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Ordering infinite utility streams: maximal anonymity

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Abstract. In ordering infinite utility streams, anonymity and Pareto are considered two basic principles. Anonymity is usually expressed by means of a group of cyclic or Pareto-compatible permutations. Maximal (for inclusion) groups of cyclic permutations involve free ultrafilters on the lattice of partitions of positive integers and are therefore nonconstructible objects. This result is in line with the conjecture of Fleurbaey and Michel (2003) and with the results of Lauwers (2006) and Zame (2007).

1 Introduction

In the literature on ordering infinite utility streams, finite anonymity (ensuring equal treatment of generations) and Pareto (ensuring sensitivity for the interests for each generation) are considered two basic principles. Infinite versions of the utilitarian and of the leximin ordering—e.g. Asheim and Tungodden (2004), Banerjee (2006), Basu and Mitra (2007), Bossert, Sprumont, and Suzumura (2007), Asheim, d'Aspremont, and Banerjee (2008), and Kamaga and Kojima (2009a,b)—do satisfy both basic principles. Also the no-dictatorship axioms of Chichilnisky (2009) have some appeal to these principles. The use of the Chichilnisky-criterion boils down to selecting a Pareto-efficient utility stream under some (finite anonymous) constraint at infinity; see Chichilnisky (2009, Thm 3).

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The imposition of finite anonymity and Pareto, however, already limits the possibilities to order the set of infinite utility streams. Lauwers (2006) and Zame (2007) show the impossibility—as conjectured by Fleurbaey and Michel (2003)—to construct a finite anonymous, Paretian, complete, and transitive binary relation on the set of infinite utility streams.

This note investigates the boundaries of combining different anonymity principles (finite anonymity, fixed step anonymity, ...) and different Pareto principles (weak Pareto, strong Pareto, ...) without insisting on completeness. Furthermore, we restrict the attention to the domain $\{0,1\}^\infty$ of infinite utility streams made up out of zeros and ones.

The motivation to consider this particular domain lies within the finite analogue. Consider the domain $[0,1]^n$ of utility streams of length n. Here, the Pareto axiom has a power equal to 2^{1-n}: the probability that the Pareto axiom is able to rank two randomly selected (uniform distribution) vectors is equal to 2^{1-n}. The combination of anonymity and Pareto has a power equal to $2/(n+1)$. However, when restricted to the set $\{0,1\}^n$ of vectors made up out of zeros and ones, the combination of anonymity and Pareto generates a complete ranking. Indeed, the combination of anonymity and Pareto boils down to counting the number of ones in such a vector. The higher this count, the higher the vector is ranked. The utilitarian, the leximin ordering, or any other ordering on the set $[0,1]^n$ that combines Pareto and anonymity share a common trunk: when restricted to the set $\{0,1\}^n$ they all coincide with the counting procedure. The combination of anonymity and Pareto has a full bite on the set $\{0,1\}^n$ of 0-1-utility vectors.

Inspired by this finite setting, we consider the domain of infinite 0-1-utility streams and focus on what different criteria may have in common. We obtain the following results. *First*, we characterize cyclic permutations as those permutations that are compatible with strong Pareto. Here, we strengthen a result by Mitra and Basu (2007). *Second*, we show that a permutation is cyclic if and only if it does not conflict with either strong or weak Pareto: if a particular anonymity-axiom is compatible with weak (resp. strong) Pareto, then it is also compatible with strong (resp. weak) Pareto. *Third*, we tackle a question posed by Mitra and Basu (2007) and we investigate maximal (for inclusion) groups within the set of cyclic permutations. The strongest anonymity condition compatible with Pareto allows for a complete ordering of the set $\{0,1\}^\infty$ but involves the existence of an ultrafilter on the lattice of partitions of the set of positive integers. Since ultrafilters are nonconstructible objects, this result is in line with those obtained by Lauwers (2006) and Zame (2007).

The next section collects preliminaries on social welfare relations, cyclic permutations, ultrafilters on sets, and ultrafilters on lattices. Lemma 1 and Corollary 1 in subsection 2.2 characterize Pareto-compatible permutations. Section 3 develops the main result: a maximal anonymity condition involves an ultrafilter on the lattice of partitions. Section 4 concentrates on a particular set of permutations: the group of fixed step permutations in the class of variable step permutations. Section 5 concludes.

2 Preliminaries

We recall the notion of a social welfare relation and of a cyclic permutation. We mainly follow Mitra and Basu (2007). Next, we recall the notions of ultrafilters on sets and on lattices. Here, we follow Halbeisen and Löwe (2001).

2.1 Social welfare relations

Let $\mathbb{N}_0 = \{1, 2, 3, \ldots\}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{Q}$ the set of rational numbers. Let $Y \subseteq \mathbb{R}$ be the set of all possible utility levels. We follow Basu and Mitra (2003) and assume that Y has at least two distinct elements, say, 0 and 1. The set $X = Y^{\mathbb{N}_0}$ collects all possible utility streams and is called the domain. An infinite utility stream x is a vector in X. Each x in X can be viewed as a map from $\mathbb{N}_0$ to Y, associating with each t in $\mathbb{N}_0$ the element x_t in Y. Each utility stream x in $\{0,1\}^{\mathbb{N}_0}$ is identified with the subset $\{\, t \mid x_t = 1 \,\}$ of $\mathbb{N}_0$. Let $\mathcal{S}$ collect all subsets of $\mathbb{N}_0$. Due to the identification of subsets of $\mathbb{N}_0$ with their indicator functions, we abuse language and say that $\mathcal{S}$ is a subset of X. Vector inequalities are denoted $\leq$, $<$, and $\ll$. Set inclusions are denoted $\subseteq$ and $\subset$.

A social welfare relation (SWR) is a reflexive and transitive binary relation in the domain X. The symmetric and the asymmetric component of the SWR $\precsim$ are denoted by $\sim$ and $\prec$. The SWR $\precsim_1$ is a subrelation to a SWR $\precsim_2$ if for each x and y in X we have (i) $x \precsim_1 y$ implies $x \precsim_2 y$ and (ii) $x \prec_1 y$ implies $x \prec_2 y$.

A permutation π on $\mathbb{N}_0$ is a one-to-one map from $\mathbb{N}_0$ to $\mathbb{N}_0$. For each x in X, the composite map $x \circ \pi$ is a map from $\mathbb{N}_0$ to Y and can be written as the infinite utility stream

$$x \circ \pi \;=\; \big(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(t)}, \ldots\big)\,.$$

Let $\mathrm{Sym}(\mathbb{N}_0)$ collect all permutations on $\mathbb{N}_0$. The set $\mathrm{Sym}(\mathbb{N}_0)$ when equipped with the composition operation becomes a group. The next definition collects three infinite versions of the Pareto axiom and one concept related to permutations.

Definition.

- A SWR $\precsim$ satisfies the *Pareto* axiom if for each x and y in X we have that $x < y$ implies $x \prec y$.
- A SWR $\precsim$ satisfies the *weak Pareto* axiom if for each x and y in X,
  - we have that $x \leq y$ implies $x \precsim y$,
  - and that $x \ll y$ implies $x \prec y$.
- A SWR $\precsim$ satisfies the *intermediate Pareto* axiom if for each x and y in X,
  - we have that $x \leq y$ implies $x \precsim y$,
  - and that $x < y$ and $x_i < y_i$ for infinitely many i in $\mathbb{N}_0$ implies $x \prec y$.

- Let $\mathcal{Q}$ be a class of permutations. A SWR $\precsim$ satisfies *$\mathcal{Q}$-anonymity* if for each π in $\mathcal{Q}$ and for each x in X we have $x \sim x \circ \pi$.

The Pareto axiom, also known as the strong Pareto axiom, postulates sensitivity in each coordinate. The intermediate Pareto axiom postulates sensitivity in each infinite set of coordinates. This intermediate version is useful when ranking subsets of $\mathbb{N}_0$, where the imposition of weak Pareto only demands that the full set $\mathbb{N}_0$ is strictly larger than the empty set $\varnothing$. The intermediate Pareto axiom occurs in Crespo et al (2009) as the infinite Pareto principle. With respect to anonymity, we only consider classes of permutations that include the group of finite permutations. Hereby, the permutation π is said to be finite if there exists a T in $\mathbb{N}_0$ such that $\pi(t) = t$ for each $t \geq T$. Let $\mathcal{Q}_{\text{fn}}$ collect all finite permutations. A SWR is said to be finite anonymous if it satisfies $\mathcal{Q}_{\text{fn}}$-anonymity. Finally, let $(\mathcal{Q}, \circ)$ be a group of permutations. The following relation is denoted by $\precsim_{\mathcal{Q}}$. For each x and y in X, we have

$$x \precsim_{\mathcal{Q}} y \qquad \text{if and only if} \qquad \text{there is a } \pi \text{ in } \mathcal{Q} \text{ such that } x \circ \pi \leq y.$$

This relation is $\mathcal{Q}$-anonymous, reflexive (the identity permutation belongs to the group $\mathcal{Q}$), and transitive (the group $\mathcal{Q}$ is closed under composition).

2.2 Cyclic permutations

Let π be a permutation on the set $\mathbb{N}_0$. The vector $(k, \pi(k), \pi^2(k), \pi^3(k), \ldots)$ is said to be the cycle generated by π on k. Each permutation can be written as a succession of cycles on disjoint sets (Hall, 1976, Chapter 5). For example, the permutation

$$\pi_1 = (1,2)(3,4)(5,6)\cdots(2n-1,2n)\cdots$$

switches the odd and even numbers, for each n in $\mathbb{N}_0$ the number $2n-1$ is mapped upon $2n$ and $2n$ is mapped upon $2n-1$. The final element in a cycle is mapped upon the first element in that cycle. The permutation

$$\pi_2 = (1)(2,3)(4,5)\cdots(2n,2n+1)\cdots$$

keeps the number 1 fixed and then switches the even and odd numbers. A permutation on $\mathbb{N}_0$ might generate a cycle of infinite length. The permutation

$$\pi_3 = (\ldots,9,7,5,3,1,2,4,6,8,\ldots)$$

maps 1 upon 2. Furthermore, π_3 maps an even number upon its even successor and an odd number upon its odd predecessor, as such $\pi_3(123) = 121$ and $\pi_3(100) = 102$. We keep the references π_1, π_2, and π_3 throughout this note.

The decomposition of a permutation into pairwise disjoint cycles is unique, except for the order in which the cycles are written, also within each cycle the numbers are allowed to be permuted cyclically. For example, the permutations $(1,2)(3)(4,5,6,7)$ and $(3)(1,2)(5,6,7,4)$ coincide.

A permutation representable by an infinite sequence of finite cycles is said to be *cyclic*. Alternatively (Mitra and Basu, 2007), the permutation π is cyclic if for each n in $\mathbb{N}_0$ there exists a k in $\mathbb{N}_0$ such that

$$\pi^k(n) \;=\; \underbrace{\pi \circ \pi \circ \cdots \circ \pi}_{k \text{ times}}(n) \;=\; n.$$

The period k might be different for different values of n. Hence, a cyclic permutation π is non-wandering in the sense that for each n in $\mathbb{N}_0$ the sequence $\pi(n)$, $\pi^2(n)$, $\pi^3(n)$, ... returns to n after a finite numbers of iterations.

Each permutation partitions the set $\mathbb{N}_0$: present the permutation as a juxta position of cycles and replace the brackets (and) by { and }. Each cyclic permutation partitions the set $\mathbb{N}_0$ into an infinite sequence of finite sets. For example, the partition induced by the permutation π_1 is equal to

$$\mathrm{Part}(\pi_1) \;=\; \Big\{ \{1,2\}\,, \{3,4\}\,, \ldots,\; \{2n-1, 2n\}\,, \ldots \Big\}.$$

The set of all cyclic permutations is denoted by $\mathcal{P}$. Obviously, finite permutations are cyclic. The next lemma highlights the main motivation to study cyclic permutations. The lemma already appeared in Mitra and Basu (2007, Lemma 1). Their proof presupposes that the domain X is sufficiently rich and uses coordinatewise convergent sequences of infinite utility streams. The proof below only uses 0-1-utility streams and therefore strengthens their result. Corollary 1 rephrases Lemma 1 in terms of subsets of $\mathbb{N}_0$.

Lemma 1. A permutation π is cyclic if and only if there is no x in X satisfying $x < x \circ \pi$.

Proof. The only-if-part is straightforward. If the permutation π is cyclic, then it can be decomposed as an infinite juxta position of permutations on finite sets. Each permutation on a finite set is unable to conflict with the Pareto principle.

The if-part (if there is no conflict with Pareto, then the permutation is cyclic) is done by contraposition. Hence, consider a permutation π with an infinite cycle at m in $\mathbb{N}_0$:

$$(\ldots, \pi^{-4}(m), \pi^{-3}(m), \pi^{-2}(m), \pi^{-1}(m), m, \pi^1(m), \pi^2(m), \pi^3(m), \pi^4(m), \ldots).$$

Relabel this cycle (let 1 denote m) to obtain the cycle π_3 and consider the following table:

$$\begin{array}{rlllllllllll}
\pi_3 = & (\ldots, & 9, & 7, & 5, & 3, & 1, & 2, & 4, & 6, & 8, & \ldots), \\
x = & (\ldots, & 0, & 0, & 0, & 0, & 0, & 1, & 1, & 1, & 1, & \ldots), \\
y = x \circ \pi_3 = & (\ldots, & 0, & 0, & 0, & 0, & 1, & 1, & 1, & 1, & 1, & \ldots).
\end{array}$$

The first line in this table is a cycle of infinite length. The second line presents an infinitely long utility stream in X. This utility stream is made up of two sequences, a sequence of 'ones' is attached to the even positions ($x_{2n} = 1$) and a sequence of zeros is attached to the odd positions ($x_{2n-1} = 0$). The final line presents the permuted utility stream $y = x \circ \pi_3$ (recall that $y_i = x_{\pi(i)}$). The utility stream y dominates x (indeed, $x_1 < y_1$). $\square$

Corollary 1. A permutation π on $\mathbb{N}_0$ is cyclic if and only if there does not exist a subset S of $\mathbb{N}_0$ satisfying the strict inclusion $S \subset \pi(S)$.

In case the domain $X = Y^{\mathbb{N}_0}$ is sufficiently rich ($\mathbb{Q} \cap [0,2] \subseteq Y$), the infinite cycle π_3 generates a stronger domination result. There exists a stream z such that $z \ll (z \circ \pi_3)$:

$$\begin{array}{rl} \pi_3 = & (\ldots, \; 9, \; 7, \; 5, \; 3, \; 1, \; 2, \; 4, \; 6, \; 8, \; \ldots), \\ z = & (\ldots, \; \tfrac{1}{9}, \; \tfrac{1}{7}, \; \tfrac{1}{5}, \; \tfrac{1}{3}, \; 1, \; 2\text{-}\tfrac{1}{2}, \; 2\text{-}\tfrac{1}{4}, \; 2\text{-}\tfrac{1}{6}, \; 2\text{-}\tfrac{1}{8}, \; \ldots), \\ z \circ \pi_3 = & (\ldots, \; \tfrac{1}{7}, \; \tfrac{1}{5}, \; \tfrac{1}{3}, \; 1, \; 2\text{-}\tfrac{1}{2}, \; 2\text{-}\tfrac{1}{4}, \; 2\text{-}\tfrac{1}{6}, \; 2\text{-}\tfrac{1}{8}, \; 2\text{-}\tfrac{1}{10}, \; \ldots). \end{array}$$

Lemma 1, thus, holds when Pareto is weakened to intermediate Pareto or weak Pareto.

Corollary 2. Let Y include $\mathbb{Q} \cap [0,2]$. A permutation π is cyclic if and only if there is no utility stream x in $X = Y^{\mathbb{N}_0}$ satisfying $x \ll x \circ \pi$.

Lemma 1 and Corollary 2 indicate that each infinite cycle conflicts with the Pareto axioms. We summarize. Let Y be sufficiently rich, let $\mathcal{Q}$ be a class of permutations, let $\precsim$ be a SWR. Then,

$\mathcal{Q}$-anonymity and Pareto are compatible	$\Longleftrightarrow$	$\mathcal{Q}$-anonymity and weak Pareto are compatible	$\Longleftrightarrow$	the class $\mathcal{Q}$ only contains cyclic permutations.

Within the class of transitive and reflexive relations, there is no trade-off between the three Pareto axioms and anonymity: if Q-anonymity is compatible with one infinite version of the Pareto axiom, then it is compatible with all three infinite versions of Pareto (as listed in Subsection 2.1, Definition). Furthermore, if $\mathcal{Q}$ is a group of cyclic permutations, then the social welfare relation $\precsim_{\mathcal{Q}}$ satisfies

- $x \sim_{\mathcal{Q}} y$ if and only if there exists a π in $\mathcal{Q}$ such that $x \circ \pi = y$, and
- $x \prec_{\mathcal{Q}} y$ if and only if there exists a π in $\mathcal{Q}$ such that $x \circ \pi < y$.

Let us verify the first item. Suppose that both $x \precsim_{\mathcal{Q}} y$ and $y \precsim_{\mathcal{Q}} x$ hold. Then there exist two permutations π and σ in $\mathcal{Q}$ such that $x \circ \pi \leq y$ and $y \circ \sigma \leq x$. Therefore,

$$x \circ \pi \circ \sigma \; \leq \; y \circ \sigma \; \leq \; x.$$

Since the permutation $\pi \circ \sigma$ is cyclic, the inequalities become equalities and $y \circ \sigma = x$.

Finally, there exist evaluations on the set of infinite utility streams that combine monotonicity and $\mathrm{Sym}(\mathbb{N}_0)$-anonymity. For example, the map

$$\liminf : Y \longrightarrow \mathbb{R} : x \longmapsto \liminf(x)$$

is $\mathrm{Sym}(\mathbb{N}_0)$-anonymous and monotonic (if $x \leq y$, then $\liminf(x) \leq \liminf(y)$). This map looks for the smallest accumulation point within a utility stream and—hence—violates Pareto: the utility streams $x = (0, 0, \ldots, 0, \ldots)$ and $y = (1, 1/2, \ldots, 1/k, \ldots)$ both converge to zero, while $x \ll y$. Also $\limsup$ is monotonic and $\mathrm{Sym}(\mathbb{N}_0)$-anonymous. Chambers (2009) provides a characterization of $\liminf$ and of $\limsup$.

2.3 Filters on sets

Let S be a set. A filter on S is a nonempty family $\mathcal{F}$ of subsets of S that satisfies

- $\varnothing$ is not in $\mathcal{F}$,
- if A and B are in $\mathcal{F}$, then $A \cap B$ is in $\mathcal{F}$,
- if A is in $\mathcal{F}$ and $A \subseteq B$, then B is in $\mathcal{F}$.

If, in addition,

- for each $A \subseteq S$, either $A \in \mathcal{F}$ or $S - A \in \mathcal{F}$,

then $\mathcal{F}$ is an ultrafilter. An ultrafilter is a filter that is maximal for inclusion. For example, the family of all cofinite subsets of S (i.e. those subsets of S whose complements are finite) is a filter on S. The family of all subsets of S that contain a given element s of S is an ultrafilter on S and is said to be principal. An ultrafilter is principal as soon it contains a finite set. An ultrafilter that is not principal is said to be free. The intersection $\cap_{\mathcal{F}} A$ of a free ultrafilter $\mathcal{F}$ is the empty set.

A family $\mathcal{F}$ of subsets of S satisfies the finite intersection property if $A_1, A_2, \ldots, A_n \in \mathcal{F}_0$ implies $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{F}_0$. If one adds to $\mathcal{F}_0$ all the sets $B \subseteq S$ that contain finite intersections $A_1 \cap A_2 \cap \ldots \cap A_n$ of elements of $\mathcal{F}_0$, then one obtains a filter $\mathcal{F}_1$. By Zorn's lemma (which is equivalent to the Axiom of Choice) there exists a maximal filter $\mathcal{F}$ on S that includes $\mathcal{F}_1$. This maximal filter $\mathcal{F}$ is an ultrafilter on S. The nonconstructiveness of free ultrafilters is well known. Jehne and Klinge (1977, p209), for example, state that free ultrafilters on the set of positive integers are so highly unconstructive that they cannot be distinguished from one another.

Let $\mathcal{F}$ be a filter on $\mathbb{N}_0$ and let each element in $\mathcal{F}$ be infinite. Consider the following relation $\precsim_{\mathcal{F}}$ on the collection $\mathcal{S}$ of subsets of $\mathbb{N}_0$. For each S and T in $\mathcal{S}$ we have

$$S \precsim_{\mathcal{F}} T \quad \text{if and only if} \quad \Big\{ t \in \mathbb{N}_0 \;\Big|\; |\, S \cap \{1, 2, \ldots, t\} \,| \;\leq\; |\, T \cap \{1, 2, \ldots, t\} \,| \Big\} \in \mathcal{F}.$$ [1]

This relation is reflexive, transitive, and Paretian. In addition, this relation satisfies an anonymity principle that is stronger than finite anonymity. In case $\mathcal{F}$ is a free ultrafilter, then the relation $\precsim_{\mathcal{F}}$ on $\mathcal{S}$ is complete.

2.4 Filters on the lattice of partitions

The notion of a filter on sets extends to a filter on a lattice of partitions. We follow Halbeisen and Löwe (2001) and recall the definitions and some results.

A partition of $\mathbb{N}_0$ is a family of pairwise disjoint nonempty sets such that their union coincides with $\mathbb{N}_0$. If A and B are two partitions of $\mathbb{N}_0$, we say that A is coarser than B (or that B is finer than A) and we write $A \sqsubseteq B$ if each piece in A is a union of pieces of B.

[1] The cardinality of a set S is denoted by $|S|$.

The coarsest partition of $\mathbb{N}_0$ (everything in one piece) is denoted by $0 = \{\mathbb{N}_0\}$, the finest partition (all pieces of which are singletons) by 1. Each partition is in between 0 and 1.

Let Ω_0 collect those partitions of $\mathbb{N}_0$ that consist out of infinitely many finite pieces. Partitions containing one (or more) infinite piece(s) are not distinguished, they are denoted by 0. We endow the class $\Omega = \Omega_0 \cup \{0\}$ with two operations $\cup$ and $\cap$. The partition $A \cup B$ is the coarsest partition in Ω that refines A and B, and the partition $A \cap B$ is the finest partition in Ω that is coarser than A and B. In case the partition $A \cap B$ contains an infinite piece, we put $A \cap B$ equal to 0. As such $(\Omega, \sqsubseteq)$ is a lattice.

A filter on the lattice $(\Omega, \sqsubseteq)$ is a collection $\mathcal{F}$ of members of Ω that satisfies

- 0 is not in $\mathcal{F}$,
- if both A and B are in $\mathcal{F}$, then $A \cap B$ is in $\mathcal{F}$,
- if B is in $\mathcal{F}$ and $B \sqsubseteq A$ (with A in Ω), then A is in $\mathcal{F}$.

A family $\mathcal{B} \subseteq \Omega$ is said to be a filter base if (i) $0 \notin \mathcal{B}$, and (ii) for each A_1 and A_2 in $\mathcal{B}$, there is a B in $\mathcal{B}$ such that $B \sqsubseteq A_1 \cap A_2$. In case $\mathcal{B}$ is a filter base, then the family $\mathcal{B}^+ = \{A \in \Omega |$ there is a B in $\mathcal{B}$ such that $B \sqsubseteq A\}$ is a filter on the lattice $(\Omega, \sqsubseteq)$. The filter $\mathcal{B}^+$ coincides with the intersection of all filters that include $\mathcal{B}$.

A filter that is maximal for inclusion is said to be an ultrafilter. Each ultrafilter $\mathcal{F}$ on $(\Omega, \sqsubseteq)$ is free, i.e. $\bigcap \{ A \,|\, A \in \mathcal{F} \} = 0$. We recall two facts on ultrafilters (Facts 2.1-2 in Halbeisen and Löwe, 2001, p321).

- A family $\mathcal{F}$ is an ultrafilter on $(\Omega, \sqsubseteq)$ if and only if for each A in Ω either $A \in \mathcal{F}$ or there is a B in $\mathcal{F}$ such that $A \cap B = 0$ (the 'either-or' being exclusive).
- If F is a family of elements of Ω with the finite intersection property (for each finite subfamily $\{ A_1, A_2, \ldots, A_n \} \subseteq F$ we have $A_1 \cap A_2 \cap \ldots \cap A_n \neq 0$), then there is an ultrafilter $\mathcal{F}$ on $(\Omega, \sqsubseteq)$ with $F \subseteq \mathcal{F}$.

The second fact is implied by Zorn's lemma. Similar to the existence of a free ultrafilter on a set, also the existence of an ultrafilter on the lattice of partitions involves the use of the Axiom of Choice in set theory. The notion "ultrafilter on a lattice" generalizes the notion "free ultrafilter on a set". The next example clarifies this statement.

Example. Each infinite subset $S = \{n_1, n_2, \ldots, n_k, \ldots\}$ of $\mathbb{N}_0$ induces a partition V_S in Ω as follows:

$$V_S \;=\; \Big\{ [1, n_1], [n_1+1, n_2], \ldots, [n_k+1, n_{k+1}], \ldots \Big\},$$

where $[i, j]$ with $i \leq j$ is a shorthand for the set $\{i, i+1, \ldots, j-1, j\} \subset \mathbb{N}_0$. Now, let $\mathcal{F}_{\mathbb{N}_0}$ be a free filter on the set $\mathbb{N}_0$. Then, the family

$$\mathcal{F}_\Omega \;=\; \Big\{ V_S \in \Omega \,\Big|\, S \in \mathcal{F}_{\mathbb{N}_0} \Big\}$$

is a filter on the lattice $(\Omega, \sqsubseteq)$. Moreover, $\mathcal{F}_\Omega$ is an ultrafilter on the lattice $(\Omega, \sqsubseteq)$ if and only if $\mathcal{F}_{\mathbb{N}_0}$ is a free ultrafilter on the set $\mathbb{N}_0$.

3 Maximal anonymity

This section develops the main result. We start with some additional notation. Let the partition $A = \{ N_1, N_2, \ldots, N_k, \ldots \}$ belong to Ω_0. We will refer to

$$\mathrm{Sym}(A) \;=\; \mathrm{Sym}(N_1) \times \mathrm{Sym}(N_2) \times \cdots \times \mathrm{Sym}(N_k) \times \cdots,$$

with $\mathrm{Sym}(N_k)$ the group of all permutations on the finite set N_k, as the symmetric group of the partition A. The group $\mathrm{Sym}(A)$ stabilizes the partition A, i.e. this group collects all the permutations with an induced partition that is equal to or finer than A. We shorten $\mathrm{Sym}(\mathrm{Part}(\pi))$ to $\mathrm{Sym}(\pi)$. A group $\mathcal{Q}$ of permutations that includes $\mathrm{Sym}(\pi)$ for each π in $\mathcal{Q}$ is said to be a *partition group*.

Let π belong to a partition group $\mathcal{Q}$ of cyclic permutations, then $\mathcal{Q}$-anonymity imposes indifference between a utility stream x, the permuted stream $y = x \circ \pi$, and all the streams obtained from x through rearrangements within the cycles of π. For example, the partition group $\mathrm{Sym}(\pi_1)$ contains each permutation of the form

$$(1,2)^{k_1}\,(3,4)^{k_2}\,\cdots(2n-1,2n)^{k_n}\,\cdots$$

with k_i either 1 or 0 (where $(a,b)^1 = (a,b)$ and $(a,b)^0 = (a)(b)$). Therefore, if we impose $\mathrm{Sym}(\pi_1)$-anonymity, then the utility streams

$$x = \underbrace{1,0}\,,\underbrace{1,0}\,,\ldots,\underbrace{1,0}\,,\ldots \qquad \text{and} \qquad y = \underbrace{0,1}\,,\underbrace{0,1}\,,\ldots,\underbrace{0,1}\,,\ldots,$$

become equally good. In addition, for each subset S of $\mathbb{N}_0$, the utility stream x_S obtained from x by switching the utilities in two subsequent positions $2n-1$ and $2n$ for each n in S, is equally good as x (and y). The move from x to x_S involves 'less' switches than the move from x towards y. In this case, indifference between x and x_S can be interpreted as a 'weaker' demand than indifference between x and y. The next lemma indicates that partition groups allow us to shift the focus from permutations towards partitions.

Lemma 2. Let σ_A and σ_B be two cyclic permutations on $\mathbb{N}_0$. Then, $\mathrm{Sym}(\sigma_B)$ contains a permutation ρ such that $\rho \circ \sigma_A$ generates the partition $\mathrm{Part}(\sigma_A) \cap \mathrm{Part}(\sigma_B)$.

Proof. Denote $A = \mathrm{Part}(\sigma_A)$ and $B = \mathrm{Part}(\sigma_B)$. We prove the lemma in case $C = A \cap B$ consists out of an infinite number of finite sets. In case the partition C contains an infinite piece, the same ideas apply.

Without loss (otherwise re-enumerate $\mathbb{N}_0$), we assume the existence of an increasing sequence $n_1, n_2, \ldots, n_k, \ldots$ in $\mathbb{N}_0$ such that the partition C can be written as

$$C \;=\; \Big\{ \underbrace{[1,n_1]}_{S}, [n_1+1,n_2], \ldots, [n_k+1,n_{k+1}], \ldots \Big\}.$$

Both A and B are finer than C. We focus on one of the pieces in C, say $S = [1, n_1]$. Again, without loss, we assume that the restriction of σ_A to S is as follows

$$\sigma_A|_S \;=\; (1,2,\ldots,k_1)(k_1+1,k_1+2,\ldots,k_2)\cdots(k_{m-1}+1,k_{m-1}+2,\ldots,n_1).$$

Denote the partition classes by $S_1 = [1, k_1], S_2 = [k_1 + 1, k_2], \ldots, S_m = [k_{m-1} + 1, n_1]$.
We construct a permutation ρ in $\mathrm{Sym}(B|_S)$ by induction. The partition $A \cap B$—when restricted to S—is equal to S. Hence, there exists a couple (ℓ_1, ℓ^1) in $S_1 \times (S - S_1)$ both belonging to one piece of B. Put $\rho(\ell_1) = (\ell^1)$. Let ℓ^1 belong to $S^1 = S_i$. Move on to the set $S_2 = S_1 \cup S^1$. Again, there exists a couple (ℓ_2, ℓ^2) in $S_2 \times (S - S_2)$ that both belong to one piece of B. Put $\rho(\ell_2) = \ell^2$. This procedure ends after m steps. Put the permutation ρ equal to $(\ell_1, \ell^1)(\ell_2, \ell^2) \cdots (\ell_m, \ell^m)$, elements of S that are not listed remain fixed.
The permutation $\rho \circ \sigma_A$ generates the cycle S in one piece. Repeat the whole construction for the other pieces in C and paste together the corresponding permutations to obtain the result. □

In general, only the relation $\mathrm{Part}(\sigma_1) \cap \mathrm{Part}(\sigma_2) \sqsubseteq \mathrm{Part}(\sigma_1 \circ \sigma_2)$ holds. For example, consider the following cyclic permutations:

$$\begin{aligned}
\sigma_1 = \quad & (1)(2,3,5,6,7,4)(8,11,13,14,15,12,10,9)\underline{(16,19,21,22,23,20,18,17)}\cdots, \\
\sigma_2 = \quad & (1,2,3)(4,8,10,11,7,5)(6)(9)(12,16,18,19,15,13)(14)(17)\underline{(20,24,26,27,23,21)(22)(25)}\cdots.
\end{aligned}$$

The representation continues by repeating the underlined cycles taking into account a shift of $+8$. Here, $\mathrm{Part}(\sigma_1) \cap \mathrm{Part}(\sigma_2) = \mathbb{N}_0$ while both compositions $\sigma_2 \circ \sigma_1$ and $\sigma_1 \circ \sigma_2$ are cyclic:

$$\begin{aligned}
&\sigma_2 \circ \sigma_1 = \pi_1 = (1,2)(3,4)(5,6)(7,8)\cdots, \text{ and} \\
&\sigma_1 \circ \sigma_2 = (1,3)(2,5)(4,11)(6,7)(8,9)(10,13)(12,19)(14,15)(16,17)(18,21)\cdots.
\end{aligned}$$

We continue with some further notation. Let $\mathcal{B}$ be a family of partitions in Ω. Let

$$\left\{ \pi \;\middle|\; \text{there is a } B \text{ in } \mathcal{B} \text{ such that } B \sqsubseteq \mathrm{Part}(\pi) \right\}$$

be the set of all permutations that stabilize an element of $\mathcal{B}$. Denote by $\mathcal{Q}_{\mathcal{B}}$ the smallest partition group that includes this set of stabilizers. If $\mathcal{B}$ is a filter base, then $\mathcal{Q}_{\mathcal{B}}$ and $\mathcal{Q}_{\mathcal{B}^+}$ coincide. Furthermore, we use $\precsim_{\mathcal{B}}$ as a shorthand for the social welfare relation $\precsim_{\mathcal{Q}_{\mathcal{B}}}$.

Proposition 1. Let $\mathcal{B}$ be a family of partitions in Ω. Then, $\mathcal{Q}_{\mathcal{B}}$ is a maximal group of cyclic permutations if and only if $\mathcal{B}^+$ is an ultrafilter.

Proof. The if-part. Let $\mathcal{B}^+$ be a filter. Then, $0 \notin \mathcal{B}$, and $\mathcal{Q}_{\mathcal{B}}$ only contains cyclic permutations. If π and ρ belong to $\mathcal{Q}_{\mathcal{B}}$, then $\mathrm{Part}(\pi) \cap \mathrm{Part}(\rho)$ belongs to $\mathcal{B}^+$. Hence, $\mathcal{Q}_{\mathcal{B}}$ is closed for composition. Next, observe that the partition induced by a permutation coincides with the partition induced by its inverse permutation. Therefore, $\mathcal{Q}_{\mathcal{B}}$ is a (partition) group of cyclic permutations.

Now, suppose that $\mathcal{B}^+$ is an ultrafilter. We have to show that $\mathcal{Q}_{\mathcal{B}}$ is maximal. Therefore, assume that the cyclic permutation π is not in $\mathcal{Q}_{\mathcal{B}}$. The induced partition $A = \mathrm{Part}(\pi)$ does not belong to the ultrafilter $\mathcal{B}^+$. Hence, there is a B in $\mathcal{B}^+$ such that $A \cap B = 0$. Lemma 2 implies the existence of a permutation in $\mathrm{Sym}(B)$ such that the composition with

π induces the partition 0. This composed permutation has an infinite cycle. Therefore, the permutation π cannot be added to $\mathcal{Q}_{\mathcal{F}}$ to generate a larger group of cyclic permutations.

The only-if-part. Let $\mathcal{Q}_{\mathcal{B}}$ be a maximal subgroup of cyclic permutations. We have to show that $\mathcal{B}^+$ is an ultrafilter. Since only cyclic permutations are involved, $0 \notin \mathcal{B}$. Next, assume that the partition A is not in $\mathcal{B}^+$. A permutation π that induces A does not belong to $\mathcal{Q}_{\mathcal{B}}$. Since the group $\mathcal{Q}_{\mathcal{B}}$ is maximal, there is a σ in $\mathcal{Q}_{\mathcal{B}}$ such that $\pi \circ \sigma$ is not cyclic. Conclude that $A \sqcap \text{Part}(\sigma) \sqsubseteq \text{Part}(\pi \circ \sigma) = 0$ with $\text{Part}(\sigma)$ in $\mathcal{B}^+$. □

Each partition group $\mathcal{G}$ of cyclic permutations defines a social welfare relation $\precsim_{\mathcal{G}}$. The other way around, each social welfare relation defines a partition group. The definition is as follows. Let $\precsim$ be a SWR in $\mathcal{S}$. The set of *permissible partitions* is defined as

$$\Pi(\precsim) \;=\; \Big\{ A \in \Omega \;\Big|\; \text{for each } \pi \text{ in } \text{Sym}(A) \text{ and for each } S \text{ in } \mathcal{S} \text{ we have } \pi(S) \sim S \Big\}.$$

If the SWR $\precsim_1$ is a subrelation to the Paretian SWR $\precsim_2$, then $\Pi(\precsim_1) \subseteq \Pi(\precsim_2)$. The next proposition investigates this link between partition groups and permissible permutations.

Proposition 2. Let the family $\mathcal{B}$ of partitions in Ω be a filter base. Then, the relation $\precsim_{\mathcal{B}}$ is reflexive, transitive, Paretian, and $\mathcal{B}$-anonymous. Furthermore, the set $\Pi(\precsim_{\mathcal{B}})$ of permissible partitions coincides with the filter $\mathcal{B}^+$.

Proof. The conditions imposed upon $\mathcal{B}$ turn $\mathcal{Q}_{\mathcal{B}}$ into a partition group of cyclic permutations. This group $\mathcal{Q}_{\mathcal{B}}$ coincides with $\mathcal{Q}_{\mathcal{B}^+}$. Mitra and Basu (2007, Proposition 3) show that for each group $\mathcal{G}$ of cyclic permutations, the relation $\precsim_{\mathcal{G}}$ is reflexive, transitive, Paretian, and $\mathcal{G}$-anonymous. Apply their result for $\mathcal{G} = \mathcal{Q}_{\mathcal{B}}$ and conclude that $\precsim_{\mathcal{B}}$ satisfies the properties as listed.

Let us now verify that $\Pi(\precsim_{\mathcal{B}})$ coincides with $\mathcal{B}^+$. The inclusion $\mathcal{B}^+ \subseteq \Pi(\precsim_{\mathcal{B}})$ is immediate. In case $\mathcal{B}^+$ is an ultrafilter also the reverse inclusion holds (otherwise there exists a cyclic permutation π outside the group $\mathcal{Q}_{\mathcal{B}}$ that keeps the indifference relation; as $\mathcal{Q}_{\mathcal{B}}$ is maximal $\mathcal{Q}_{\mathcal{B}} \cup \{\pi\}$ generates noncyclic permutations and a contradiction is obtained).

There remains one single statement to be proved: the inclusion $\Pi(\precsim_{\mathcal{B}}) \subseteq \mathcal{B}^+$ under the assumption that $\mathcal{B}^+$ is not an ultrafilter. We show this inclusion by contradiction and assume $A \notin \mathcal{B}^+$. There exists an ultrafilter $\mathcal{F}$ that extends $\mathcal{B}$ and does not contain A (in the family $\mathcal{A}$ of all filters which do not contain A each chain has a maximal element, so by Zorn's lemma $\mathcal{A}$ has a maximal element that appears to be an ultrafilter). The relation $\precsim_{\mathcal{B}}$ is a subrelation to $\precsim_{\mathcal{F}}$, and $A \notin \Pi(\precsim_{\mathcal{F}})$. Hence, A does not belong to $\Pi(\precsim_{\mathcal{B}})$. □

Theorem. Let $\mathcal{B}$ be a family of subsets in Ω. If the relation $\precsim_{\mathcal{B}}$ in $\mathcal{S}$ is complete and Paretian, then $\mathcal{B}^+$ is an ultrafilter on $(\Omega, \sqsubseteq)$.

Proof. By definition, the relation $\precsim_{\mathcal{B}}$ is transitive and finite anonymous. If the relation $\precsim_{\mathcal{B}}$ is also complete and Paretian, then the collection $\mathcal{B}^+$ contains the partition 1 (reflexivity), is closed for intersection (transitivity), and does not contain the partition 0 (Pareto). Hence, $\mathcal{B}^+$ is a filter and $\mathcal{Q}_{\mathcal{B}}$ is a partition group of cyclic permutations.

As $\precsim_{\mathcal{B}}$ is complete, the group $\mathcal{Q}_{\mathcal{B}}$ is maximal in $\mathcal{P}$. Indeed, otherwise there is a (partition) group $\mathcal{Q}$ such that $\mathcal{Q}_{\mathcal{B}} \subset \mathcal{Q} \subset \mathcal{P}$. The relation $\precsim_{\mathcal{Q}}$ is reflexive, transitive, Paretian, and strictly extends the relation $\precsim_{\mathcal{B}}$ (here, we use the statement on permissible partitions in Proposition 2). As the social welfare relation $\precsim_{\mathcal{B}}$ is assumed to be complete we arrive at a contradiction. Hence, the group $\mathcal{Q}_{\mathcal{B}}$ is maximal. From Proposition 1 we learn that $\mathcal{B}^+$ is an ultrafilter. $\square$

In a finite setting the counting procedure is well defined and is generated by the combination of Pareto and anonymity. The above theorem shows that this counting procedure fails when moving towards the infinite set $\mathbb{N}_0$. An anonymity condition strong enough to generate (in combination with Pareto) a complete 'counting' relation, involves the existence of a free ultrafilter on the lattice $(\Omega, \sqsubseteq)$ and hence involves nonconstructive methods. Alternatively, consider a social welfare relation in a set X of infinite utility streams such that its restriction to the set $\mathcal{S}$ is generated by the Pareto axiom in combination with an anonymity axiom (based upon a partition group). Then, this restricted relation either is incomplete or it generates a free ultrafilter on the lattice $(\Omega, \sqsubseteq)$ in which case there is no explicit description available.

4 Fixed and variable step permutations

Mitra and Basu (2007, Section 5) argue in favor of a particular group of cyclic permutations, to wit, the group of fixed step permutations.[2] This section considers the group of fixed step permutations within the set of variable step permutations and rephrases the previous results towards this particular setting.

For each n in $\mathbb{N}_0$, the partition

$$F_n = \Big\{ [1, n], [n+1, 2n], \ldots, [kn+1, (k+1)n], \ldots \Big\}$$

is said to be fixed step. A permutation π for which the partition $\mathrm{Part}(\pi)$ is finer than F_n (i.e. $F_n \sqsubseteq \mathrm{Part}(\pi)$), for some n, is said to be fixed step. The partition group $\mathcal{Q}_{\mathrm{fx}}$ of fixed step permutations is not maximal and the corresponding social welfare relation $\precsim_{\mathrm{fx}} = \precsim_{\mathcal{Q}_{\mathrm{fx}}}$ on $\mathcal{S}$ is not complete. For example, $\precsim_{\mathrm{fx}}$ is unable to compare the sets $\{1, 6, 15, 28, \ldots\}$ and $\{3, 10, 21, 36, \ldots\}$.

For each infinite subset $S = \{n_1, n_2, \ldots, n_k, \ldots\}$ of $\mathbb{N}_0$, the partition

$$V_S = \Big\{ [1, n_1], [n_1+1, n_2], \ldots, [n_k+1, n_{k+1}], \ldots \Big\}$$

is said to be variable step. A fixed step partition F_n is a particular example of a variable step partition. A permutation that generates a variable step partition is said to be variable step. Let the set $\mathcal{Q}_{\mathrm{var}}$ collect all variable step permutations. Finite permutations are fixed step, fixed step permutations are variable step, and variable step permutations are cyclic;

[2]Fixed step permutations have been proposed by Lauwers (1997) and by Fleurbaey and Michel (2003).

$\mathcal{Q}_{\rm fn} \subset \mathcal{Q}_{\rm fx} \subset \mathcal{Q}_{\rm var} \subset \mathcal{P}$. On the other hand, the next lemma reveals that $\mathcal{Q}_{\rm var}$ is not a group. Nevertheless, Lemmas 3 and 4 provide further arguments in favor of the set of variable step permutations.

Lemma 3. The set $\mathcal{Q}_{\rm var}$ of variable step permutations generates the group $\mathrm{Sym}(\mathbb{N}_0)$ of all permutations on $\mathbb{N}_0$. In particular, each permutation in $\mathbb{N}_0$ can be decomposed into two variable step permutations.

Proof. Let $\pi : \mathbb{N}_0 \to \mathbb{N}_0$ be a permutation. We construct two variable step permutations σ and τ such that $\pi = \sigma \circ \tau$. The construction is done via subsequent extensions of two permutations on finite sets of increasing length.

Put $\sigma(1) = \pi(1)$ and $\tau(1) = 1$. Let $t_1 > \pi(1)$ and extend σ to a permutation on the set $T_1 = [1, t_1]$. Define τ on the set T_1 such that $\sigma \circ \tau$ coincides with π (when restricted to the domain of the currently defined composition $\sigma \circ \tau$). Let $t_2 > \max\{\rho^{-1}(T_1) \cup \tau(T_1)\}$ and extend τ to a permutation on the set $T_2 = [1, t_2]$. Then extend σ to the set T_2 such that $\sigma \circ \tau$ coincides with π. Let $t_3 > \max \sigma(T_2)$ and extend σ to a permutation on the set $T_3 = [1, t_3]$, and so forth.

The permutation σ satisfies $\sigma(T_{2k+1}) = T_{2k+1}$ for each k in $\mathbb{N}_0$. Similarly, the permutation τ satisfies $\tau(1) = 1$ and $\tau(T_{2k}) = T_{2k}$ for each k in $\mathbb{N}_0$. Therefore the permutations σ and τ both belong to $\mathcal{Q}_{\rm var}$. □

Hence, combining variable step anonymity and transitivity upon a relation implies the imposition of $\mathrm{Sym}(\mathbb{N}_0)$-anonymity. To illustrate this lemma further, we mention that the infinite cycle π_3 coincides with $\pi_1 \circ \pi_2$. Therefore, a partition group of cyclic permutations cannot contain both π_1 and π_2. Each anonymity condition based upon a group of variable step permutations, strong enough to generate a complete ranking on $\mathcal{S}$, and weak enough to allow for Pareto, should impose either π_1-anonymity or π_2-anonymity (the either-or being exclusive). Next, we show that each cyclic permutation can be rewritten (after a re-numbering of $\mathbb{N}_0$) as a variable step permutation.

Lemma 4. Each cyclic permutation is conjugated to a variable step permutation.

Proof. Let π be in $\mathcal{P}$. We have to find a permutation $\tilde{\pi}$ in $\mathcal{Q}_{\rm var}$ and a permutation σ of the set $\mathbb{N}_0$, such that $\pi = \sigma^{-1} \circ \tilde{\pi} \circ \sigma$. Present π as a product of cycles:

$$\pi \;=\; (a_{11}, \ldots, a_{1k_1})(a_{21}, \ldots, a_{2k_2}) \cdots (a_{n1}, \ldots, a_{nk_n}) \cdots .$$

Then, it suffices to check the identity $\tilde{\pi} = \sigma \circ \pi \circ \sigma^{-1}$ with

$$\tilde{\pi} \;=\; (1, \ldots, k_1)(k_1 + 1, \ldots, k_2) \cdots (k_{n-1} + 1, \ldots, k_n) \cdots ,$$

and $\sigma : a_{ij} \mapsto k_{i-1} + j$ with $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, k_i - k_{i-1}$. We put $k_0 = 0$. □

Both lemmas attract the focus upon extensions of the group $\mathcal{Q}_{\rm fx}$ of fixed step permutations within the set $\mathcal{Q}_{\rm var}$ of variable step permutations. Consider a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}_0$. Define the following set of variable step permutations:

$$\mathcal{Q}_{\mathcal{F}} \;=\; \Big\{ \pi \;\Big|\; \text{there is a } S \text{ in } \mathcal{F} \text{ such that } V_S \sqsubseteq \mathrm{Part}(\pi) \Big\}.$$

We now reformulate the results of the previous section in terms of filters on $\mathbb{N}_0$.

Proposition 3.[3] Let $\mathcal{F}$ be a family of subsets of $\mathbb{N}_0$. The set $\mathcal{Q}_{\mathcal{F}}$ is a maximal (partition) group of cyclic permutations if and only if the smallest filter $\mathcal{F}^+$ that includes the family $\mathcal{F}$ is a free ultrafilter on the set $\mathbb{N}_0$.

In case the family $\mathcal{F}$ contains the sets $\{n, 2n, \ldots, kn, \ldots\}$ for each n in $\mathbb{N}_0$, then $\mathcal{Q}_{\mathcal{F}}$ includes the set $\mathcal{Q}_{\text{fx}}$ of fixed step permutations and the social welfare relation $\precsim_{\mathcal{Q}_{\mathcal{F}}}$ on $\mathcal{S}$ is fixed step anonymous.

Recall the social welfare relation $\precsim_{\mathcal{F}}$ on $\mathcal{S}$ with $\mathcal{F}$ a free ultrafilter on $\mathbb{N}_0$ defined by

$$S \precsim_{\mathcal{F}} T \quad \text{if and only if} \quad \left\{ t \in \mathbb{N}_0 \,\middle|\, |\, S \cap \{1, 2, \ldots, t\} \,| \;\leq\; |\, T \cap \{1, 2, \ldots, t\} \,| \right\} \in \mathcal{F}.$$

The relation $\precsim_{\mathcal{F}}$ coincides with $\precsim_{\mathcal{Q}_{\mathcal{F}}}$. Hence, in terms of social welfare relations we obtain:

Corollary 3. Let $\mathcal{F}$ be a family of subsets of $\mathbb{N}_0$. Then, the relation $\precsim_{\mathcal{Q}_{\mathcal{F}}}$ on $\mathcal{S}$ is complete and Paretian if and only if $\mathcal{F}^+$ is a free ultrafilter on $\mathbb{N}_0$.

Conclusion

Anonymity and Pareto are two important principles. In a finite context, these principles induce the counting procedure. Infinite versions of these principles that are strong enough to allow for comparing the "sizes" of different sets, unavoidably involve nonconstructive objects. This negative result obviously persists when ranking infinite utility streams rather than subsets of $\mathbb{N}_0$. The larger part of the recent literature on the ranking of infinite utility streams starts from this kind of impossibilities and looks for incomplete criteria with appealing properties.

The negative results and the (highly) incomplete criteria indicate the limitations of combining appealing properties in "one single" criterion and imply an invitation to look for new procedures to rank infinite utility streams. Inspired by the result of Chichilnisky (2009, Thm 3), I propose a two-step procedure (obviously, further research is needed). In a *first* step, the focus is on completeness and on anonymity. Here, the Pareto condition is weakened to a monotonicity condition which requires that $x \leq y$ implies $x \precsim y$. This monotonicity condition is compatible with completeness, constructibility, and (any form of) anonymity. A typical criterion would consider the limiting behavior of the utility streams. In a *second* step, one further investigates the set of "optimal" utility streams obtained in the first step. On this restricted domain, a stronger Pareto principle should be employed.

The ultimate goal of a social welfare relation on the set of infinite utility streams is its application in policies that involve the very long run. A two-step procedure as described above takes care of this very long run without ignoring the short run.

[3] Marc Fleurbaey (2006) establishes the result mentioned in this corollary.

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