimal winning coalition $T \subseteq S$ such that $t_{i1}^* > 0$. Since for all $i \notin T$, $t_{i1}^* = 0$, we deduce that $t_{i0}^* = 0$ for all such i as none of these legislators is pivotal. Furthermore since the simple game is not oligarchic, there is a minimal winning coalition different from T. Let T' be any such coalition and $t_{i1}^{**} = 0$ for all $i \in (N \backslash T) \cap T'$ and must be minimal winning, as any additional offer would be useless but accepted. We deduce from that observation and the previous claim that $t_{i1}^{**} = t_{i1}^* - \varepsilon$ for some small enough for all $i \in T \cap T'$. From the construction, legislators in T' vote for the reform. Since the cost of t_1^{**} is smaller than the cost attached to the strategy t_1^*, we contradict our assumption that (t_0^*, t_1^*) is a Nash equilibrium.

3.2 Stackelberg Equilibrium

In this section, we examine the equilibrium path which arises when, instead of moving simultaneously, the two lobbyists move in sequence. Following Banks (2000), Diermeier and Myerson (1999) and Groseclose and Snyder (1996), we assume that lobby 1, the advocate for change, must make the first move and announce its offers first, and lobby1's offers are known to lobby 0 when lobby 0 makes its offers to induce legislators to oppose the bill. In what follows, we refer to this equilibrium as the Stackelberg equilibrium[20].

[20] In this game, there is a second mover advantage.

The subgame-perfect equilibrium of this sequential version of the lobbying game can be easily described. Let $t_1 = (t_{11}, t_{21}, ..., t_{n1}) \in \Re^n_+$ be lobby 1's offers. Lobby 0 will find it profitable to make a counter offer if there exists a blocking coalition S such that:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i W_1\right) < \sum_{i \in S} \alpha^i W_0 + W_0.$$

Indeed, in such case, there exists a vector $t_0 = (t_{10}, t_{20}, ..., t_{n0})$ of offers such that:

$$t_{i1} + \alpha^i W_1 < t_{i0} + \alpha^i W_0 \text{ for all } i \in S \text{ and } \sum_{i \in S} t_{i0} < W_0.$$

The first set of inequalities implies that legislators in S will vote against the bill while the last one simply says that the operation is beneficial from the perspective of lobby 0. Therefore , if lobby 1 wants to make an offer that cannot be cancelled out by lobby 0, it must satisfy the list of inequalities:

$$\sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right) < W_0 \text{ for all } S \in \mathcal{B}.$$

The cheapest offer t_1 meeting these constraints is the solution of the following linear programming problem:

$$\begin{gathered} \underset{t_1}{Min} \sum_{i \in N} t_{i1} \\ \text{subject to the constraints} \\ \sum_{i \in S} \left(t_{i1} + \alpha^i \Delta W\right) \geq W_0 \text{ for all } S \in \mathcal{B} \\ \text{and } t_{i1} \geq 0 \text{ for all } i \in N. \end{gathered} \tag{3}$$

Lobby 1 will find it profitable to offer the optimal solution t_1^* of problem (3) if the optimal value of this linear programming problem is less than W_1. It is then important to be able to compute this optimal value. To do so, we first introduce the following definition from combinatorial theory.

Definition 1 *Let $H = (N, \mathcal{H})$ be an arbitrary hypergraph. A fractional cover of H is a* *vector $t \in \Re^n$ such that:*

$$\begin{aligned} \sum_{i \in S} t_i &\geq 1 \textit{ for all } S \in \mathcal{H} \\ \textit{and } t_i &\geq 0 \textit{ for all } i \in N. \end{aligned} \tag{4}$$

A fractional matching of H is a vector $\gamma \in \Re^{\#\mathcal{H}}$ such that:

$$\begin{aligned} \sum_{S \in \mathcal{H}_i} \gamma(S) &\leq 1 \textit{ for all } i \in N \\ \textit{and } \gamma(S) &\geq 0 \textit{ for all } S \in \mathcal{H}. \end{aligned} \tag{5}$$

A fractional cover t^* minimizing $\sum_{i \in N} t_i$ subject to the constraints (4) is called an optimal fractional cover and $\psi^*(H) \equiv \sum_{i \in N} t_i^*$ is called the fractional covering number. A fractional matching γ^*maximizing $\sum_{S \subseteq N} \gamma(S)$ subject to the constraint (5) is called an optimal fractional matching and $\mu^*(H) \equiv \sum_{S \subseteq N} \gamma^*(S)$ is called the fractional matching number.

It is well known that $\psi^*(H) = \mu^*(H) \geq 1$[21]. The following result summarizes the equilibrium analysis of the sequential game.

Proposition 3 *(i) Either $W_1 \geq \sum_{S \in \mathcal{B}} \gamma(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W\right]$ for all fractional matchings γ of $(N, \mathcal{B})$ and then lobby 1 offers an optimal solution t_1^* to problem (3) and lobby 0 offers nothing and so the bill is passed.*

(ii) Or $W_1 < \sum_{S \in \mathcal{B}} \gamma(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W\right]$ for at least one fractional matching γ of $(N, \mathcal{B})$ and then both lobbyists promise nothing and so the bill is not passed.

Proof. Let v^* be the optimal value of problem (3). From the duality theorem of linear programming, v^* is the optimal value of the following linear programming problem:

$$\underset{\gamma}{Max} \sum_{S \in \mathcal{B}} \gamma(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W\right]$$

subject to the constraints

$$\sum_{S \in \mathcal{B}_i} \gamma(S) \leq 1 \text{ for all } i \in N$$

$$\text{and } \gamma(S) \geq 0 \text{ for all } S \in \mathcal{B}.$$

[21] See for instance theorem 5.5. in Füredi (1988).

The conclusion follows. ■

This result leads to several interesting conclusions. If $W_0 - \sum_{i \in S} \alpha^i \Delta W \leq 0$ for all $S \in \mathcal{B}$, then $\gamma = 0$ is a solution and therefore $v^* = 0$. We are in case (i) but lobby 1 promises nothing. If instead, $W_0 - \sum_{i \in S} \alpha^i \Delta W > 0$ for at least one $S \in \mathcal{B}$, then $v^* > 0$. Note further that for any fractional matching γ:

$$\begin{aligned}
\sum_{S \in \mathcal{B}} \gamma(S) \left[W_0 - \sum_{i \in S} \alpha_i \Delta W \right] &= W_0 \sum_{S \in \mathcal{B}} \gamma(S) - \Delta W \sum_{S \in \mathcal{B}} \gamma(S) \sum_{i \in S} \alpha^i \\
&= W_0 \sum_{S \in \mathcal{B}} \gamma(S) - \Delta W \sum_{i \in N} \alpha^i \sum_{S \in \mathcal{B}_i} \gamma(S) \\
&\geq W_0 \sum_{S \in \mathcal{B}} \gamma(S) - \Delta W \sum_{i \in N} \alpha^i .
\end{aligned}$$

When γ is an optimal fractional matching, we then obtain:

$$\sum_{S \in \mathcal{B}} \gamma(S) \left[W_0 - \sum_{i \in S} \alpha^i \Delta W \right] \geq W_0 \mu^*(\mathcal{B}) - \Delta W \sum_{i \in N} \alpha^i$$

and therefore

$$v^* + \Delta W \sum_{i \in N} \alpha^i \geq W_0 \mu^*(\mathcal{B}). \tag{6}$$

After simplifications, we deduce that if:

$$\frac{W_1}{W_0} \leq \frac{\mu^*(\mathcal{B}) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i}, \tag{7}$$

then, we are in case (ii). Inequality (7) is simply a sufficient condition for case (ii) to prevail. It is also necessary for any problem where it can be shown that all the coordinates of t_1^*, the solution to problem (3), are strictly positive. Indeed, in that case, we deduce from the complementary slackness condition, that:

$$\sum_{S \in \mathcal{B}_i} \gamma(S) = 1 \text{ for all } i \in N,$$

and (6) becomes an equality. This leads to the question: when is it the case that the cheapest strategy of lobby 1 consists of bribing the whole legislature? We offer an answer to that question in the case where n is odd i.e. $n = 2k - 1$ for some integer $k \geq 2$ and $(N, \mathcal{W})$

is the majority game.

Proposition 4 *Let $t_1^* = (t_{11}^*, t_{21}^*, ..., t_{n1}^*)$ be an optimal offer by lobby 1. Then, there exists an integer m^* such that $t_{i1}^* > 0$ and $t_{i1}^* + \alpha^i \Delta W = t_{j1}^* + \alpha^j \Delta W$ for all $i, j = 1, ..., m^*$. Further, either $\frac{W_0}{k} > \alpha^k \Delta W$ and m^* is determined as the unique smallest integer m such that $\frac{W_0}{k} \leq \Delta W \alpha^m$ if any and $m^* = n$ otherwise. Or $\frac{W_0}{k} \leq \alpha^k \Delta W$ and m^* is the smallest value of $m \leq k-1$ such that: $W_0 < \Delta W \left[\sum_{l=m+1}^{k} \alpha^l + m\alpha^{m+1}\right]$.*

Proof. Assume without loss of generality that $\alpha^1 \leq \alpha^2 \leq .... \leq \alpha^n$. Let $t_1^* = (t_{11}^*, t_{21}^*, ..., t_{n1}^*)$ be an optimal solution to problem (3) and $N^* \equiv \{i \in N : t_{i1}^* > 0\}$.

Claim 1: $t_{i1}^* + \alpha^i \Delta W = t_{j1}^* + \alpha^j \Delta W$ for all $i, j \in N^*$.

Assume on the contrary that $t_{i1}^* + \alpha^i \Delta W < t_{j1}^* + \alpha^j \Delta W$ for some $i, j \in N^*$. Then:

$$\sum_{l \in S} \left(t_{l1} + \alpha^l \Delta W\right) > W_0 \text{ for all } S \subseteq N \text{ such that } \#S = k \text{ and } i \in S. \tag{8}$$

Indeed if

$$\sum_{l \in \widehat{S}} \left(t_{l1} + \alpha^l \Delta W\right) = W_0 \text{ for some } \widehat{S} \subseteq N \text{ such that } \#S = k \text{ and } i \in S$$

then, we would obtain

$$\sum_{l \in \left(\widehat{S} \setminus \{i\}\right) \cup \{j\}} \left(t_{l1} + \alpha^l \Delta W\right) < W_0$$

contradicting our assumption that t_1^* is a solution to problem (3). Let $t_1^{**} = (t_{11}^{**}, t_{21}^{**}, ..., t_{n1}^{**})$ be such that $t_{l1}^{**} = t_{l1}^*$ for all $l \neq i$ and $t_{i1}^{**} = t_{i1}^{**} - \varepsilon$ for some $\varepsilon > 0$. If ε is selected small enough, it follows from inequalities (8) that t_1^{**} meets the constraints of problem (3). Since further $\sum_{i \in N} t_{i1}^{**} < \sum_{i \in N} t_{i1}^*$, we contradict our assumption that t_1^* is a solution to problem (3).

Claim 2: $\alpha^i < \alpha^j$ for all $i \in N^*$ and $j \notin N^*$

Assume on the contrary that $\alpha^i \geq \alpha^j$ for some $i \in N^*$ and $j \notin N^*$. Then as in claim 1:

$$\sum_{l \in S} \left(t_{l1} + \alpha^l \Delta W\right) > W_0 \text{ for all } S \subseteq N \text{ such that } \#S = k \text{ and } i \in S.$$

Indeed, if:

$$\sum_{l\in\widehat{S}}\left(t_{l1}+\alpha^{l}\Delta W\right)=W_0 \text{ for some } \widehat{S}\subseteq N \text{ such that } \#S=k \text{ and } i\in S$$

then, we would obtain:

$$\sum_{l\in\widehat{S}\backslash\{i\}}\left(t_{l1}+\alpha^{l}\Delta W\right)+\alpha^{j}\Delta W<W_0$$

contradicting our assumption that t_1^* is a solution to problem (3). The conclusion proceeds as in claim 1.

From claims 1 and 2, we deduce than an optimal strategy t_1^* is described by an integer $m^*\equiv\#N^*$ and a real number $t^*\equiv t_{l1}^*+\alpha^{l}\Delta W$ for all $l\in N^*$ such that $t^*<\alpha^{j}\Delta W$ for all $j=m^*+1,....,n$ and $t^*-\alpha^{j}\Delta W>0$ for all $j=1,.......,m^*$.

Consider first the case where $m^*\geq k$. The most severe constraint is attached to the coalition $S=\{1,...,k\}$ and it takes the form:

$$t^*k\geq W_0.$$

Solving this for equality gives:

$$t^*=\frac{W_0}{k},$$

and a total cost for lobby 1 equal to:

$$\left(\frac{W_0}{k}\right)m^*-\Delta W\sum_{l=1}^{m^*}\alpha^{l}.$$

Since $\alpha^1\leq\alpha^2\leq.......\leq\alpha^n$, the function $\left(\frac{W_0}{k}\right)m-\Delta W\sum_{l=1}^{m}\alpha^{l}$ is concave as a function of m. Therefore, there is a unique value m^* of m for which the expression above reaches its maximal value. We obtain: $t^*=\frac{W_0}{k}\leq\Delta W\alpha^{m^*+1}$ and $t_{l1}^*=t^*-\alpha^{l}\Delta W>0$ for all $l=1,.......,m^*$.

Consider now the case where $m^*<k$. The most severe constraint is still attached to the

coalition $S = \{1, ..., k\}$ and it takes now the form:

$$t^* m^* + \Delta W \sum_{l=m^*+1}^{k} \alpha^l \geq W_0.$$

Solving this for equality leads to:

$$t^* = \frac{W_0 - \Delta W \sum_{l=m^*+1}^{k} \alpha^l}{m^*},$$

and a total cost for lobby 1 equal to:

$$W_0 - \Delta W \sum_{l=1}^{k} \alpha^l.$$

This solution is valid iff:

$$t^* \leq \Delta W \alpha^{m^*+1},$$

i.e.

$$W_0 < \Delta W \sum_{l=m^*+1}^{k} \alpha^l + \Delta W m^* \alpha^{m^*+1}.$$

Since the function $\Delta W \sum_{l=m+1}^{k} \alpha^l + \Delta W m \alpha^{m+1}$ is increasing in m and takes the value $k \Delta W \alpha^k$ when $m + 1 = k$, we are left with two cases.

Either $\frac{W_0}{k} > \Delta W \alpha^k$ and m^* is determined as the unique smallest integer m such that $\frac{W_0}{k} \leq \Delta W \alpha^m$ if any and $m^* = n$ otherwise. Or $\frac{W_0}{k} \leq \Delta W \alpha^k$. Then let $\underline{m}$ be the smallest value of $m \leq k - 1$ such that

$$W_0 < \Delta W \left[\sum_{l=m+1}^{k} \alpha^l + m \alpha^{m+1} \right].$$

Since $\frac{W_0}{k} \leq \alpha^k \Delta W$, $\underline{m}$ is well defined. On the other hand, since

$$t^* - \alpha^j \Delta W > 0 \text{ for all } j = 1, ..., m^*,$$

we must have

$$\frac{W_0 - \Delta W \sum_{l=m^*+1}^{k} \alpha^l}{m^*} \geq \Delta W \alpha^{m^*},$$

and therefore $m^* = \underline{m}$. ■

We deduce from proposition 4 that if:

$$\frac{W_0}{k} > \Delta W \alpha^n \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^n}$$

then lobby 1's cheapest offer would consist of bribing all the legislators. The corresponding cost is $\frac{nW_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l$ and lobby 1 will therefore find it profitable to do so iff:

$$W_1 \geq \frac{nW_0}{k} - \Delta W \sum_{l=1}^{n} \alpha^l \text{ i.e. } \frac{W_1}{W_0} \geq \frac{\left(\frac{2k-1}{k}\right) + \sum_{i \in N} \alpha^i}{1 + \sum_{i \in N} \alpha^i},$$

i.e. inequality (7) since $\mu^*(\mathcal{B}) = 2 - \frac{1}{k}$. For lobby 1 to bribe at least a majority of legislators, it is necessary and sufficient that:

$$\frac{W_0}{k} > \Delta W \alpha^k \text{ i.e. } \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^k}.$$

It will bribe a minimal majority if:

$$1 + \frac{1}{k\alpha^{k+1}} \leq \frac{W_1}{W_0} < 1 + \frac{1}{k\alpha^k}.$$

The corresponding cost is $W_0 - \Delta W \sum_{l=1}^{k} \alpha^l$ and lobby 1 will therefore always find it profitable to do so. At the other extreme, if:

$$W_0 < \Delta W \sum_{l=1}^{k} \alpha^l$$

then, lobby 1 does not offer any bribe.

One noteworthy feature of inequality (7) is that it establishes a connection between our problem and classical problems in the combinatorics of sets. The number $\mu^*(\mathcal{B}) = \psi^*(\mathcal{B})$ describes an important feature of the decision making process in the legislature. It suggests that the larger this combinatorial invariant is, the larger the efficiency threshold for the reform is, i.e. the efficient policy to be the equilibrium policy. In particular[22], from

[22] This is the main result of Diermeier and Myerson (1999). They call $\mu^*(\mathcal{B})$ the hurdle factor of the legislature but do not point out the connection with the combinatorics of sets.

proposition 1, we see that if $\alpha^1 = ... = \alpha^n = 0$, then lobby 1 will be active iff:

$$\frac{W_1}{W_0} \geq \mu^*(\mathcal{B}).$$

The literature on hypergraphs contains many results on the exact value or bounds of $\mu^*(\mathcal{B})$. Some of these results[23] make use of the covering and matching numbers of a hypergraph $H = (N, \mathcal{H})$, denoted respectively $\mu(\mathcal{H})$ and $\psi(\mathcal{H})$, and defined as the fractional covering and matching numbers except for the fact that the vector t and γ are constrained to be integer valued. We have the following inequalities:

$$\mu(\mathcal{H}) \leq \mu^*(\mathcal{H}) = \psi^*(\mathcal{H}) \leq \psi(\mathcal{H}).$$

Computing the fractional covering number of a hypergraph is NP-complete. The calculation of $\mu^*(\mathcal{H})$ is straightforward for some simple hypergraphs. For instance, when $S \in \mathcal{H}$ iff $\#S \geq q$, then $\mu^*(\mathcal{H}) = \frac{n}{q}$.

Another important class of hypergraphs is the following. Let $(N_r, \mathcal{H}_r)_{1 \leq r \leq R}$ be a family of R hypergraphs with $N_r \cap N_t = \varnothing$ for all $r, t = 1, ..., R$ with $r \neq t$. Let $(N, \mathcal{H})$ be such that $N = \cup_{r=1}^{R} N_r$ and $S \in \mathcal{H}$ iff $S \cap N_r \in \mathcal{H}_r$ for all $r = 1, ..., R$. This is the definition of a multicameral legislature as defined by Diermeier and Myerson (1999): a reform is approved if it is approved in all the different R chambers according to the rules (possibly different) in use in any of the chambers. It is easy to show that:

$$\mu^*(\mathcal{H}) = \sum_{r=1}^{R} \mu^*(\mathcal{H}_r).$$

This multicameral system is a special case of a compound simple game. Let $\left(\{1, ..., R\}, \widetilde{\mathcal{H}}\right)$ be a hypergraph on the set of chambers: $\widetilde{\mathcal{H}}$ describes the power of coalitions of chambers (definition of Diermeier and Myerson (1999) corresponds to the case where $\widetilde{\mathcal{H}} = \{\{1, ..., R\}\}$ i.e. each chamber has a veto power). In the general case, $S \in \mathcal{H}$ iff:

$$\{r \in \{1, ..., R\} : S \cap N_r \in \mathcal{H}_r\} \in \widetilde{\mathcal{H}}.$$

The computation of $\mu^*(\mathcal{H})$ is now more intricate. If $\left(\{1, ..., R\}, \widetilde{\mathcal{H}}\right)$ is uniform as well as

[23] Among which the important Lovasz's inequality (1975).

$(N_r, \mathcal{H}_r)$ for all $r = 1, ..., R$, then $(N, \mathcal{H})$ is also uniform. Füredi (1981)'s inequality gives an upper bound on $\mu^*(\mathcal{H})$. In the case where $\left(\{1, ..., R\}, \widetilde{\mathcal{H}}\right)$ and $(N_r, \mathcal{H}_r)$ for all $r = 1, ..., R$ are the simple majority games, then if $\# N_r = \# N_{r'} \equiv m$ for all $r, r' = 1, ..., R$, we can show that if m and R are large integers, then $\mu^*(\mathcal{H}) \simeq 4$.

Another important case corresponds to projective planes. An $(r, \lambda)-$design is a hypergraph $(N, \mathcal{H})$ such that for all $i \in N$, $\# \{S \in \mathcal{H} : i \in S\} = r$ and for all $\{i, j\} \subseteq N$, $\# \{S \in \mathcal{H} : \{i, j\} \subseteq S\} = \lambda$. It is called symmetric if $\# N = \# \mathcal{H}$ a projective plane of order n, $PG(2, n)$ is a symmetric $(n + 1, 1)$ design. It can be shown[24] that $\mu^*(\mathcal{H}) = n + \frac{1}{n+1}$.

We conclude this section by considering the case of a weighted majority game. Let $\omega_i > 0$ be the weight attached to legislator[25] i. A coalition S is in $\mathcal{W}$ iff $\sum_{i \in S} \omega_i > \frac{\sum_{i \in N} \omega_i}{2}$. We denote by $\omega \equiv (\omega_1, ..., \omega_n)$ the simple game (will be called a representation) defined in that way and assume from now that it is strong. It is important to note that the same game may admit several representations. Isbell (1956)[26] has demonstrated that there exists a unique (up to multiplication by a constant) representation ω such that $\sum_{i \in S} \omega_i = \sum_{i \in T} \omega_i$ for all $S, T \in \mathcal{W}_m$. This representation is called the homogeneous representation of the simple game; the homogeneous representation ω for which $\sum_{i \in N} \omega_i = 1$ is called the homogeneous normalized representation. Consider the cooperative game with transferable utility (N, V) defined as follows:

$$V(S) = \begin{cases} 1 \text{ if } S \in \mathcal{W} \\ 0 \text{ if } S \notin \mathcal{W} \end{cases}.$$

Peleg (1968) has demonstrated[27] that the normalized homogeneous representation of $(N, \mathcal{W})$ coincides with the nucleolus x of (N, V). Since the game (N, V) is simple, only minimal winning coalitions matter in considering the vector of excesses. Therefore, x verifies:

$$x \in \underset{y \in S_n}{ArgMax} \underset{S \in \mathcal{W}_m}{Min} \sum_{i \in S} y_i,$$

where $S_n \equiv \left\{y \in \Re^n_+ : \sum_{i=1}^n y_i = 1\right\}$. Let:

$$C^* \equiv \underset{y \in S_n}{Max} \underset{S \in \mathcal{W}_m}{Min} \sum_{i \in S} y_i.$$

[24] See for instance, Füredi (1988).

[25] In such setting, it is more relevant to consider the players as being the leaders of the different parties rather than legislators acting on an individual basis.

[26] See also the generalization by Ostmann (1987).

[27] See also, Peleg and Rosenmüller (1992).

Then, the following simple assertion is true[28]:

Proposition 5 *Let* $(N, \mathcal{W})$ *be a strong weighted majority game. Then,*

$$\mu^*(\mathcal{W}) = \frac{1}{C^*}.$$

Proof. By definition of C^*, there exists $y \in \Re^n_+$ such that

$$\sum_{i=1}^{n} y_i = 1 \text{ and } \sum_{i \in S} y_i \geq C^* \text{ for all } S \in \mathcal{W}_m.$$

Therefore the vector z such that $z_i \equiv \frac{y_i}{C^*}$ for all $i = 1, ..., n$ verifies

$$\sum_{i=1}^{n} z_i = \frac{1}{C^*} \text{ and } \sum_{i \in S} z_i \geq 1 \text{ for all } S \in \mathcal{W}_m$$

implying that $\mu^*(\mathcal{W}) \leq \frac{1}{C^*}$.
Assume that $\mu^*(\mathcal{W}) < \frac{1}{C^*}$. This means that there exist a vector $z \in \Re^n_+$ such that

$$\sum_{i=1}^{n} z_i = \mu^*(\mathcal{W}) \text{ and } \sum_{i \in S} z_i \geq 1 \text{ for all } S \in \mathcal{W}_m.$$

Therefore the vector y such that $y_i \equiv \frac{z_i}{\mu^*(\mathcal{W})}$ for all $i = 1, ..., n$ verifies

$$\sum_{i=1}^{n} y_i = 1 \text{ and } \sum_{i \in S} y_i \geq \frac{1}{\mu^*(\mathcal{W})} \text{ for all } S \in \mathcal{W}_m.$$

Since $\frac{1}{\mu^*(\mathcal{W})} > C^*$, this contradicts our definition of C^*. ■

Proposition 5, combined with Peleg's result, provides a nice and simple way to calculate $\mu^*(\mathcal{W})$ for strong weighted majority games. The task amounts to discovering the weight of each minimal winning coalition in the normalized homogeneous representation and to take the inverse of that number. As an illustration consider a legislature with 4 parties where the

[28] The solutions x to the above problem constitute what is called the least core in abstract cooperative game theory. We would like to thank R. Holzman for calling that simple but useful argument and the connection to the least core to our attention. After completing our paper, we have discovered, while reading Montero (2005), that Young (1978) did reach the same conclusion. He also points out the relevance of the least core and the nucleolus to predict offers in lobbying games sharing similarities with ours.

number of representatives of each party is described by the vector $\omega = (49, 17, 17, 17)$. This leads to the strong weighted majority game[29]:

$$S \in \mathcal{W}_m \text{ iff } S = \{1,2\}, \{1,3\}, \{1,4\} \text{ or } \{2,3,4\}.$$

The normalized homogeneous representation is here $\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$. It follows that $\mu^*(\mathcal{W}) = \frac{5}{3}$.

Besides the knowledge of $\mu^*(\mathcal{W})$, it is of interest to know how the amount of money $\mu^*(\mathcal{W})W_0$ is allocated across parties. The answer to this question is provided by an optimal fractional cover t^*. When the simple game is symmetric, not surprisingly, all the legislators receive the same amount. When the game is not symmetric, then the question arises of how t_i^* is related to the power of legislator i. There is an extensive literature on the measurement of the power of players in simple games with a prominent place occupied by the Banzhaf index (1965, 1968) and the Shapley-Shubik index (1954). The intuition that the payoffs of the legislators are proportional to any of these power measures has been challenged by several authors[30]. In our context, as we have just seen, the relative shares of the legislators correspond to the nucleolus of the normalized simple game or equivalently to the weights in the homogeneous representation.

3.3 The Majority Game with Three Legislators

In this subsection, we illustrate how our results apply to the simplest simple game of interest: the majority game among three legislators with similar weights. The following proposition follows from propositions 2 and 4 and the fact that here $\mu^*(\mathcal{W}) = \frac{3}{2}$. As before, we assume without loss of generality that: $\alpha_1 \leq \alpha_2 \leq \alpha_3$.

Proposition 6 *(i) A Nash equilibrium in pure strategies exists iff* $\Delta W(\alpha_1 + \alpha_2) \leq W_0$*. At equilibrium, none of the lobbyists makes offers.*

(ii) In the Stackelberg equilibrium, lobby 2 never makes offers and

Either (ii1) $\Delta W(\alpha_1 + \alpha_2) \geq W_0$ *in which case, lobby 1 does not make offers.*

Or (ii1) $\Delta W(\alpha_1 + \alpha_2) \leq W_0 < 2\Delta W \alpha_2$*, in which case lobby 1 offers* $W_0 - \Delta W(\alpha_1 + \alpha_2)$ *to legislator 1 and nothing to legislators 2 and 3.*

[29] Such a simple game is called an apex game.

[30] For instance, in a legislative setting described as a bargaining game a la Baron-Ferejohn (1989), Snyder, Ting and Ansolabehere. 2005 show that the expected equilibrium payoffs of the legislators are proportional to their weights. Montero (2005) also found a rationale for the nucleolus in a somewhat comparable bargaining setting.

Or (ii3) $2\Delta W\alpha_2) \leq W_0 < 2\Delta W\alpha_3$*, in which case lobby 1 offers* $\frac{W_0}{2} - \Delta W\alpha_1$ *to legislator 1,* $\frac{W_0}{2} - \Delta W\alpha_2$ *to legislator 2 and nothing to legislator 3.*

Or (ii4) $W_0 \geq 2\Delta W\alpha_3$*, in which case lobby 1 makes an offer to all three legislators if* $W_1 \geq \frac{3W_0}{2} - \Delta W(\alpha_1 + \alpha_2 + \alpha_3)$ *and makes no offer otherwise.*

Instead of calculating the Stackelberg equilibrium, we could consider, as suggested by Grossman and Helpman (2001) the possibility for the lobbyists to randomize over their monetary transfers. The determination of a Nash equilibrium in mixed strategies is an alternative way to handle the non-existence problem described in the preceding section.

Proposition 7 *Assume that* $\Delta W(\alpha_1+\alpha_2) \leq W^0$*. Then, a mixed-strategy equilibrium exists with the following features. Lobby* 0 *makes offers to legislators* $i = 1, 2$ *with the probability that the offer to legislator* i *does not exceed* x *given by:*

$$F_i^0(x) = \begin{cases} 1 - \frac{\widetilde{W_0}}{2W_1} \text{ for } x \in [0, \Delta W\alpha_i), \\ \frac{W_1 - \widetilde{W_0}/2 + x - \Delta\alpha_i}{W^1} \text{ for } x \in \left[\Delta W\alpha_i, \Delta W\alpha_i + \widetilde{W_0}/2\right] \end{cases}.$$

Lobby 1 *makes an offer just to one of the legislators* $i = 1, 2$ *with equal probability. The probability that the offer to legislator* i *is less or equal to* x *is*

$$F_i^1(x) = \frac{x + \Delta W\alpha_{-i}}{W_0 - (x + \Delta\alpha_i)} \text{ for } x \in \left[0, \widetilde{W_0}/2\right], \text{ where } \widetilde{W_0} = W_0 - \sum_{i=1}^{2} \Delta W\alpha_i.$$

Proof. A priori any legislator i prefers to vote for policy 1, and to make him indifferent between two policies lobby 0 has to pay $\Delta W\alpha_i$. We show that strategies described above form an equilibrium.

First, consider the choice of lobby 1. Given that lobby 0 is bribing agents $i = 1, 2$, it is enough for lobby 1 to bribe just one of these agents to get policy 1. Since the highest amount that lobby 0 can offer to legislator i is $\Delta\alpha_i + \widetilde{W_0}/2$, lobby 1 will never offer more than $\widetilde{W_0}/2$. It remains to show that lobby 1 randomizes among the transfers on the interval $\left[0, \widetilde{W_0}/2\right]$, i.e. the group is indifferent among these alternatives.

Suppose that lobby 1 makes positive offer x to legislator 1 and zero to legislator 2. Then expected payoff of lobby 1 given the behavior of lobby 0 is $E\left[U_1 \left| x, 0\right.\right] = W_1 F_i^0(x + \Delta\alpha_1) - x$. Since it is equal to $W^1 - \widetilde{W_0}/2$, $E\left[U_1 \left| x, 0\right.\right]$ does not depend on x. It is clear that $E\left[U_1 \left| x, 0\right.\right] = E\left[U_1 \left| 0, x\right.\right]$. Therefore, lobby 1 achieves the same expected payoff for different contribution levels, and it is also indifferent between bribing legislator 1 or bribing legislator 2.

Now, let us consider the behavior of lobby 0. It needs to buy at least two votes (simple majority) to get its preferred outcome. The cheapest way is to bribe legislators 1 and 2. For lobby 0, it is a waste of resources to offer to legislator $i = 1, 2$ a positive bribe less than $\Delta\alpha_i$, since in that case the legislator would prefer policy 1. If lobby 1 does not make an offer to legislator i, then by offering $\Delta\alpha_i$ lobby 0 makes legislator i indifferent between the two policies. Amount $W_0 - \sum_{n=1}^{K+1} \Delta\alpha_n$ can be divided equally between legislators $i = 1, 2$. Thus, the maximum possible offer is calculated as $\Delta\alpha_i + \widetilde{W}_0/2$. Then, it is necessary to show that lobby 0 is indifferent among the bribes (x, y) with $x \in \left[0, \Delta\alpha_1 + \widetilde{W}_0/2\right]$ and $y \in \left[0, \Delta\alpha_2 + \widetilde{W}_0/2\right]$.

Suppose that lobby 0 offers x and y respectively to legislator $i = 1, 2$. Given the equilibrium strategy of lobby 1 expected payoff of lobby 0 is calculated as

$$E\left[U_0 \left| x, y\right.\right] = \frac{1}{2}\left(W_0 - x - y\right) F_1^1(x - \Delta\alpha_1) - \frac{1}{2} y \left(1 - F_1^1(x - \Delta\alpha_1)\right) + \\ + \frac{1}{2}\left(W_0 - x - y\right) F_2^1(y - \Delta\alpha_2) - \frac{1}{2} x \left(1 - F_2^1(y - \Delta\alpha_2)\right) = 0.$$

So, $E\left[U_0 \left| x, y\right.\right]$ does not depend on x and y, i.e. lobby 0 achieves the same expected payoff for any pair of offers x and y, each of which is less or equal to $\Delta\alpha_i + \widetilde{W}_0/2$. Thus, lobby 0 is willing to randomize in the described manner as well as lobby 1. ■

Note that in this Nash equilibrium in mixed strategies, lobby 0 gets an expected payoff equal to 0, which is the same as if it had no opportunity to bid for votes. Thus, it gets no surplus. In contrast, lobby 1 earns a positive surplus equal to $W_1 - \frac{\widetilde{W}_0}{2}$. The outcome is random. The efficient policy is chosen if the offer of lobby 0 to a contested legislator i does not exceed the offer of lobby 1 by an amount equal to $\Delta W\alpha_i$. Otherwise, the inefficient policy is chosen. The probability P of selecting the efficient outcome is given by:

$$\begin{aligned} P &\equiv \Pr\left(\text{Lobby 1 wins}\right) \qquad (9) \\ &= \frac{W_1 + \frac{\widetilde{W}_0}{2}}{W_1} + \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_1}{W_1} \ln \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_1}{\widetilde{W}_0 + \Delta W\alpha_1} + \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_2}{W_1} \ln \frac{\frac{\widetilde{W}_0}{2} + \Delta W\alpha_2}{\widetilde{W}_0 + \Delta W\alpha_2}. \end{aligned}$$

The computation of P proceeds stepwise. The probability of lobby 1's success, given the fact that it offers x to legislator 1 is $\Pr\left(M^1 \text{ wins} \left| x, 0\right.\right) = F_1^0(x + \Delta\alpha_1)$. Using the distribution of x it is easy to show that:

$$\Pr\left(\text{lobby 1 wins}\,|\text{legislator 1 is bribed}\right) = \tag{10}$$
$$\frac{\Delta\alpha_2}{\widetilde{W_0}+\Delta\alpha_2}\frac{W_1-\frac{\widetilde{W_0}}{2}}{W_1}+\int_0^{\frac{\widetilde{W_0}}{2}}\frac{W_1-\frac{\widetilde{W_0}}{2}+x}{W_1}\frac{\widetilde{W_0}+2\Delta\alpha_2}{\left(\widetilde{W_0}+\Delta\alpha_2-x\right)^2}dx.$$

We compute similarly the probability that lobby 1 wins given that legislator 2 is bribed. Note further that:

$$\frac{\partial P}{\partial\alpha_1}=\frac{\Delta}{2W_1}\ln\frac{(W_0+\Delta\alpha_1-\Delta\alpha_2)(W_0-\Delta\alpha_1)}{(W_0-\Delta\alpha_1+\Delta\alpha_2)(W_0-\Delta\alpha_2)}+\frac{\Delta^2}{2W_1(W_0-\Delta\alpha_1)}.$$

It follows that if $\alpha_1=\alpha_2$, then $\frac{\partial P}{\partial\alpha_1}>0$ since the first term is zero. If $\alpha_1=0$, then

$$\frac{\partial P}{\partial\alpha_1}=\frac{\Delta}{2W_1}\left(\ln\frac{W_0}{W_0+\Delta\alpha_2}+\frac{\Delta}{W_0}\right).$$

4 Political Uncertainty

In this section, we analyze the lobbying game under political uncertainty in the case where the simple game is the majority game with an odd number $n=2k+1$ of legislators. we assume that the types α_i of the legislators are independently and identically distributed from a continuous cumulative distribution function[31] F with bounded support $[\underline{\alpha},\bar{\alpha}]$ where $0\leq\underline{\alpha}<\bar{\alpha}$. We denote by f the probability density function, which is assumed to be strictly positive on the whole interval $[\underline{\alpha},\bar{\alpha}]$. Finally, we assume that the hazard rate $\frac{F}{f}$ is increasing and that the hazard rate $\frac{1-F}{f}$ is decreasing.

4.1 The Optimal Strategy of Lobby 0 when Lobby 1 is Inactive

We first consider the case where lobby 1 is inactive[32] i.e. $T_1^*=0$. This is an important benchmark to start our exploration of the competition between the two lobbies. In such a context, the strategic interaction with the other lobby disappears and the game becomes

[31] Therefore, the probability that any legislator has a type less than or equal to some α is $F(\alpha)$.

[32] Some authors simply ignore the existence of several lobbies and the competitive aspect resulting from that. One possible justification in our context is to argue that lobby 1 cannot overcome its own collective action difficulties and act efficiently with respect to its own global stake.

merely an agency problem with adverse selection where the principal is lobby 0 and the agents are the n legislators. The conflict of interest arises from our assumption that, without compensation, legislators would vote for policy 1. The contractual problem faced by lobby 0 amounts to the selection of a vector $T_0 \in \Re_+^n$ conditional on verifiable information. Given our observability assumptions, this information consists of the n-dimensional vector of individual votes. In principle, lobby 0 could make the payment to legislator i contingent upon the votes of other legislators as well or a general statistic depending upon the whole profile of votes. We assume here that the reward to legislator i is simply based on his own vote: legislator i receives t_{i0} if and only if he voted against the bill. This excludes, for instance, the ingenious contractual solution of Dal Bo (2002) where a given legislator is paid only in the event where his vote has been decisive.

The rest of this section is devoted to a complete analysis of this principal-agent(s) problem i.e. to a characterization of the main features of the optimal strategy T_0^*. We will denote by n_0^* the number of legislators who have been promised to receive bribes by lobby 0 in the optimal strategy i.e.:

$$n_0^* \equiv \# \{i \in N : t_{i0}^* > 0\} .$$

This is an important feature of the strategy as it provides an answer to the question: how large is the supermajority bought by lobby 0? A second feature is the total amount of dollars paid by lobby 0. From its perspective, this is a risky prospect, as it does not know for sure what will be the behavioral response of the legislators. Therefore, the amount $M_0^* \equiv \sum_{i \in N} t_{i0}^*$ just represents the upper bound of the range of this random variable. Other parameters of interest are the first E_0^* and second V_0^* moments of this random variable. The expected rate of return of this "investment" is then given by:

$$\frac{W_0 - E_0^*}{E_0^*}.$$

The third and last feature of the strategy that deserves to be investigated is the distribution of M_0^* across legislators. We have seen in section 3 that, when the simple game is not symmetric i.e. when some legislators are more powerful than others, i.e. when they are not perfect substitutes, we should expect some differentials in the way they will be treated by the lobbies. However, when the game is symmetric, they are all offered the same amount. Our assumption that the legislators are all identical ex ante together with the fact that the majority game is symmetric suggest that it will happen here too. This is not straightforward and calls for a proof, as the behavioral responses of the legislature following any possible history of offers is now more complicated. In cases where uniformity across the bribed legislators is shown to be optimal, we can, without loss of generality, limit ourselves to strategies defined by two dimensions: an integer n_0^* and a real number t_0^*.

4.1.1 The Voting Subgame(s)

Given any profile of offers T_0, a Bayesian strategy for legislator i in the continuation voting subgame is a mapping σ_i from the set of types $[\underline{\alpha}, \bar{\alpha}]$ into $\{0, 1\}$: $\sigma_i(T_0, \alpha_i) = 0$ means that legislator i votes for the status quo when T_0 is the vector of standing offers and his type is α_i.

A key determinant of legislator i strategic evaluation is the probability p_i of being pivotal. Legislator i of type α_i with an offer equal to t_{i0} votes for the status quo if and only if

$$t_{i0} + p_i \alpha_i W_0 \geq p_i \alpha_i W_1.$$

The Bayesian decision rule is therefore described by a cut point $\widehat{\alpha_i}$: legislator i votes for the status quo if his type α_i is below the cutpoint and votes for the reform otherwise. The cut point $\widehat{\alpha_i}$ is defined as

$$\widehat{\alpha_i} = \max \left\{ \underline{\alpha}, \min \left\{ \frac{t_{i0}}{p_i \Delta W}, \bar{\alpha} \right\} \right\}. \tag{11}$$

Let $N_0 \equiv \{i \in N : t_{i0} > 0\}$. Under the restriction that offers are uniform i.e. $t_{i0} \equiv t_0$ for all $i \in N_0$, all legislators in N_0 face the same decision problem. Hereafter, we will restrict our attention to symmetric equilibria i.e. we assume that these legislators use the same decision rule. We will denote by $\widehat{\alpha}$ the cut point describing this strategy and by p the probability of being pivotal for any of them. For the legislators outside N_0, voting for the reform is a dominant strategy

For any legislator i in N_0 the probability p of being pivotal is simply the probability that exactly k other legislators vote for 0. Since the legislators in $N \setminus N_0$ always vote for the reform, this is the probability of the event that exactly k legislators from $N_0 \setminus \{i\}$ vote for the status quo. Given the cut point $\widehat{\alpha}$, it is possible to write down explicitly the formula for p:

$$p = p(t_0, n_0, \widehat{\alpha}) = B_k \left[n_0 - 1, F(\widehat{\alpha}) \right], \tag{12}$$

where $B_k [n, q] = C_n^k q^k (1 - q)^{N-k}$ denotes the probability of the event k for a binomial random variable with parameters n and q. The pivotal probability depends upon the voting strategies played by the other legislators. The equilibrium pivotal probability will be the solution of (12) when $\widehat{\alpha}$ is the equilibrium cut point. Since the equilibrium cut point is itself dependent upon the equilibrium pivotal probability, we are left with an existence issue which is covered by the following proposition[33].

Proposition 8 *For any given $t_0 \geq 0$ and n_0, the continuation voting subgame has two interior symmetric equilibria $\underline{\alpha} < \widehat{\alpha}_L < \widehat{\alpha}_R < \overline{\alpha}$ and $\overline{\alpha}$ as a corner equilibrium. The low cut point equilibrium $\widehat{\alpha}_L$ Pareto dominates[34] the two other equilibria.*

Proof. The proof of the first assertion is divided into two cases.

(i) $n_0 = k + 1$, i.e. the lobby offers positive transfers to a simple majority of voters.

In this case the unique cut-off level exists. Applying (12) one gets that $p = F^k(\widehat{\alpha})$. Substituting it into (11) it follows that for $t \in (\Delta\bar{\alpha}, \infty)$ $\widehat{\alpha} = \bar{\alpha}$, and for $t \in [0, \Delta\bar{\alpha}]$ the cut point $\widehat{\alpha}$ is defined by

$$\widehat{\alpha} F^k(\widehat{\alpha}) = t/\Delta. \tag{13}$$

From assumptions on the distribution function it follows that the LHS of this equality is strictly increasing function of $\widehat{\alpha}$, therefore $\widehat{\alpha}$ is uniquely defined by (13). One can see that $\widehat{\alpha}$ is increasing function of t.

(ii) $n_0 > k + 1$, i.e. the number of voters receiving positive offers from the lobby is more than a simple majority.

In this case there can be 3, 2 or 1 equilibrium cut-off levels. From (12) the probability of being pivotal is

$$C^k_{n_0 - 1} F^k(\widehat{\alpha})(1 - F(\widehat{\alpha}))^{n_0 - 1 - k}.$$

First, let us consider function $\widehat{\alpha} F^k(\widehat{\alpha})(1 - F(\widehat{\alpha}))^{n_0 - 1 - k}$. One can see that on the interval $[\underline{\alpha}, \bar{\alpha}]$ it is nonnegative: it is equal to zero at $\underline{\alpha}$ and $\bar{\alpha}$, and it is strictly positive elsewhere on the interval. It has exactly one maximum at $\alpha^*_{n_0} \in (\underline{\alpha}, \bar{\alpha})$, where $\alpha^*_{n_0}$ is defined from

$$\frac{\partial}{\partial \alpha}\left[F^k(\alpha)(1 - F(\alpha))^{n_0 - 1 - k}\right] = 0.$$

[33] A game with similar features has been examined by Palfrey and Rosenthal (1985) as describing the decision to vote in an election given that voters incur a private cost to do so. In their model voters compare this cost to the expected differential benefit. They also face the issue of multiplicity of equilibria.

[34] Some warning is needed about what we mean by Pareto dominance. Precisely, we refer to unanimity in restriction to the coalition N_0 of legislators. It represents a way to solve the coordination issue faced by this subset of players.

To see that $\alpha^*_{n_0}$ is uniquely defined on the interval $(\underline{\alpha}, \bar{\alpha})$ let us rewrite the derivative as

$$\alpha F(\alpha)(1 - F(\alpha)) \left[\frac{1}{\alpha} + k \frac{f}{F}(\alpha) - (n_0 - 1 - k) \frac{f}{1 - F}(\alpha) \right].$$

From the assumptions it follows that the function in the brackets is monotonically decreasing, and for $\alpha \to \underline{\alpha}$ it approaches $+\infty$ and for $\alpha \to \bar{\alpha}$ it approaches $-\infty$. Therefore, it can be equal to zero exactly at one point $\alpha^*_{N_0} \in (\underline{\alpha}, \bar{\alpha})$. For convenience let

$$t^*_0 = C^k_{n_0 - 1} \Delta \alpha^* F^k(\alpha^*)(1 - F(\alpha^*))^{n_0 - 1 - k}.$$

From (11), (12) if $t_0 = t^*_0$ $\widehat{\alpha} = \alpha^*_{N_0}$, if $t_0 \in [0, t^*_0)$ there are two solutions for $\widehat{\alpha}_L$ and $\widehat{\alpha}_R$ defined by

$$\alpha F^k(\alpha)(1 - F(\alpha))^{n_0 - 1 - k} = \frac{t_0}{C^k_{n_0 - 1} \Delta} \tag{14}$$

and for all $t_0 \in (0, \infty)$ there is also solution $\widehat{\alpha} = \bar{\alpha}$.

Consider now the second assertion. The expected utility of agent n is

$$U_n(\alpha_n, \widehat{\alpha}) = \begin{cases} P^1(\widehat{\alpha}) \alpha_n W_1 + (1 - P^1(\widehat{\alpha})) \alpha_n W_0, & \text{for } \alpha_n \geq \widehat{\alpha} \\ P^0(\widehat{\alpha}) \alpha_n W_0 + (1 - P^0(\widehat{\alpha})) \alpha_n W_1 + t_0, & \text{for } \alpha_n \leq \widehat{\alpha} \end{cases},$$

where $P^1 = \Pr(\text{at least } k \text{ from the other } n_0 - 1 \text{ agents choose } 1)$ and

$P^0 = \Pr(\text{at least } k \text{ from the other } n_0 - 1 \text{ agents choose } 0)$.

First, let us consider the case $\alpha_n \leq \widehat{\alpha}$. Expected utility can be written as $U_n(\alpha_n, \widehat{\alpha}) = \alpha_n W_1 - \Delta P^0(\widehat{\alpha}) \alpha_n + t_0$.

Probability P^0 can be written as

$$P^0(\widehat{\alpha}) = \sum_{i=k}^{n_0 - 1} C^i_{n_0 - 1} F^i(\widehat{\alpha}) \left(1 - F(\widehat{\alpha})\right)^{n_0. - 1 - i}.$$

From lemma 1 it follows that

$$\frac{\partial P^0}{\partial \widehat{\alpha}} = f(\widehat{\alpha})(n_0 - k) C^{k-1}_{n_0 - 1} F^k(\widehat{\alpha})(1 - F(\widehat{\alpha}))^{n_0 - 1 - k} \geq 0.$$

Thus, $P^0(\widehat{\alpha})$ is increasing and expected utility is decreasing in $\widehat{\alpha}$. Therefore $U_n(\alpha_n, \widehat{\alpha}_L) \geq U_n(\alpha_n, \widehat{\alpha}_R)$, i.e. in equilibrium $\widehat{\alpha}_L$ utility of each agent n is at least as high as in equilibrium

$\widehat{\alpha}_R$. The case $\alpha_n \geq \widehat{\alpha}$ is similar. ■

In solving backward the whole game, we solve each terminal voting subgames following a pair (t_0, n_0) by considering the equilibrium $\widehat{\alpha}_L = \widehat{\alpha}_L(t_0, n_0)$ which will be denoted simply $\widehat{\alpha} = \widehat{\alpha}(t_0, n_0)$ without risk of confusion.

4.1.2 The Optimal Offer of Lobby 0

We are now in position to investigate the two dimensions of the optimal strategy of lobby 0. Given t_0 and N_0, the probability of accepting the bribe by any legislator in N_0 is simply $F(\widehat{\alpha})$ and the probability of success for the lobby 0 is

$$G(t_0, n_0) = \sum_{j=k+1}^{n_0} B_j \left[n_0, F(\widehat{\alpha})\right].$$

Therefore, the expected payoff of the lobby 0 is

$$\Pi(t_0, n_0) = G(t_0, n_0)W_0 - n_0 F(\widehat{\alpha})t_0.$$

The following proposition describes the optimal amount of the offer t_0 when lobby 0 buy a minimal winning coalition.

Proposition 9 *When $n_0 = k+1$, the equilibrium offer t_0^* is uniquely defined:*
for $W_0 \in [0, N_0\underline{\alpha}\Delta]$, $t_0^* = 0$;
for $W_0 \in \left(n_0\underline{\alpha}\Delta, n_0\bar{\alpha}\Delta + \frac{\Delta W}{f(\bar{\alpha})}\right)$, $t_0^* = aF^k(a)\Delta W < \Delta W$ where $a \in (\underline{\alpha}, \bar{\alpha})$ is the unique solution to the equation: $W^0 - n_0 a\Delta W = \frac{F(a)}{f(a)}\Delta W$
for $W_0 \in \left[n_0\bar{\alpha}\Delta W + \frac{\Delta W}{f(\bar{\alpha})}, +\infty\right)$ $t^* = \Delta W\bar{\alpha}$.

Proof. In this case the expected payoff of the lobby is defined by

$$\Pi(k+1, t_0) = F^{k+1}(\widehat{\alpha})\left(W_0 - (k+1)\Delta\widehat{\alpha}\right).$$

Since the cut-off level $\widehat{\alpha}(t_0)$ is increasing function, it is possible to substitute for t_0 from the second stage problem and to maximize with respect to $\widehat{\alpha}$:

$$\frac{\partial \Pi}{\partial \widehat{\alpha}} = (k+1)F^k(\widehat{\alpha})\left[f(\widehat{\alpha})(W_0 - (k+1)\Delta\widehat{\alpha}) - \Delta F(\widehat{\alpha})\right] = 0.$$

First, consider $W_0 - (k+1)\widehat{\alpha}\Delta = \frac{F(\widehat{\alpha})}{f(\widehat{\alpha})}\Delta$. By assumption $\frac{F}{f}$ is increasing, therefore the RHS is increasing function of $\widehat{\alpha}$, and the LHS is decreasing. Therefore, these two functions can intersect at most once on the interval $(\underline{\alpha}, \bar{\alpha})$. It is easy to see that interior solution $a \in (\underline{\alpha}, \bar{\alpha})$ exists only if $W_0 - (k+1)\Delta\underline{\alpha} > 0$ and $W_0 - (k+1)\bar{\alpha}\Delta < \frac{\Delta}{f(\bar{\alpha})}$.

Summing up, there are three cases:

If $(k+1)\underline{\alpha}\Delta < W_0 < (k+1)\bar{\alpha}\Delta + \frac{\Delta}{f(\bar{\alpha})}$ the cut point is $a \in (\underline{\alpha}, \bar{\alpha})$ defined by

$$W_0 - (k+1)a\Delta = \frac{F(a)}{f(a)}\Delta \text{ and}$$

$\frac{\partial \Pi}{\partial \widehat{\alpha}} > 0$ for $\widehat{\alpha} < a$ and $\frac{\partial \Pi}{\partial \widehat{\alpha}} < 0$ for $\widehat{\alpha} > a$.

In case $0 < W_0 < (k+1)\underline{\alpha}\Delta$ the cut point is $\underline{\alpha}$ since $\frac{\partial \Pi}{\partial \widehat{\alpha}} < 0$ on the whole interval $(\underline{\alpha}, \bar{\alpha})$.

If $W_0 - (k+1)\bar{\alpha}\Delta > \frac{\Delta}{f(\bar{\alpha})}$ the cut point is $\bar{\alpha}$, since $\frac{\partial \Pi}{\partial \widehat{\alpha}} > 0$ on $(\underline{\alpha}, \bar{\alpha})$. ■

Of course, it is not necessarily optimal for lobby 0 to buy a minimal winning coalition. It may prefer to buy a supermajority. Given the fact that the function Π is continuous with respect to t_0 and that n_0 takes a finite number of values, an optimal strategy is always well defined. It remains however that the derivation of general results concerning this policy are difficult to derive. The results that follow offer some preliminary insights in some more structured settings.

Proposition 10 *Assume that F is the uniform distribution on the interval $[0,1]$ and that $n = 3$. Then there exists $\lambda \in \left]\frac{2}{3}, 1\right[$ such that:*

(i) $n_0 = 2$ iff $W_0 \geq \frac{38}{7}\Delta W$.

(ii) If $W_0 \in [0, \lambda\Delta W]$, then $M_0^* = 3t(\alpha_1)$ where

$$\alpha_1 \equiv \frac{3\Delta W + W_0 - \sqrt{9\Delta W^2 - 10\Delta W W_0 + (W_0)2}}{8\Delta W}.$$

(iii) If $W_0 \in \left[\lambda\Delta, \frac{38}{7}\Delta\right]$, then $M_0^* = \frac{8}{9}\Delta W$.

(iv) If $W_0 \in \left]\frac{38}{7}\Delta, +\infty\right[$, then $M_0^* = 2\Delta W$.

Proof. (i) $n_0 = 2$. $\Pi(k+1, \alpha) = \alpha^2(W_0 - 2\Delta\alpha)$.

For $W_0 \in [0, 3\Delta]$ function $\Pi(k+1, \alpha)$ reaches its maximum at $a = W_0/(3\Delta)$. The corresponding transfer $t^* = t(a) = \frac{(W_0)^2}{9\Delta}$ and $\Pi(k+1, a) = \frac{(W_0)^3}{27\Delta^2} < \Delta$;

For $W_0 \in (3\Delta, +\infty)$ function $\Pi(k+1, \alpha)$ is increasing on the whole interval, therefore the maximum point is 1. The corresponding transfer $t^* = \Delta$ and $\Pi(k+1, 1) = W_0 - 2\Delta > \Delta$.

(ii) $n_0 = 3$.

In this case $\alpha^* = 2/3$.
$\Pi(k+2,\alpha) = \alpha^3 W_0 + 3(1-\alpha)\Pi(k+1,\alpha)$.
$\Pi(k+2,\alpha^*) = \frac{4}{9}\left(\frac{5}{3}W_0 - \frac{4}{3}\Delta\right)$.
The roots of the equation $\frac{\partial \Pi(k+2,\alpha)}{\partial \alpha} = 0$ are defined by

$$\alpha_{1,2} = \frac{3\Delta + W_0 \pm \sqrt{9\Delta^2 - 10\Delta W_0 + (W_0)^2}}{8\Delta}. \tag{15}$$

If the discriminant is non-negative $\Pi(k+2,\alpha)$ reaches its maximum at α_1 (the smallest root) and its minimum at α_2 (the largest root).

$\frac{\partial \Pi(k+2,\alpha)}{\partial \alpha} > 0$ on the whole interval $[0,\alpha^*]$ in the following two cases: for $W_0 \in [\Delta, 9\Delta]$ since the discriminant is non-positive, and for $W_0 \in (9\Delta, +\infty)$ since $\alpha_1 > 1$.

Therefore, for $W_0 \in (\Delta, +\infty)$ function $\Pi(k+2,\alpha)$ reaches its maximum at α^*. The optimal transfer $t^* = \frac{8}{27}\Delta$ and $\Pi(k+2,\alpha^*) = \frac{4}{27}(5W_0 - 4\Delta)$.

It remains to consider the three cases in turn.

· $W_0 \in (0,\Delta)$. Function $\Pi(k+2,\alpha)$ reaches its maximum on the interval $[0,\alpha^*]$ either at α_1 or α^* and $\Pi(k+1,\alpha)$ reaches its maximum on $[0,1]$ at a. One can check that $\alpha_1 \geq a$ and $\Pi(k+1,\alpha) \leq \Pi(k+2,\alpha)$ on $[0,\alpha_1]$ (the proof is provided in a more general case in the next section). Therefore, $n_0 = 3$ and maximum point is either α_1 or α^*. More precisely, for $W_0 \in (0, 2/3\Delta)$ maximum point is α_1 since $\alpha_2 > \alpha^*$. Since α_2 is decreasing in W^0 for $W^0 \geq 2/3\Delta$ minimum point $\alpha_2 < \alpha^*$.

· $W_0 \in (\Delta, 3\Delta)$. It is always the case that $\Pi(k+1,a) \leq \Pi(k+2,\alpha^*)$.

· $W_0 \in (3\Delta, +\infty)$. It is necessary to compare $\Pi(k+1,1)$ and $\Pi(k+2,\alpha^*)$. It follows that $\Pi(k+1,1) \geq \Pi(k+2,\alpha^*)$ for $W_0 \geq \frac{38}{7}\Delta$ and the opposite inequality is true otherwise. ■

The strategy of lobby 0 described in proposition 10 displays an interesting feature. Not surprisingly, the larger is the stake W_0, the more money the lobby spends to buy votes. What is more intriguing however, is that this money is spent on fewer legislators, i.e. the size of the coalition to which offers are made becomes smaller. These are two equilibrium predictions in the above special setting. They suggest the following two general questions.

· Is it the case, that lobbying activities are normal goods i.e. exhibiting positive income effects?

· Is it the case that the size of the coalition of legislators approached by the lobby decreases as the stake becomes larger?

We strongly suspect that the answers to these two questions are positive when F is the uniform distribution on the interval $[0,1]$. More precisely, we think that the following assertion is true but have not been able to prove it in full generality for the moment.

There exist thresholds $\overline{w}(m)$, $m = k+1, ..., n_0$, such that $\overline{w}(m)$ is decreasing in m and for $W_0 \in [\overline{w}(m)\Delta, \overline{w}(m-1)\Delta]$ $n_0 = m$. That is, starting with small values of W_0 the lobbying group prefers to bribe all members of the committee ($n_0 = n$) and with the increase of W_0 it bribes less and less members, bribing just a simple majority for rather large values.

Figures 3,4,5, and 6 suggest the plausibility of such pattern. The following two propositions are additional pieces of evidence in defense of that conjecture.

Proposition 11 *Assume that F is the uniform distribution on the interval $[0,1]$. Let $\alpha^*_{n_0}$ denotes the optimal cut point when lobby 0 restricts itself to a coalition of size n_0. The following statements hold true:*

(i) $\alpha^*_{n_0}$ is decreasing with respect to n_0.

(ii) If $W_0 \geq (k+2)\Delta W$, then for any $n_0 > k+1$, the function $\Pi(n_0, \alpha)$ is increasing in α on the whole interval $\left[0, \alpha^*_{n_0}\right]$.

(iii) $\Pi(n_0+1, \alpha) > \Pi(n_0, \alpha)$ for any $\alpha \in \left[0, \alpha^*_{n_0+1}\right]$ and such that $\Pi(n_0, \alpha) \geq 0$.

Proof. (i) For the case of uniform distribution $\alpha^*_{n_0} = \frac{k+1}{n_0}$.

(ii) The expected payoff can be written as

$$\Pi(k+1, \alpha) = \alpha^{k+1} \left(W^0 - (k+1)\,\Delta\alpha\right).$$

For $n_0 > k+1$

$$\begin{aligned}\Pi(n_0, \alpha) \;&=\; C^{k+1}_{n_0} \alpha^{k+1} (1-\alpha)^{n_0-k-1} \left(W^0 - (k+1)\,\Delta\alpha\right) + \\ & \quad W^0 \sum_{i=k+2}^{n_0} C^i_{n_0} \alpha^i (1-\alpha)^{n_0-i} \quad \text{or}\end{aligned}$$

$$\Pi(n_0, \alpha) = C^{k+1}_{n_0} (1-\alpha)^{n_0-k-1}\, \Pi(k+1, \alpha) + W^0 \sum_{i=k+2}^{n_0} C^i_{n_0} \alpha^i (1-\alpha)^{n_0-i}. \tag{16}$$

Taking derivative of (16) with respect to α we get

$$\begin{aligned}\frac{\partial \Pi(n_0, \alpha)}{\partial \alpha} \;&=\; C^{k+1}_{n_0} (1-\alpha)^{n_0-k-1} \frac{\partial \Pi(k+1, \alpha)}{\partial \alpha} - \\ & \quad - (n_0 - k - 1)\, C^{k+1}_{n_0} (1-\alpha)^{n_0-k-2}\, \Pi(k+1, \alpha) + \\ & \quad + W^0 \frac{\partial}{\partial \alpha} \left(\sum_{i=k+2}^{n_0} C^i_{n_0} \alpha^i (1-\alpha)^{n_0-i} \right).\end{aligned}$$

Substituting for $\Pi(k+1,\alpha)$ from (16) and using lemma 1 of appendix to simplify the last term one obtains

$$\frac{\partial \Pi(n_0,\alpha)}{\partial \alpha} = C_{n_0}^{k+1}(1-\alpha)^{n_0-k-2} \times \times \left[(1-\alpha)\frac{\partial \Pi(k+1,\alpha)}{\partial \alpha} + (n_0-k-1)(k+1)\Delta\alpha^{k+2}\right]. \qquad (17)$$

It follows that $\frac{\partial \Pi(n_0,\alpha)}{\partial \alpha} \geq 0$ for $W^0 \geq (k+2)\Delta$ since $\frac{\partial \Pi(k+1,\alpha)}{\partial \alpha} \geq 0$. One can also notice from (17) that if $a(n_0)$ is a maximum point of $\Pi(n_0,\alpha)$ then necessarily $a(n_0) \geq a(k+1)$ for any $n_0 > k+1$.

(iii)

$$\begin{aligned}\Pi(n_0+1,\alpha) - \Pi(n_0,\alpha) &= \\ &= C_{n_0}^{k+1}(1-\alpha)^{n_0-k-1}\Pi(k+1,\alpha)\frac{(k+1)-(n_0+1)\alpha}{n_0-k} + \\ &+ W^0\sum_{i=k+2}^{n_0} C_{n_0}^{i}\alpha^i(1-\alpha)^{n_0.-i-1}\frac{i-(n_0+1)\alpha}{n_0+1-i} + W^0\alpha^{n_0+1}.\end{aligned}$$

One can notice that $i-(n_0+1)\alpha \geq 0$ for any $\alpha \leq \alpha^*(n_0+1)$ and $i \geq k+1$.

From (16) it follows that $\Pi(n_0,\alpha) \geq 0$ is equivalent to

$$C_{n_0}^{k+1}(1-\alpha)^{n_0-k-1}\Pi(k+1,\alpha) \geq -W^0\sum_{i=k+2}^{n_0} C_{n_0}^{i}\alpha^i(1-\alpha)^{n_0-i}.$$

Substituting this into the previous expression one gets

$$\Pi(n_0+1,\alpha) - \Pi(n_0,\alpha) \geq$$

$$W^0\sum_{i=k+2}^{n_0} C_{n_0}^{i}\alpha^i(1-\alpha)^{n_0-i-1}\left[\frac{(n_0+1)}{n_0+1-i}(1-\alpha) - \frac{(n_0+1)}{n_0+1-k}(1-\alpha)\right] + W^0\alpha^{n_0+1} > 0.$$

■

Remark 1 *It can be shown that for* $W^0 < (k+2)\Delta$ *function* $\Pi(n_0,\alpha)$ *can have one interior maximum and one interior minimum. From (17)* $\frac{\partial \Pi(n_0,\alpha)}{\partial \alpha} = 0$ *is equivalent to* $\Delta(n_0+1)\alpha^2 - (W^0+(k+2)\Delta)\alpha - W^0 = 0$. *It follows that maximum point* $a(n_0)$ *is increasing in* n_0 *and minimum point is decreasing.*

The next proposition shows that if lobby 0 buys a minimal winning coalition, then the stake must be larger than some minimal threshold. More precisely

Proposition 12 *Assume that F is the uniform distribution on the interval $[0,1]$. A necessary condition for $n_0^* = k+1$ is:*

$$W_0 \geq \frac{e-1}{e-2}(k+1)\Delta W$$

Proof. In order to get the result I compare

$$\max_{\alpha\in[0,1]} \Pi(k+1,\alpha) \text{ and } \max_{\alpha\in\left[0,\alpha^*_{N_0}\right]} \Pi(k+2,\alpha).$$

First, one can notice that

$$\frac{\partial\Pi(k+1,\alpha)}{\partial\alpha} = (k+1)\alpha^k \left(W^0 - (k+2)\Delta\alpha\right).$$

Therefore, $\Pi(k+1,\alpha)$ is increasing on $[0,1]$ if $W^0 \geq (k+2)\Delta$ and otherwise it has maximum at $a(k+1) = \frac{W^0}{(k+2)\Delta}$. Second, if $\Pi(k+2,\alpha)$ has maximum at some $a(k+2) < \alpha^*_{n_0}$, then necessarily $a(k+2) > a(k+1)$. It follows from the fact that

$$\frac{\partial\Pi(k+2,\alpha)}{\partial\alpha} = (k+2)(1-\alpha)\frac{\partial\Pi(k+1,\alpha)}{\partial\alpha} + (k+2)(k+1)\Delta\alpha^{k+2}.$$

Therefore, $\frac{\partial\Pi(K+2,\alpha)}{\partial\alpha} = 0$ if and only if

$$(1-\alpha)\frac{\partial\Pi(k+1,\alpha)}{\partial\alpha} = -(k+1)\Delta\alpha^{k+2} < 0.$$

· Suppose that $W^0 \geq (k+2)\Delta$.

Then according to the previous result function $\Pi(k+1,\alpha)$ reaches its maximum at $\alpha = 1$ and $\Pi(k+2,\alpha)$ - at $\alpha^*_{n_0}$.

$$\Pi(k+1,1) = W^0 - (k+1)\Delta \text{ and}$$

$$\Pi(k+2, \alpha^*(+2)) = W^0 \left(\left(\frac{k+1}{k+2}\right)^{k+1} + \left(\frac{k+1}{k+2}\right)^{k+2} \right) - (k+1)\, \Delta \left(\frac{k+1}{k+2}\right)^{k+2}.$$

Let x be defined as $x = K + 2 \geq 3$ then

$$\Pi(k+1, 1) - \Pi(k+2, \alpha^*(k+2)) =$$
$$W^0 \left[1 - \frac{2x-1}{x-1}\left(1 - \frac{1}{x}\right)^x\right] - (k+1)\Delta \left[1 - \left(1 - \frac{1}{x}\right)^x\right].$$

For $x \geq 3$ function $\phi(x) = \left(1 - \frac{1}{x}\right)^x$ is positive and increasing, moreover,

$$\lim_{x \to \infty} \phi(x) = 1/e.$$

Function

$$g(x) = \frac{1 - \phi(x)}{1 - \frac{2x-1}{x-1}\phi(x)}$$

is decreasing, $g(x) > 0$ for $x \geq 3$ and

$$\lim_{x \to \infty} g(x) = \frac{e-1}{e-2}.$$

Thus,

$$\Pi(k+1, 1) - \Pi(k+2, \alpha^*(k+2)) > 0$$
$$\text{iff } W^0 > (k+1)\Delta g(k+2) \text{ and } \frac{e-1}{e-2} < g(x) < 19/7.$$

· Now, suppose that the opposite condition is satisfied, i.e. $W^0 \leq (k+2)\Delta$.

Then $\Pi(k+1, \alpha)$ reaches its maximum at $a(k+1) < 1$. Function $\Pi(k+2, \alpha)$ is either increasing on the whole interval $\left[0, \alpha^*_{n_0}\right]$ or it reaches its maximum at $a(k+2) < \alpha^*_{n_0}$.

If $\Pi(k+2, \alpha)$ has maximum at $a(k+2)$ then $a(k+2) > a(k+1)$, and from proposition 11

$$\Pi(k+2, a(k+2)) - \Pi(k+1, a(k+1)) \geq \Pi(k+2, a(k+1)) - \Pi(k+1, a(k+1)) > 0.$$

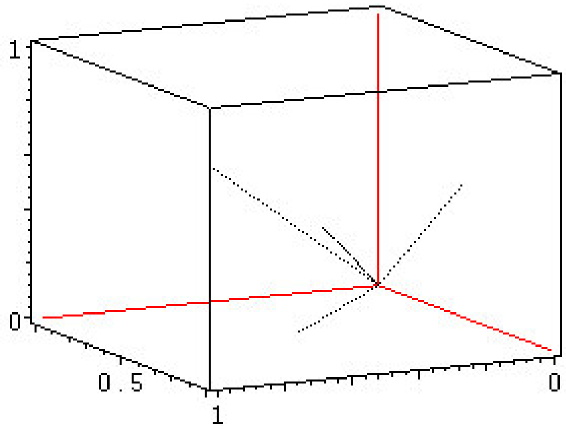

Figure 1: set of pure strategies.

If $\Pi(k+2,\alpha)$ is increasing on the whole interval, then the following is true

$$\Pi(k+2,\alpha^*(k+2)) - \Pi(k+1,a(k+1)) \leq \Pi(k+2,\alpha^*(k+2)) - \Pi(k+1,1) < 0$$

if $W^0 > (k+1)\Delta g(k+2)$. ■

4.2 The Optimal Strategy of Lobby 0 when Lobby 1 is active

In this section, we return to the game theoretical framework i.e. we take into account the lobbying or counterlobbying strategy of lobby 1. As in the preceding section, we disregard the possibility for a lobby to offer different offers to those who receive offers and we denote by (t_0, n_0) and (t_1, n_1) the respective (pure) strategies of lobby 0 and lobby 1. It should observed that the two-player game describing this competition is quite unusual as the sets of pure strategies of the players are non convex subsets of the $(n-1)$-dimensional unitary simplex[35] as illustrated on figure 1. This implies that the equilibrium analysis will be rather intricate and requires a specific treatment. We first explore the nature of the best response functions of the two lobbies.

Given a profile $((t_0, n_0), (t_1, n_1))$ of lobbying strategies, the legislators are partitioned

[35]This follows, of course from our uniformity assumption.

into three groups:

· The group N_0 incudes the legislators who have received an offer exclusively from lobby 0.

· The group N_{01} incudes the legislators who have received an offer from both lobbies.

· the group $N \setminus (N_0 \cup N_{01})$ of legislators who did not receive any offer.

Note that at equilibrium, no legislator will receive a positive offer from lobby 1 exclusively as this would be from its perspective a wasteful investment.

4.2.1 The Voting Subgame(s)

As before, in solving continuation voting subgames, we restrict our attention to symmetric equilibria i.e. all legislators in similar positions follow the same decision rule. The legislators in group N_0 will be described by the cut point $\widehat{\alpha}$ while those in N_{01} will be described by the cut point $\widehat{\beta}$. Let us denote by p^0 and p^{01} the probability of being pivotal for a legislator in N_0 and N_{01} respectively. Then, the cut points are defined as

$$\widehat{\alpha} = \max\left\{\underline{\alpha}, \min\left\{\frac{t_0}{p^0 \Delta W}, \bar{\alpha}\right\}\right\}, \tag{18}$$

and

$$\widehat{\beta} = \max\left\{\underline{\alpha}, \min\left\{\frac{t_0 - t_1}{p^{01} \Delta W}, \bar{\alpha}\right\}\right\}. \tag{19}$$

The dominant strategy of legislators in $N \setminus (N_0 \cup N_{01})$ is to vote for the reform.

For any legislator i in $N_0 \cup N_{01}$ the probability of being pivotal is simply the probability that exactly k other agents vote for 0. It is equal to the probability that exactly k agents from $(N_0 \cup N_{01}) \setminus \{i\}$ vote for 0. Given the cut points $\widehat{\alpha}$ and $\widehat{\beta}$, it is possible to write down explicitly the formula for the probabilities of being pivotal:

$$p^0(N_0, N_{01}, \widehat{\alpha}, \widehat{\beta}) = \sum_{r=\max(0,k-n_{01})}^{\min(k,n_0-1)} B_r\left[n_0 - 1, F(\widehat{\alpha})\right] B_{k-r}\left[n_{01}, F(\widehat{\beta})\right]. \tag{20}$$

A similar expression is derived for $p^{01}(N_0, N_{01}, \widehat{\alpha}, \beta)$. The continuation voting subgames are more complicated to analyze than before, as there are now a pair of equations and two variables $\widehat{\alpha}$ and $\widehat{\beta}$ to be determined.

4.2.2 The Optimal Strategies of Lobbies 0 and 1

We offer some preliminary but incomplete insights in the case where F is the uniform distribution on the interval $[0,1]$ and $n=3$. There are seven possible cases according to the number of agents belonging to each of three groups N_0, N_{01} and $N\setminus(N_0\cup N_{01})$. They are denoted by 201, 300, 210, 120, 111, 021, 030. For example,case 201 describes the situation in which two legislators receive an offer from the lobbying group 0 and the third one does not receive an offer at all. The first two regimes, namely 201 and 300 bring us back to the previous situation, in which only lobbying group M^0 is active.

We are primarily interested in the best response of lobby 0 to the strategy of lobby 1. The following table describes the possible regimes r for lobby 0 given (t_1, n_1).

$(\mathbf{1},\mathbf{t_1})$	$(\mathbf{2},\mathbf{t_1})$	$(\mathbf{3},\mathbf{t_1})$
210	120	030
111	021	

For each r it is possible to calculate the best response $t_0(t_1)$ and also $\Pi_r(t_1)=\Pi_r^0(t_0(t_1),t_1)$. Then, given (n_1,t_1), the reaction $t_0(t_1)$ of lobby 0 maximizes $\Pi_r(t_1)$.

Case 1: 210.

Pivotal probabilities are defined by

$$p^0=\alpha\,(1-\beta)+\beta\,(1-\alpha)\ \text{ and}$$

$$p^{01}=2\alpha\,(1-\alpha)\,.$$

Then, the second stage solutions $\alpha\,(t^0,t^1)$ and $\beta\,(t^0,t^1)$ are defined from the following system

$$\begin{cases}\alpha=\max\left\{0,\min\left\{\frac{t^0}{\Delta[\alpha(1-\beta)+\beta(1-\alpha)]},1\right\}\right\}\\ \beta=\max\left\{0,\min\left\{\frac{t^0-t^1}{2\alpha(1-\alpha)\Delta},1\right\}\right\}\end{cases}. \tag{21}$$

Suppose that $\alpha,\beta\in(0,1)$. Eliminating β from the system, one can express t^0 through α and t^1:

$$t^0=2\Delta\alpha^2\,(1-\alpha)-t^1\,(1-2\alpha)\,. \tag{22}$$

Lobbying group 0 maximizes:

$$\Pi_{210}^0 = \left(\alpha^2 + 2\alpha\beta\left(1-\alpha\right)\right)W^0 - t^0\left(2\alpha+\beta\right). \tag{23}$$

We take the grid for t^1 consisting of 100 points and for each value of t^1 calculate t^0, β and Π_{210}^0 a functions of α. On the interval $[0,1]$, equation (22) may define two solutions for $t^0(\alpha)$ one of which is decreasing in α and the other is decreasing. It is more intuitive that a larger amount of bribe from lobby 0 corresponds to a higher probability for a legislator to accept the bribe and to vote against the reform. Therefore we consider the increasing solution.

We maximize the function Π_{210}^0 with respect to α on the interval where $t^0(\alpha)$ is positive and increasing. Below the graphs of the functions (22) and (23) are different values of t^1 and $W^0 = \Delta = 1$. As we can see the function $\Pi_{210}^0(\alpha)$ may have two local maxima on the considered interval. For small values of t^1, the function $\Pi_{210}^0(\alpha)$ reaches its maximum in the right boundary of the interval (figure 10). With an increase of t^1, the point where the maximal is reached moves to the left: first to the higher and the then to the lower local maximum point (figures 11, 12). We can have a situation where the maximum is reached at both points.

Case 2: 120. It is symmetric with respect to α and β to the previous case. Therefore, the analysis is very similar.

Case 3: 111. It cannot appear at the equilibrium since the system for the second-stage solution is consistent if and only if $t^1 = 0$:

The system for the second-stage solution is consistent if and only if $t^1 = 0$:

$$\begin{cases} \alpha = \max\left\{0, \min\left\{\frac{t^0}{\Delta\beta}, 1\right\}\right\} \\ \beta = \max\left\{0, \min\left\{\frac{t^0 - t^1}{\Delta\alpha}, 1\right\}\right\} \end{cases}.$$

Then, we are back to case 201.

Case 4: 021.

The second-stage solution $\beta(t^0, t^1)$ is defined by $\beta^2 = \frac{t^0 - t^1}{\Delta}$ for $\frac{t^0 - t^1}{\Delta} \in (0,1)$; $\beta = 0$ or 1 if $\frac{t^0 - t^1}{\Delta} < 0$ or $\frac{t^0 - t^1}{\Delta} > 1$ respectively.

At the first stage each lobbying group maximizes its expected payoff:

$$\Pi_{021}^0 = \beta^2 W^0 - 2\beta t^0.$$

Case 5: 030.

The second-stage solution is given by:

$$2\beta^2 (1-\beta) = \frac{t^0 - t^1}{\Delta}.$$

LHS is increasing in t^0 and decreasing in t^1 on the interval $[0, 2/3]$. Similar to the previous case, we take the grid for t^1 and express t^0 as a function of β. The expected payoff of lobby 0 is given by:

$$\Pi^0_{030} = \left(2\beta^2 (1-\beta) + \beta^3\right) W^0 - 3\beta t^0.$$

It is maximized for $\beta \in \left[0, \frac{2}{3}\right]$ since the function $t^0(\beta)$ is increasing on this interval and decreasing on $\left[\frac{2}{3}, 1\right]$.

5 Appendix

5.1 A Technical Lemma

Lemma 1 *Function $U(n_0, m, \alpha) = \sum_{k=m}^{n_0} C^k_{n_0} F(\alpha)^k (1-F(\alpha))^{n_0-k}$,*

where $m < n_0$ is increasing with respect to α and n_0. More precisely,

$$\frac{\partial U(n_0, m, \alpha)}{\partial \alpha} = (n_0 + 1 - m) f(\alpha) C^{m-1}_{n_0} F^{m-1}(\alpha) (1-F(\alpha))^{n_0-m} \tag{24}$$

and

$$U(n_0+1, m, \alpha) - U(n_0, m, \alpha) = C^{m-1}_{n_0} F^m(\alpha) (1-F(\alpha))^{n_0+1-m}. \tag{25}$$

Proof.

$$\begin{aligned}\frac{\partial U(n_0, m, \alpha)}{\partial \alpha} &= f(\alpha) \sum_{i=m}^{n_0-1} C^i_{n_0} \left[k F^{i-1}(\alpha) (1-F(\alpha))^{n_0-i} - (n_0 - i) F^i(\alpha) (1-F(\alpha))^{n_0-i-1}\right] + \\ &\quad + n_0 f(\alpha) F^{n_0-1}(\alpha).\end{aligned}$$

Substituting for

$$k C^i_{n_0} = (n_0 - (i-1)) C^{i-1}_{n_0},$$

one gets that this is equivalent to

$$f(\alpha)\sum_{i=m}^{n_0-1}(n_0-(i-1))\,C_{n_0}^{i-1}F^{i-1}(\alpha)\,(1-F(\alpha))^{n_0-i}-$$
$$f(\alpha)\sum_{i=m}^{n_0-1}C_{n_0}^{i}\,(n_0-i)\,F^{i}(\alpha)\,(1-F(\alpha))^{n_0-i-1}+n_0f(\alpha)F^{n_0-1}(\alpha).$$

In the two sums all terms except the first and the last ones are cancelled out. Thus,

$$\frac{\partial U(n_0,m,\alpha)}{\partial\alpha}=(n_0+1-m)f(\alpha)C_{n_0}^{m-1}F^{m-1}(\alpha)\,(1-F(\alpha))^{n_0-m},$$

which proves (24).

Next,

$$\begin{aligned}U(n_0+1,m,\alpha)-U(n_0,m,\alpha) &= \sum_{i=m}^{n_0+1}C_{n_0+1}^{i}F^{i}(\alpha)\,(1-F(\alpha))^{n_0+1-i}-\\ &\quad-\sum_{i=m}^{n_0}C_{n_0}^{i}F^{i}(\alpha)\,(1-F(\alpha))^{n_0-i}.\end{aligned}$$

After applying

$$C_{n_0+1}^{i}=C_{n_0}^{i}+C_{n_0}^{i-1}$$

the difference is equal to

$$F(\alpha)^{n_0+1}+\sum_{i=m}^{n_0}C_{n_0}^{i}F^{i}(\alpha)\,(1-F(\alpha))^{n_0+1-i}+$$
$$\sum_{i=m}^{n_0}C_{n_0}^{i-1}F^{i}(\alpha)\,(1-F(\alpha))^{n_0+1-i}-\sum_{i=m}^{n_0}C_{n_0}^{i}F^{i}(\alpha)\,(1-F(\alpha))^{n_0-i}.$$

After summing up the first and the third sums it boils down to

$$F(\alpha)^{n_0+1}-\sum_{i=m}^{n_0}C_{n_0}^{i}F^{i+1}(\alpha)\,(1-F(\alpha))^{n_0-i}+\sum_{i=m}^{n_0}C_{n_0}^{i-1}F^{i}(\alpha)\,(1-F(\alpha))^{n_0+1-i}.$$

One can notice that in the two sums all the terms except the first and the last ones are cancelled out. Therefore, we get (25). ■

5.2 Figures

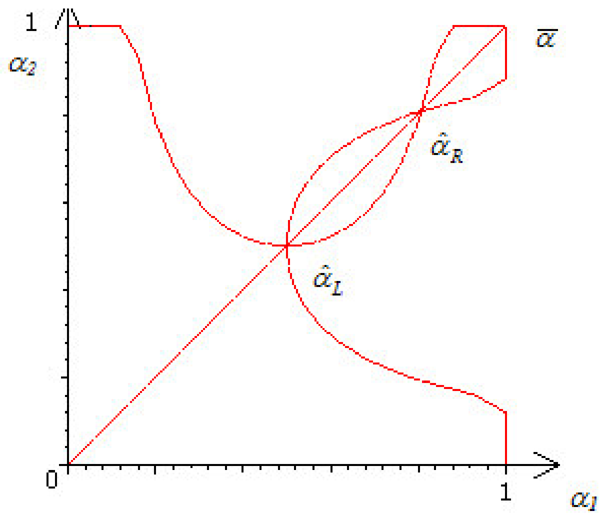

Figure 2: reaction curves $\alpha_1^0(\alpha_2^0)$ and $\alpha_2^0(\alpha_1^0)$ for $\alpha_1^0 = \alpha_2^0 = \alpha_3^0 = \alpha^0$.

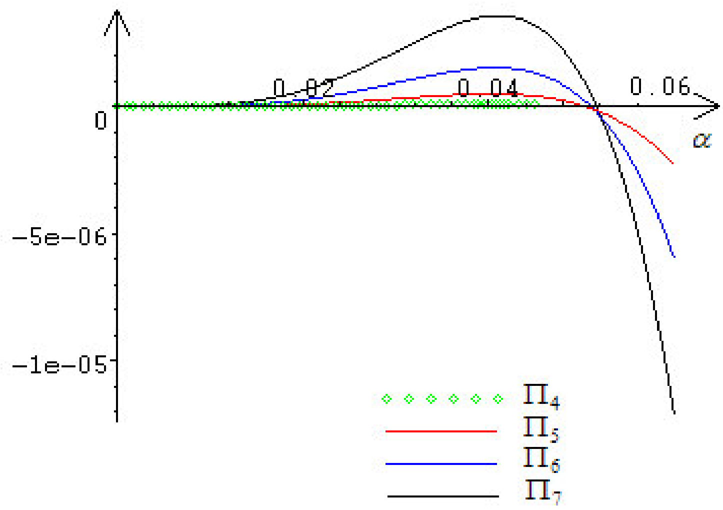

Figure 3: $W_0 = 0.2\Delta$, $n_0 = 7$.

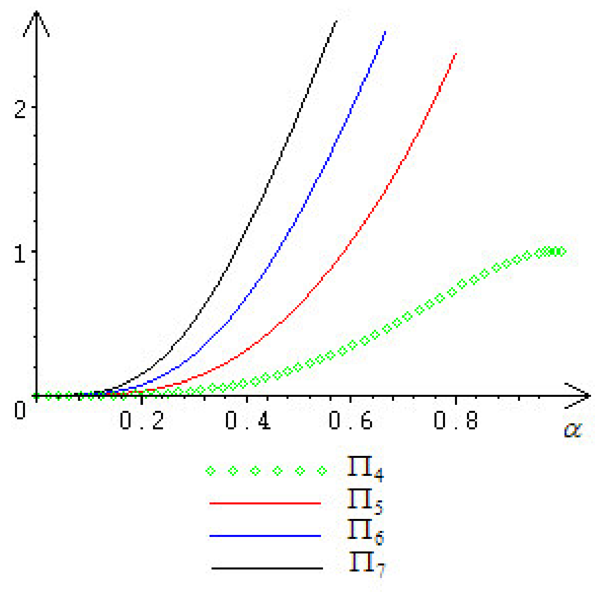

Figure 4: $W_0 = 5\Delta$, $n_0 = 7$.

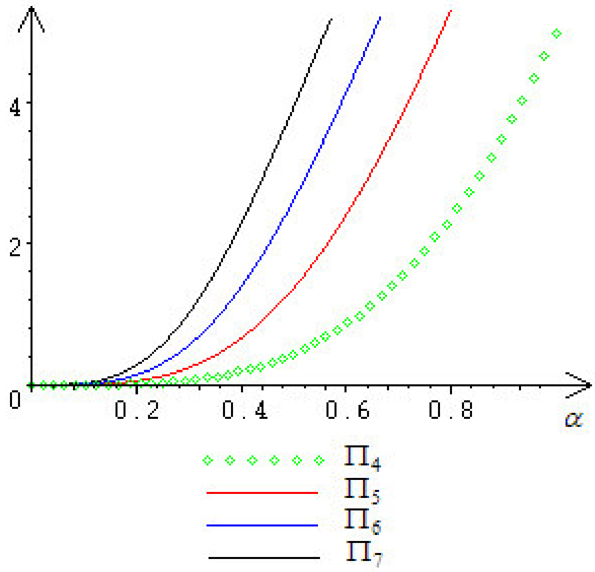

Figure 5: $W_0 = 9\Delta$, $n_0 = 5$.

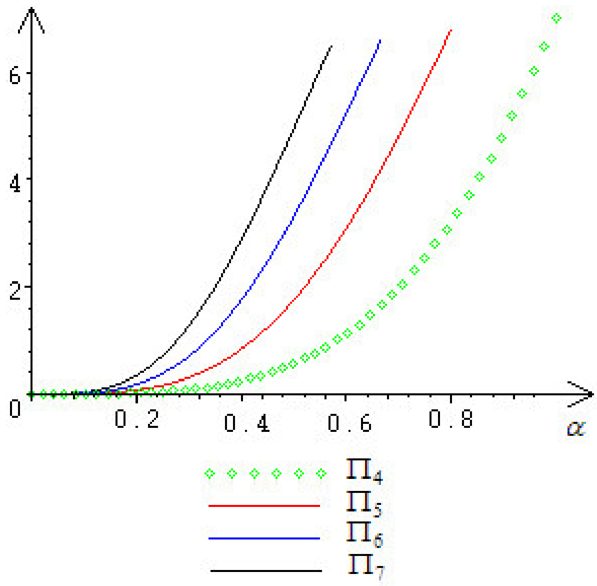

Figure 6: $W_0 = 11\Delta$, $n_0 = 4$ (simple majority).

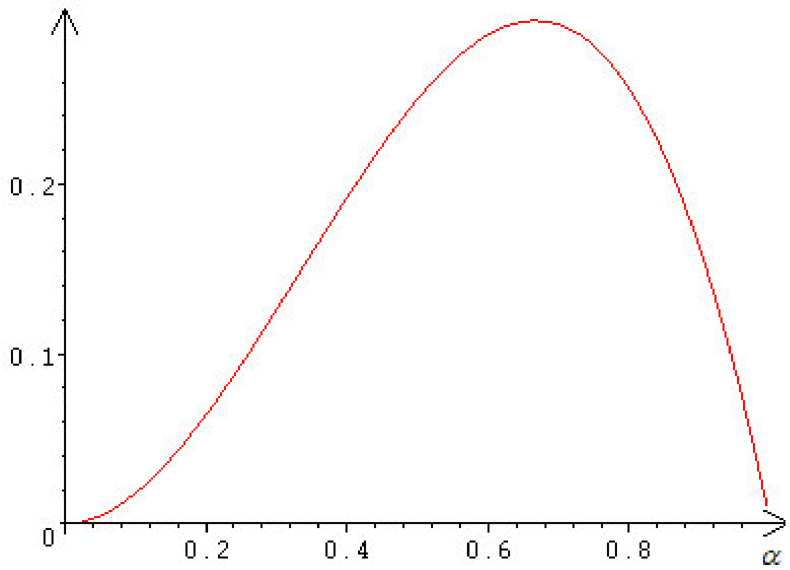

Figure 7: $t^0(\alpha)$ for $t^1 = 0.0001$.

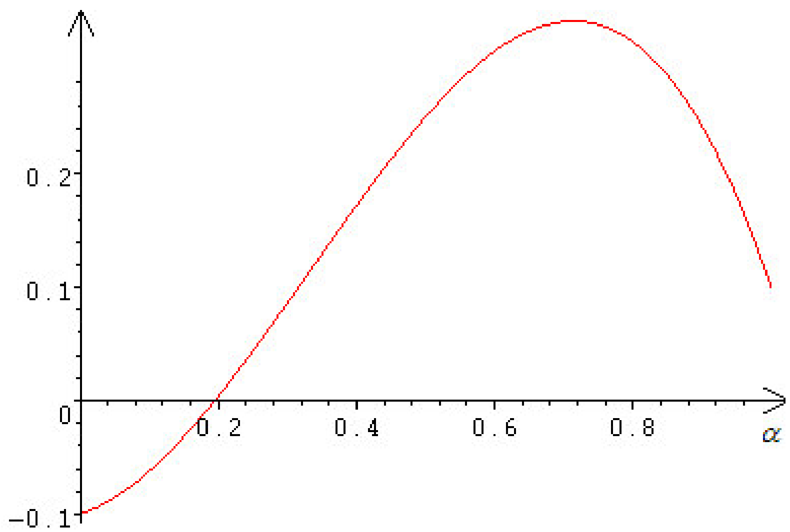

Figure 8: $t^0(\alpha)$ for $t^1 = 0.1$.

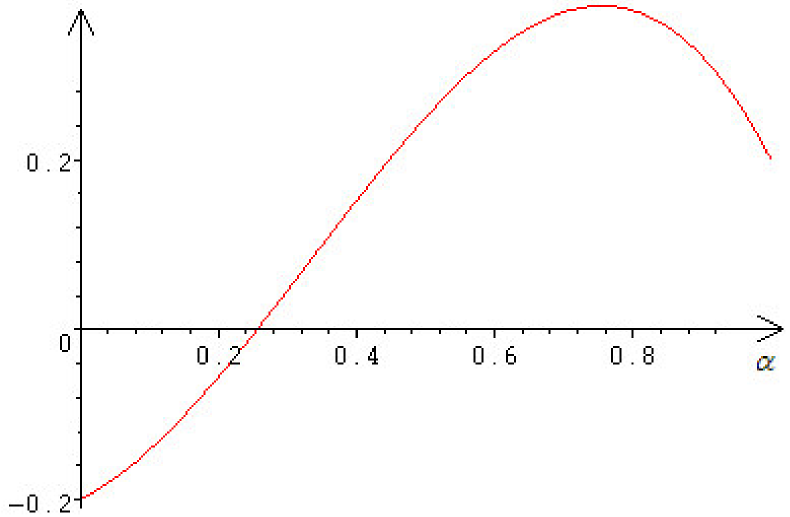

Figure 9: $t^0(\alpha)$ for $t^1 = 0.2$.

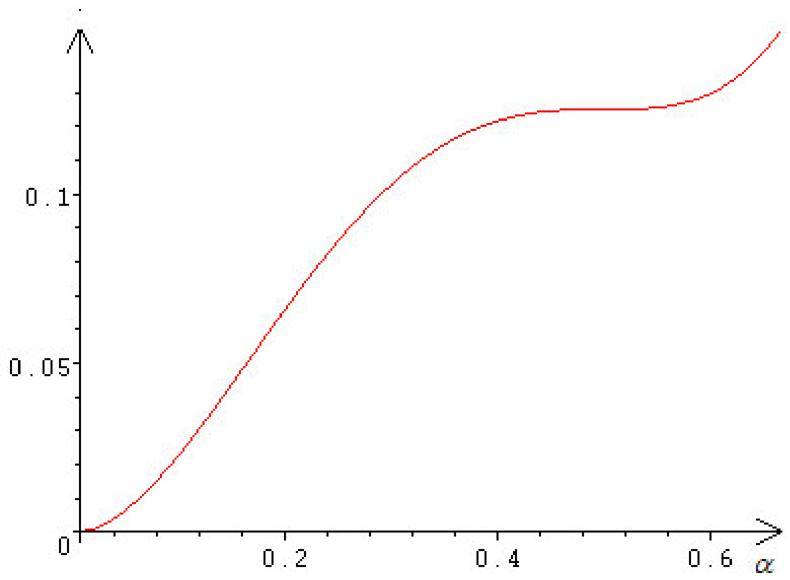

Figure 10: $\Pi^0_{210}(\alpha)$ for $t^1 = 0.0001$.

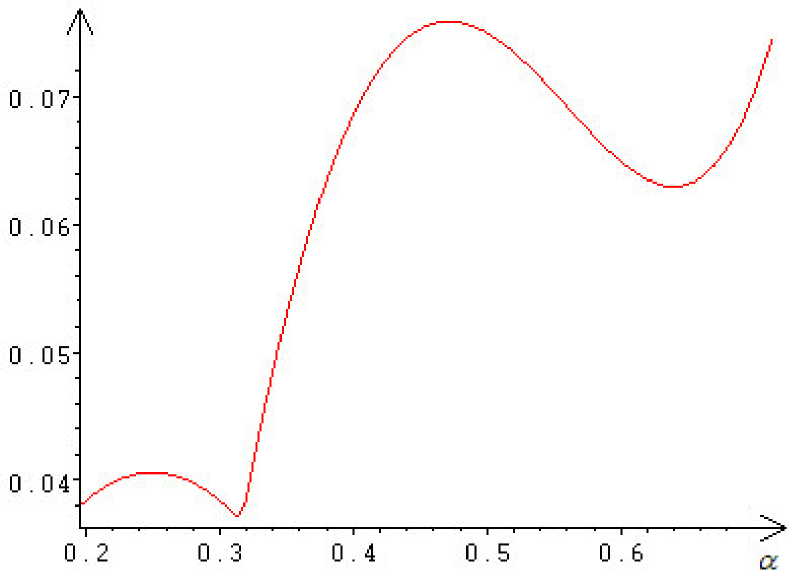

Figure 11: $\Pi^0_{210}(\alpha)$ for $t^1 = 0.1$.

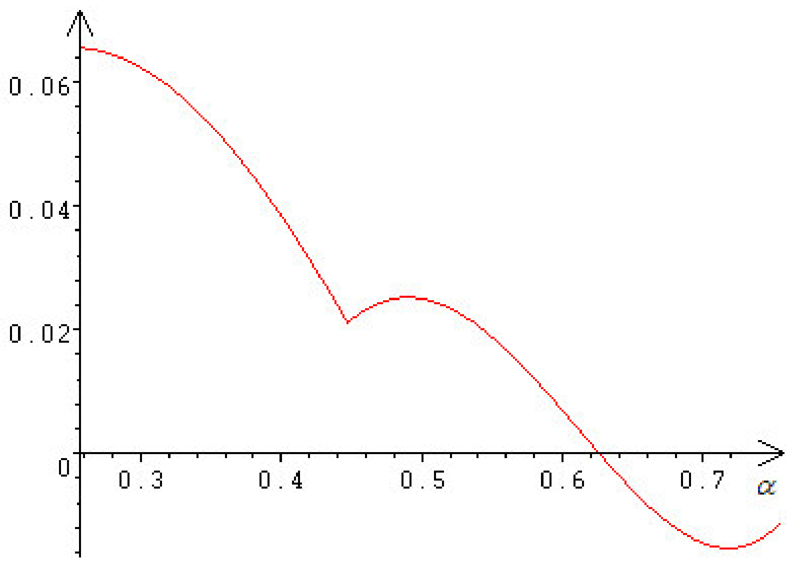

Figure 12: $\Pi^0_{210}(\alpha)$ for $t^1 = 0.2$.

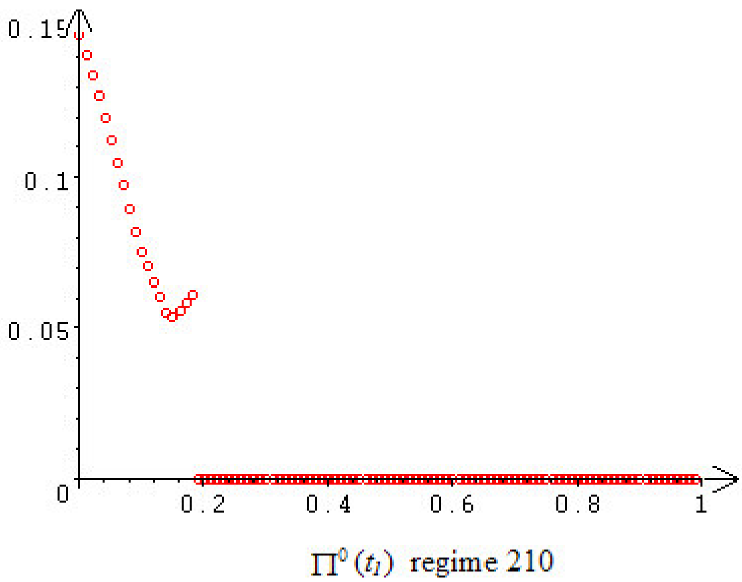

Figure 13: expected payoff of lobby 0 in regime $(\mathbf{1}, \mathbf{t}_1)$.

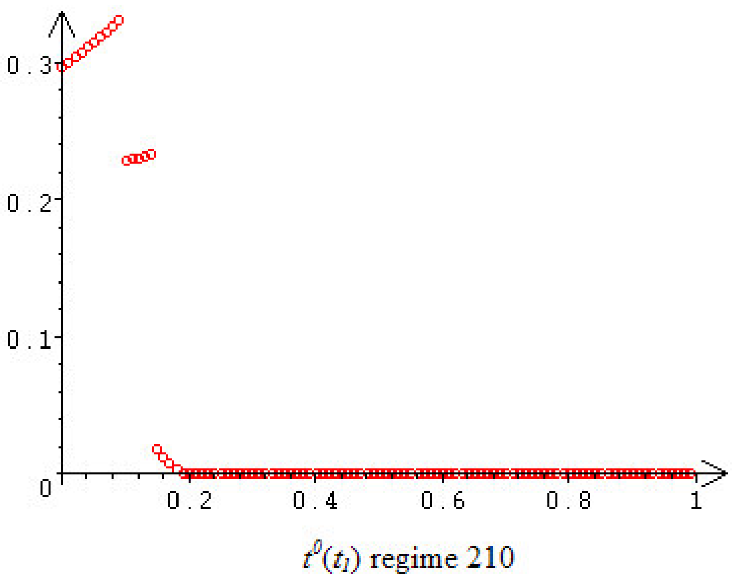

Figure 14: reaction of lobby 0 in regime $(\mathbf{1}, \mathbf{t}_1)$.

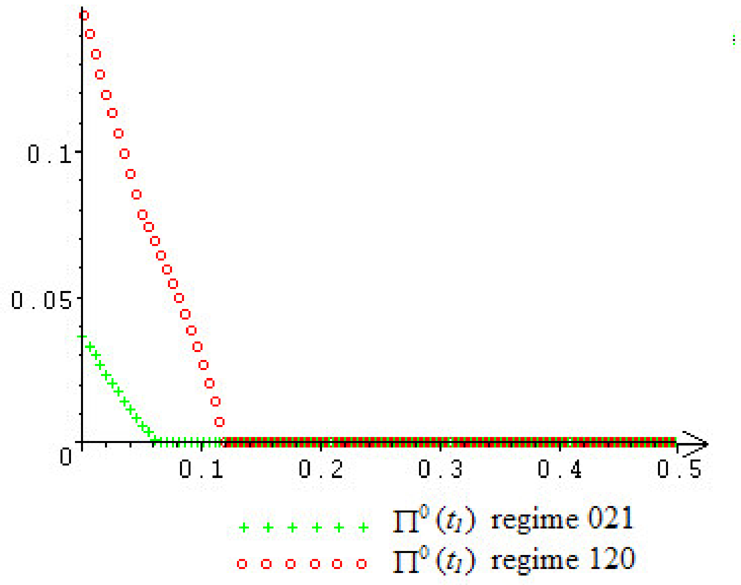

Figure 15: expected payoff of lobby 0 in regime $(\mathbf{2}, \mathbf{t}_1)$.

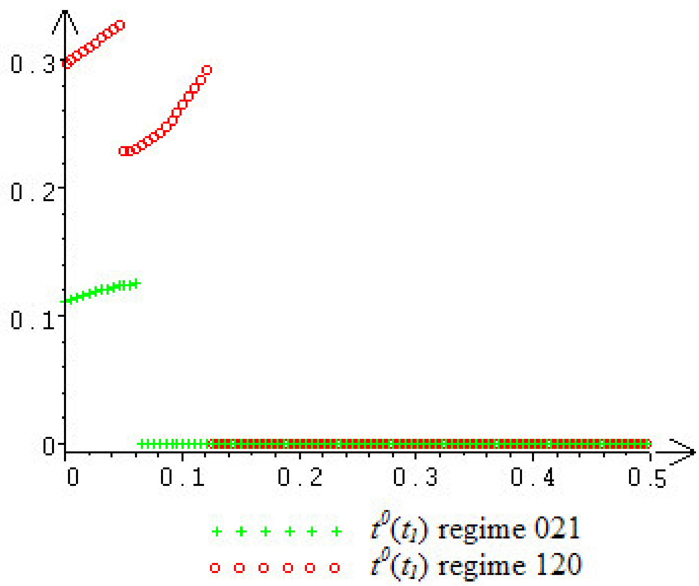

Figure 16: reaction of lobby 0 in regime $(\mathbf{2}, \mathbf{t}_1)$.

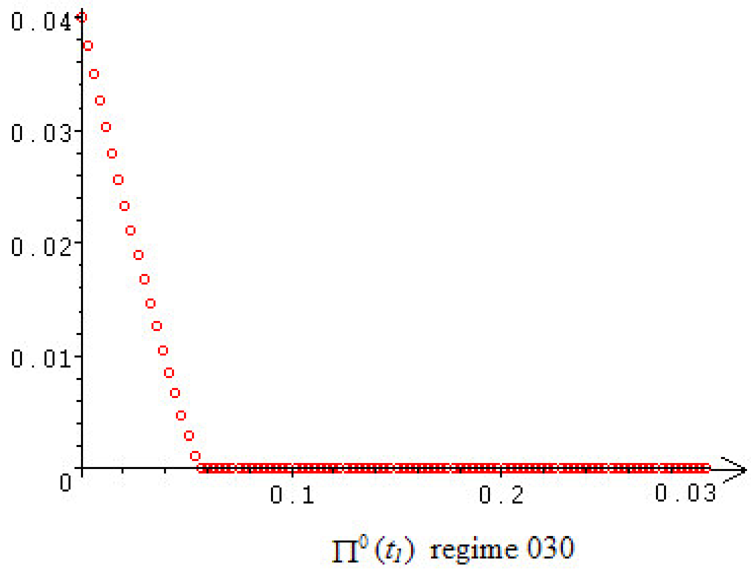

Figure 17: expected payoff of lobby 0 in regime $(\mathbf{3}, \mathbf{t}_1)$.

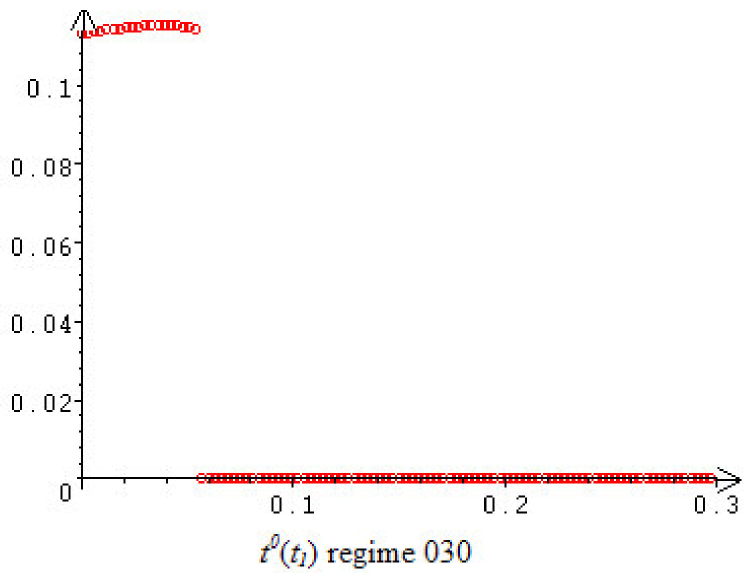

Figure 18: reaction of lobby 0 in regime $(\mathbf{3}, \mathbf{t}_1)$.

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