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DISCUSSION
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Revealed Preference tests for Weak Separability: an integer programming approach*

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Abstract

We focus on the revealed preference conditions that characterize the collection of finite data sets that are consistent with the maximization of a weakly separable utility function. From a theoretical perspective, we show that verifying these revealed preference conditions is a difficult problem, i.e. it is NP-complete. From a practical perspective, we present an integer programming approach that can verify the revealed preference conditions in a straightforward way, which is particularly attractive in view of empirical analysis. We demonstrate the versatility of this integer programming approach by showing that it also allows for testing homothetic separability and weak separability of the indirect utility function. We illustrate the practical usefulness of the approach by an empirical application to Spanish household consumption data.

Keywords: weak separability, revealed preference, integer programming

JEL-classification: C14, C60, D01, D10.

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The universe cannot be dealt with in one stroke and so a bit has to be broken off and treated as if the rest did not matter.

Afriat, 1969

1 Introduction

We focus on the revealed preference conditions for consistency of a finite data set with the maximization of a weakly separable utility function. Our main contribution is twofold. First, we show that verification of these revealed preference conditions is a difficult problem. In particular, the problem is NP-complete, which essentially means that it cannot be solved in polynomial time. As we will discuss below, this actually motivates our second contribution. Specifically, we show that the revealed preference conditions can be verified by means of elementary integer programming procedures, which are easily implemented in practice. We demonstrate the versatility of this integer programming approach by showing that it can also assess homothetic separability and weak separability of the indirect utility function. Finally, we illustrate the approach by applying it to a Spanish panel data set.

Weak separability of the utility function is a frequently used assumption in theoretical and applied demand analysis. A group of goods is said to be separable if the marginal rate of substitution between any two goods in the group is independent from the quantities consumed of any good outside this group (Leontief, 1947; Sono, 1961). Weak separability has several convenient implications.[1] First of all, it allows for representing consumption in terms of two stage budgeting. This means that, in order to determine the demanded quantities of the goods in the separable group, it suffices to know the prices of the goods in this group and the total within group expenditures. Further, weak separability is a crucial condition for the construction of group price and quantity indices. Such aggregates can be useful, for example, to compute group cost of living indices for welfare analysis. Finally, from an empirical point of view, weak separability significantly reduces the number of parameters of the demand system to be estimated in practical applications.

Considering these advantages for both theoretical and empirical work, an important issue concerns empirically testing the validity of the separability assumption (prior to effectively imposing it). In the literature, there are two approaches to test for weak separability. The most popular approach uses econometric techniques to verify certain parameter restrictions given a specific demand model. Although this approach is fairly flexible in terms of the demand model that is used, it also poses a number of problems. First, in order to verify weak separability, it is necessary to estimate a full demand system without any restriction on the cross price elasticities. As such, most tests of weak separability will have a degrees of freedom problem in the sense that too many parameters must be estimated given the amount of data. Next, if the hypothesis of weak separability is rejected, it is impossible to verify whether this implies a rejection of weak separability or, instead, a rejection of the specific functional form imposed on demand a priori. In other words, if the null hypothesis of weak separability is rejected, this may well be due to

[1] See also Deaton and Muellbauer (1980) for a more thorough discussion

the use of a wrong functional form rather than a nonseparable utility structure per se.[2] Finally, most econometric tests for separability are based on separability of the indirect utility function (i.e. separability in prices), which does not imply separability of the direct utility function (i.e. separability in quantities) unless the subutility function is homothetic. As such, a rejection of weak separability can also reveal a failure of this homotheticity assumption.

An alternative, less known approach to test weak separability is based on revealed preference theory. In several seminal contributions to the literature, Afriat (1969), Varian (1983) and Diewert and Parkan (1985) developed revealed preference conditions that characterize the collection of data sets that are rationalizable by a (weakly) separable utility function. The revealed preference approach remedies different problems associated with the econometric approach. First, the revealed preference conditions can meaningfully be applied to data sets with only two observations, which avoids the degrees of freedom problem discussed above. Further, the revealed preference approach abstains from imposing a specific functional form on the utility functions. As such, the tests are insensitive to model misspecification. Finally, the revealed preference approach does not require additional assumptions like homotheticity of the subutility function or separability of the indirect utility function (although such additional assumptions can be imposed and tested; see below).

Unfortunately, the revealed preference conditions have the drawback that they take the form of a set of nonlinear, quadratic inequalities, which are very hard to verify. In order to avoid this problem, several heuristics have been proposed that provide separate sufficient and necessary conditions for data consistency with weak separability (see Section 2 for an overview). The lack of an efficient algorithm to verify the revealed preference conditions raises the question whether such an algorithm exists at all. In this study, we show that the answer is no. In particular, we prove that the verification of the revealed preference conditions for weak separability is an NP-complete problem.[3] This NP-completeness result implies that it is impossible to find a polynomial time algorithm that verifies whether a data set is consistent with the maximization of a weakly separable utility function (unless one can prove P= NP). This indicates that we should better look for a widely applied and (for moderately sized problems) reasonably quick non-polynomial time algorithm to verify the revealed preference conditions. Given this, we present an easy-to-implement (non-polynomial time) integer programming procedure to verify the revealed preference conditions.[4] Our approach exploits the equivalence of the generalized axiom of revealed preference (GARP) and a set of mixed integer inequalities. Such an integer pro-

[2]In this respect, it is important to know that imposing separability conditions on a particular functional form might lead to additional difficulties. In particular, Blackorby, Primont, and Russel (1978) showed that testing for separability using several econometric specifications based on local approximations of the true model (i.e. flexible functional forms) is actually equivalent to testing a much stronger condition. For example, it turns out impossible to test separability for the translog model without imposing the much more stringent assumption of additive separability. Barnett and Choi (1989) confirmed this result by means of Monte Carlo simulations.

[3]We refer to Garey and Johnson (1979) for an introduction into the theory of NP-completeness.

[4]In private communication, Per Hjertstrand pointed out to us that, in an unpublished working paper, he independently developed a closely similar integer programming procedure; see Hjerstrand (2011b). We thank him for providing us with the reference. In a sense, Hjertstrand's procedure is complementary to ours as it uses an integer programming formulation of the Afriat inequalities in statement (ii) of our following Theorem 2, while our approach builds on an integer programming formulation of the GARP condition (iii) in Theorem 2 (see also Theorem 4).

gramming approach has proven very useful in the literature that applies revealed preference theory to collective consumption models, which studies the behavior of multi-person households, and in the literature that investigates the testable implications of general equilibrium models.[5] We extend the insights from this literature to the model of utility maximization with a weakly separable utility function.

From a theoretical point of view, the core motivation for adopting an integer programming approach is that this is a widely accepted and a well known approach to handle NP-complete problems. Besides this, we also have a number of other motivations. First of all, our approach allows for problems with an arbitrary number of observations. Second, any mixed integer program can be solved in finite time. Hence, our approach implies the possibility to verify in finite time the necessary and sufficient conditions for a given data set to be consistent with maximization of a weakly separable utility function. A final and important argument pro our integer programming approach is that it provides a versatile framework for analyzing testable implications of different model specifications: we will show that our approach can easily accommodate for homotheticity of the subutility functions, and that we can readily adjust our integer programming procedure to test for separability of the indirect utility function.

We demonstrate the practical usefulness of our approach by applying it to data drawn from the Encuesta Continua de Presupestos Familiares (ECPF), a Spanish household survey. In this application we first compare the empirical performance of the four alternative model specifications that were also mentioned above: the standard utility maximization model, the model that additionally imposes weak separability, the homothetic separability model, and the model that assumes a weakly separable indirect utility function. Specifically, following a recent proposal of Beatty and Crawford (2010), we evaluate these different model specifications in terms of their 'predictive success'. Next, another main focus is on evaluating the computational speed of our integer programming approach for substantially large data sets. To this end, we will consider a preference homogeneity assumption that parallels an assumption often used in econometric demand analysis. This will allow us to conduct our separability tests on data sets that bring together information on multiple (similar) households.

Section 2 introduces the revealed preference conditions for rationalizability under a weakly separable utility function. Section 3 gives the NP-completeness result and presents our integer programming approach. Section 4 discusses our empirical application. Section 5 concludes.

2 Revealed preferences conditions

To set the stage, we briefly recapture the known revealed preference conditions for the standard utility maximization model and for the model that additionally imposes weak separability on the utility function. The results in this section will be useful for our discussion in the following sections.

[5] See Cherchye, De Rock, Sabbe, and Vermeulen (2008) and Cherchye, De Rock, and Vermeulen (2011a) for integer programming characterizations of collective consumption models and Cherchye, Demuynck, and De Rock (2011b) for integer programming characterizations of general equilibrium models.

Standard utility maximization. Consider a finite data set $D = \{\mathbf{p}_t; \mathbf{x}_t\}_{t \in T}$, which consists of strictly positive price vectors $\mathbf{p}_t \in \mathbb{R}^n_{++}$ and non-negative consumption bundles $\mathbf{x}_t \in \mathbb{R}^n_+$ for consumption observations t in a (finite) set T. This data set D is said to be *rationalizable* if there exists a well-behaved (i.e. increasing, continuous and concave) utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that, for all observations $t \in T$,

$$\mathbf{x}_t \in \arg\max_{\mathbf{x}} u(\mathbf{x}) \qquad \text{s.t. } \mathbf{p}_t\mathbf{x} \leq \mathbf{p}_t\mathbf{x}_t.$$

In other words, for each t it must be the case that the consumed bundle $\mathbf{x}_t$ maximizes the utility function u over the set of all affordable consumption bundles.

Next, consider the following concepts. The direct revealed preference relation R^D over the set $\{\mathbf{x}_t\}_{t \in T}$ is defined by $\mathbf{x}_t R^D \mathbf{x}_v$ if $\mathbf{p}_t\mathbf{x}_t \geq \mathbf{p}_t\mathbf{x}_v$. In words, we have that $\mathbf{x}_t R^D \mathbf{x}_v$ if $\mathbf{x}_t$ was chosen while $\mathbf{x}_v$ was also affordable. The indirect revealed preference relation R is the transitive closure of the relation R^D; $\mathbf{x}_t R \mathbf{x}_v$ if there exist bundles $\mathbf{x}_w, \mathbf{x}_r, \ldots, \mathbf{x}_m$ such that $\mathbf{x}_t R^D \mathbf{x}_w$, $\mathbf{x}_w R^D \mathbf{x}_r, \ldots, \mathbf{x}_m R^D \mathbf{x}_v$. Finally, we say that $\{\mathbf{p}_t, \mathbf{x}_t\}_{t \in T}$ satisfies the Generalized Axiom of Revealed Preferences (GARP) if for all $\mathbf{x}_t R \mathbf{x}_v$ it is not the case that $\mathbf{p}_v\mathbf{x}_v > \mathbf{p}_v\mathbf{x}_t$. In words, if $\mathbf{x}_t$ is indirectly revealed preferred to $\mathbf{x}_v$, then it is not the case that $\mathbf{x}_v$ was more expensive then $\mathbf{x}_t$ when $\mathbf{x}_v$ was bought.

Using these concepts, we can state the following result, which is probably the single most important theorem in revealed preference theory.

Theorem 1. *[Varian (1982), based on Afriat (1967)]*
The following statements are equivalent:

(i) The data set $D = \{\mathbf{p}_t, \mathbf{x}_t\}_{t \in T}$ is rationalizable,

(ii) The data set $D = \{\mathbf{p}_t, \mathbf{x}_t\}_{t \in T}$ satisfies GARP,

(iii) There exist strict positive numbers λ_t and nonnegative numbers U_t such that, for all $t, v \in T$,

$$U_t - U_v \leq \lambda_v \mathbf{p}_v(\mathbf{x}_t - \mathbf{x}_v).$$

Condition (ii) states that GARP is necessary and sufficient for rationalizability. Condition (iii) provides an equivalent characterization of utility maximization in terms of so-called Afriat inequalities. Intuitively, these Afriat inequalities allow us to obtain an explicit construction of the utility levels and the marginal utility of income associated with each observation t: they define a utility level U_t and a marginal utility of income λ_t (associated with the observed income $\mathbf{p}_t\mathbf{x}_t$) for each observed $\mathbf{x}_t$.

Theorem 1 provides two methods to verify whether a data set is rationalizable. The first method was originally suggested by Varian (1982) and focuses on verifying the GARP condition. The method consists of three steps, which comply with the three steps in the definition of GARP. The first step constructs the relation R^D from the data set $D = \{\mathbf{p}_t, \mathbf{x}_t\}_{t \in T}$. A second step computes the transitive closure of R. Here, Varian suggests using Warshall (1962)'s algorithm, which provides an efficient procedure for computing transitive closures. The third step verifies if $\mathbf{p}_v\mathbf{x}_v \leq \mathbf{p}_v\mathbf{x}_t$ whenever $\mathbf{x}_t R \mathbf{x}_v$. If this is the case, the data set satisfies GARP and is, therefore, rationalizable. Due to its efficiency, this procedure is very popular in applied work. The second method verifies the rationalizability conditions by testing feasibility of the corresponding Afriat inequalities (i.e. condition (iii)). These inequalities are linear in the unknowns U_t and λ_t, which implies that their feasibility can be verified using elementary linear programming methods. We refer to Afriat (1967) and Diewert (1973) for discussions of this method.

Weak separability. To introduce the notion of weak separability, we first partition the set of goods $N = \{1, \ldots, n\}$ in two groups. Accordingly, we can split any given consumption bundle into two separate bundles. The first bundle $\mathbf{x}$ contains all consumption quantities of the goods from the first group and the second bundle $\mathbf{y}$ corresponds to the goods in the second group. In this way, we write the consumption bundle by $(\mathbf{x}, \mathbf{y})$. Likewise, we can split any price vector into a price vector of all goods in the first group $\mathbf{p}$ and a vector of prices for the goods in the second group $\mathbf{q}$. Now, consider a data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t \in T}$. We say that this data set is *rationalizable by weak separability* if there exist a well-behaved utility function u and a well-behaved subutility function s such that, for all observations $t \in T$,

$$(\mathbf{x}_t, \mathbf{y}_t) \in \arg\max_{\mathbf{x},\mathbf{y}} u(\mathbf{x}, s(\mathbf{y})) \qquad \text{s.t. } \mathbf{p}_t\mathbf{x} + \mathbf{q}_t\mathbf{y} \leq \mathbf{p}_t\mathbf{x}_t + \mathbf{q}_t\mathbf{y}_t.$$

Varian (1983) provides the following characterization of behavior that is rationalizable by weak separability.

Theorem 2. *[Varian (1983)]*
The following statements are equivalent:

(i) The data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t \in T}$ is rationalizable by weak separability.

(ii) For all $t \in T$ there exist a nonnegative numbers S_t and strict positive numbers δ_t such that, for all t, $v \in T$,

$$S_t - S_v \leq \delta_v \mathbf{q}_v(\mathbf{y}_t - \mathbf{y}_v), \tag{ii.1}$$

$$\{\mathbf{p}_t, 1/\delta_t; \mathbf{x}_t, S_t\}_{t \in T} \textit{satisfies GARP.} \tag{ii.2}$$

(iii) For all $t \in T$, there exist nonnegative numbers S_t and U_t and strict positive numbers δ_t and λ_t such that, for all $t, v \in T$,

$$S_t - S_v \leq \delta_v \mathbf{q}_v(\mathbf{y}_t - \mathbf{y}_v), \tag{iii.1}$$

$$U_t - U_v \leq \lambda_v \left[\mathbf{p}_v(\mathbf{x}_t - \mathbf{x}_v) + \frac{1}{\delta_v}(S_t - S_v)\right]. \tag{iii.2}$$

In contrast to the conditions in Theorem 1, the conditions in this theorem are not easily verified. The main problem is that for the verification of (ii.2), the 'prices' $1/\delta_t$ and the corresponding 'quantities' S_t, which must satisfy condition (ii.1), are unobserved. This is also reflected in condition (iii.2), which can be rewritten as a set of quadratic inequalities.

The literature brings forward several methods to test the weak separability conditions. Probably the best known alternative is Varian (1983)'s three step procedure. In the first step, this method tests GARP consistency of the data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t\in T}$. If the data fail GARP, they are not rationalizable and, hence, we can reject weak separability.[6] By contrast, if the data satisfy GARP, the second step tests whether the data set $\{\mathbf{q}_t, \mathbf{y}_t\}_{t\in T}$ satisfies GARP. This GARP condition is equivalent to condition (ii.1). If GARP consistency is rejected in this second step then, again, the data set is not rationalizable by weak separability. Finally, the third step verifies GARP of a data set $\{\mathbf{p}_t, 1/\delta_t^*; \mathbf{x}_t, V_t^*\}_{t\in T}$ for some specific values δ_t^* and S_t^* that satisfy condition (ii.1). If for this last step GARP is not rejected, then we conclude that the data are consistent with weak separability.

Unfortunately, Varian (1983)'s test is not an exact one. In particular, it is possible that a data set is rationalizable by a weakly separable utility function while the algorithm effectively rejects weak separability. Simulation results indicate that this may actually occur quite frequently (see, for example, Barnett and Choi (1989); Fleissig and Whitney (2003); Hjerstrand (2009)). The problem is that the third step of the procedure fixes the values of both δ_t^* and V_t^* in an arbitrary way. In this respect, however, certain values may be more probable than others. This idea provides the intuition behind the linear program developed by Fleissig and Whitney (2003). In particular, these authors determine the values of $1/\delta_t^*$ and V_t^* based on the theory of superlative index numbers (see Diewert (1976, 1978)). A superlative index number provides an exact index number for some order approximation of the underlying (in casu homogeneous) utility function s. However, this test is again only sufficient but not necessary. An alternative testing strategy is explored by Swofford and Whitney (1994) and Fleissig and Whitney (2008), who use nonlinear programming methods to solve (iii.1) and (iii.2) simultaneously. Alas, nonlinear programming problems are generally difficult to solve. The most important problem with the solution algorithms is that they do not always yield an optimal solution: most algorithms search for local optima, which need not be globally optimal. Generally, finding a global optimum requires a fine grid search over the set of initial values. But even a very fine grid search cannot exclude that weak separability is rejected while the assumption effectively holds. We refer to Hjerstrand (2009) for a Monte Carlo comparison of the different test procedures cited in this paragraph.

3 Main results

This section contains our main results. First, we show that the problem of rationalizing a data set by a weakly separable utility function is a difficult (= NP-complete) problem. Next, we show how the above revealed preference conditions for rationalizability by a weakly separable utility function can be restated as a mixed integer programming (MIP) problem. MIP is a frequently used and widely accepted approach to handle NP-complete problems. Further, we show the flexibility of our approach by deriving MIP conditions for two related rationalizability problems. First, we consider the specific case where the subutility function is homothetic. Next, we focus on the case where separability is imposed on the indirect utility function, i.e. the case

[6] If the data set does not satisfy GARP, then it is not rationalizable by a utility function. As such, it is also not rationalizable by a weakly separable utility function.

of weak separability in prices. These two cases are interesting because they are widely used in econometric analysis of separability (see also our discussion in the Introduction).

The NP-completeness result. Consider a data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t\in T}$. For any such data set, we can ask the question whether this data set is rationalizable by a weakly separable utility function (i.e. whether it satisfies the conditions of Theorem 2: either (ii.1)-(ii.2) or (iii.1)-(iii.2)). Basically, this decision problem asks for testing rationalizability by weak separability by an arbitrary data set. The following theorem shows that this problem is NP-complete. The proof is given in Appendix A.

Theorem 3. *The question whether a given data set is rationalizable by a weakly separable utility function is an* NP*-complete problem.*

This result considers the general case without any restriction on the number of goods or observations. Of course, it does not rule out specific instances for which verification of the rationalizability conditions might be performed efficiently. Nevertheless, our result does indicate that it is highly improbably that the problem of weakly rationalizability can be solved by means of an efficient algorithm (like, for example, linear programming).

Essentially, Theorem 3 implies that one should not waste time trying to construct a polynomial time algorithm that verifies rationalizability by weak separability (unless one has taken up the ambitious task of showing that P= NP. In turn, this suggests considering easy-to-implement non-polynomial time algorithms for tackling the testing problem. Therefore, we next propose a widely used method called Mixed Integer Programming (MIP). MIP problems look like standard linear programming problems except that certain variables are restricted to be integer valued (in our case either 0 or 1).

The mixed integer program. We proceed by translating conditions (ii.1) and (ii.2) to an integer programming setting. The basic idea is to notice that condition (ii.2) is equivalent to the condition that the data set $\{\delta_t\mathbf{p}_t, 1; \mathbf{x}_t, S_t\}_{t\in T}$ satisfies GARP. Indeed, this equivalence follows from normalizing the price of good S_t to unity for all observations.[7] This new GARP condition can be transformed into the following mixed integer linear program. (The proof of Theorem 4 explains the equivalence between GARP consistency of the data set $\{\delta_t\mathbf{p}_t, 1; \mathbf{x}_t, S_t\}_{t\in T}$ and feasibility of the conditions (cs.2)-(cs.5).)

CS.WS For all $t, v \in T$, there exist numbers $S_t, U_t \in [0, 1[$, $\delta_t \in]0, 1]$ and binary variables $X_{t,v} \in \{0, 1\}$ such that, for all observations t and $v \in T$,[8]

$$S_t - S_v \leq \delta_v\mathbf{q}_v(\mathbf{y}_t - \mathbf{y}_v), \tag{cs.1}$$

$$U_t - U_v < X_{t,v}, \tag{cs.2}$$

$$(X_{t,v} - 1) \leq U_t - U_v, \tag{cs.3}$$

$$\delta_t\mathbf{p}_t(\mathbf{x}_t - \mathbf{x}_v) + (S_t - S_v) < X_{t,v}A_t, \tag{cs.4}$$

$$(X_{t,v} - 1)A_v \leq \delta_v\mathbf{p}_v(\mathbf{x}_t - \mathbf{x}_v) + (S_t - S_v). \tag{cs.5}$$

[7] This can always be done, by noticing that this does not alter the GARP conditions.

[8] The strict inequalities in cs.2 and cs.4 are difficult to handle. Therefore, in practice, we use a weak inequality and subtract a very small but fixed number from the right hand side.

Here, we let A_t be some fixed and large number (larger than $\mathbf{p}_t\mathbf{x}_t + 1$). First of all, observe that the restriction of S_t, U_t and δ_t to the unit interval is harmless as it is possible to rescale both variables without changing the revealed preference conditions (ii.1) and (ii.2). Condition (cs.1) then reproduces condition (ii.1). Hence, we only need to show that conditions (cs.2)-(cs.5) are equivalent to the condition that $\{\delta_t\mathbf{p}_t, 1; \mathbf{x}_t, S_t\}_{t\in T}$ satisfies GARP. This is captured by the following result. The proof is given in Appendix B.

Theorem 4. *The data set $D = \{\delta_t\mathbf{p}_t, 1; \mathbf{x}_t, S_t\}_{t\in T}$ satisfies GARP if and only if conditions* (cs.2)-(cs.5) *have a solution.*

Homothetic and indirect weak separability. The above MIP formulation is very flexible in terms of incorporating additional (separable) preference structure. We illustrate this by considering two special cases. The first case requires that the subutility function s is homothetic. The second case requires separability of the indirect utility function.

A data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t\in T}$ is *rationalizable by homothetic separability* if there exist a well-behaved utility function u and a well-behaved and homothetic subutility function s such that, for all observations $t \in T$,

$$(\mathbf{x}_t, \mathbf{y}_t) \in \arg\max_{\mathbf{x},\mathbf{y}} u(\mathbf{x}, s(\mathbf{y})) \qquad \text{s.t. } \mathbf{p}_t\mathbf{x} + \mathbf{q}_t\mathbf{y} \leq \mathbf{p}_t\mathbf{x}_t + \mathbf{q}_t\mathbf{y}_t.$$

The following theorem characterizes the data sets that are homothetic weakly separable. The result directly follows from combining Varian (1983)'s rationalizability conditions for a homothetic utility function with Theorem 2.[9]

Theorem 5. *The following statements are equivalent:*

(i) The data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t\in T}$ is rationalizable by homothetic separability.

(ii) For all $t \in T$ there exist nonnegative numbers U_t and strict positive numbers S_t such that, for all $t, v \in T$,

$$S_t - S_v \leq \frac{S_v}{\mathbf{q}_v\mathbf{y}_v}\mathbf{q}_v(\mathbf{y}_t - \mathbf{y}_v),$$
$$\left\{\frac{S_t}{\mathbf{q}_t\mathbf{y}_t}\mathbf{p}_t, 1; \mathbf{x}_t, \mathbf{y}_t\right\}_{t\in T} \quad \textit{satisfies GARP.}$$

In other words, to impose homotheticity of the subutility function, we only need to add the additional (linear) restriction that $\delta_t = S_t/\mathbf{q}_t\mathbf{y}_t$. As such, by substituting in the MIP problem CS.WS each occurrence of δ_t by $S_t/\mathbf{q}_t\mathbf{y}_t$ (or by imposing the additional restriction that $\delta_t = S_t/\mathbf{q}_t\mathbf{y}_t$), we obtain a MIP formulation for homothetic weak separability.

As a final result we state the revealed preference conditions for indirect weak separability. First of all, let us normalize the prices $\mathbf{p}_t$ and $\mathbf{q}_t$ such that, for all t, $\mathbf{p}_t\mathbf{x}_t + \mathbf{q}_t\mathbf{y}_t = 1$. Then, we say that the data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t\in T}$ is *rationalizable by indirect weak separability* if there

[9]See also Varian (1983) for a similar result.

exist a well-behaved (i.e. decreasing, convex and continuous) indirect utility function v and a well-behaved indirect subutility function w such that, for all observations $t \in T$,

$$\{\mathbf{p}_t, \mathbf{q}_t\} \in \arg\min v(\mathbf{p}, w(\mathbf{q})) \qquad \text{s.t. } \mathbf{p}\mathbf{x}_t + \mathbf{q}\mathbf{y}_t \leq 1. \tag{1}$$

The next theorem gives a characterization of data sets that are rationalizable in terms of an indirect weakly separable utility function. The result is obtained by combining the result in Theorem 2 with Brown and Shannon (2000) 's rationalizability conditions for an indirect utility function.[10]

Theorem 6. *The following statements are equivalent:*

(i) The data set $D = \{\mathbf{p}_t, \mathbf{q}_t; \mathbf{x}_t, \mathbf{y}_t\}_{t \in T}$ is rationalizable by indirect weak separability.

(ii) For all $t \in T$ there exist nonnegative numbers V_t and W_t and strict positive numbers λ_t and δ_t such that, for all $t, v \in T$,

$$W_t - W_v \geq -\delta_v \mathbf{y}_v(\mathbf{q}_t - \mathbf{q}_v), \tag{iv.1}$$

$$V_t - V_v \geq -\lambda_v \left(\mathbf{x}_v(\mathbf{p}_t - \mathbf{p}_v) + \frac{1}{-\delta_v}(W_t - W_v) \right). \tag{iv.2}$$

Similarly to before, we can show that the rationalizability conditions in Theorem 6 are equivalent to the following set of MIP constraints (see Appendix C for a proof):

CS.WSI There exist numbers $W_t, W_v, V_t, V_v \in [0, 1[$, $\delta_t \in]0, 1]$ and binary variables $X_{t,v} \in \{0, 1\}$ such that, for all $t, v \in T$,

$$W_t - W_v \leq \delta_v \mathbf{y}_v(\mathbf{q}_t - \mathbf{q}_v), \tag{csi.1}$$

$$V_t - V_v < X_{t,v}, \tag{csi.2}$$

$$(X_{t,v} - 1) \leq V_t - V_v, \tag{csi.3}$$

$$\delta_t \mathbf{x}_t(\mathbf{p}_t - \mathbf{p}_v) + (W_t - W_v) < X_{t,v} A_t, \tag{csi.4}$$

$$(X_{t,v} - 1)A_v \leq \delta_v \mathbf{x}_v(\mathbf{p}_t - \mathbf{p}_v) + (W_t - W_v). \tag{csi.5}$$

Again, A_t is a fixed number larger than $\mathbf{p}_t\mathbf{x}_t + 1$.

4 Empirical Application

We applied our integer programming tests to data drawn from the Encuesta Contunua de Presopuestos Familieares (ECPF) Survey. The ECPF is a quarterly budget survey (1985–1997) that interviews about 3200 Spanish households on their consumption expenditures. For each household, the survey provides consumption observations for a maximum of eight consecutive quarters. See Browning and Collado (2001) and Crawford (2010) for a more detailed explanation of this data set. We exclude all households with less then eight observations. In the end,

[10]See also Hjerstrand and Swofford (forthcoming) for a similar result.

this obtains a panel with 1585 households. The data set covers consumption decisions for 15 (nondurable) goods: (i) food and non-alcoholic drinks at home, (ii) alcohol, (iii) tobacco, (iv) energy at home, (v) services at home, (vi) nondurables at home, (vii) nondurable medicines, (viii) medical services, (ix) transportation, (x) petrol, (xi) leisure, (xii) personal services, (xiii) personal nondurables, (xiv) restaurant and bars and (xv) traveling holiday. We follow Blundell, Browning, and Crawford (2007) and define the separable group to include all goods except food (i.e. the separable group contains all goods except (i), (ii) and (xiv)). This separability assumption is frequently used in empirical analysis of consumption behavior.

Test results: pass rates and power. Table 1 reports the pass rates of the revealed preference tests. About 90% of the households (1431 out of 1585) satisfy the revealed preference condition for the standard utility maximization model (i.e. the conditions in Theorem 1). By contrast, only 870 households (or 55%) satisfy the revealed preference conditions for rationalizability by weak separability (as given by Theorem 2). Remarkably, none (!) of the households satisfy the conditions for rationalizability by homothetic separability (see Theorem 5). This already indicates that weak separability and, to a much greater extent, homothetic weak separability are rather stringent assumptions.[11] Finally, 1242 households (or 79%) pass the rationalizability conditions for indirect weak separability (see Theorem 6), which is substantially more than for the other separability assumptions.

Our diverging results for weak separability and indirect weak separability can seem surprising to some, as one may have expected these two assumptions to be about equally stringent. Still, our pass rate results suggest that the first assumption has a better empirical fit than the second one for our sample of households. In a sense, this may be a useful result from the perspective of econometric applications, which often invoke indirect weak separability (see our discussion in the Introduction). Our results reveal that observed behavior is largely consistent with such indirect separability.

Table 1: Pass rate and Power (in percentages)

model	pass rate	power					
		mean	min	1st quartile	median	3rd quartile	max
general ut. max.	90.2	11.1	0	0.3	06.9	20.1	61.0
weak separability	51.4	48.0	4.2	31.4	48.2	64.0	97.4
homothetic separability	0	99.9	99.8	100	100	100	100
indirect separability	78.4	15.8	0	0.8	11.6	27.0	82.2

[11] Given this failure of homothetic separability for our sample, one may want to test whether homotheticity holds at the subutility level, as a necessary (but not sufficient) condition for homothetic separability. This can tell us whether it is actually homotheticity of the subutility function that is strongly rejected, even without additionally imposing weak separability at the aggregate utility level. This test can be performed by simply applying Varian (1983)'s test to the data set $\{\mathbf{q}_t, \mathbf{y}_t\}_{t \in T}$. When applied to our data set, we found that, indeed, none of the households satisfies this weaker test.

Importantly, to meaningfully compare the different models, one should not merely consider the corresponding pass rates. For example, as the weak separability model is nested within the standard utility maximization model, the former model will have a lower pass rate than the latter model by construction. Indeed, Bronars (1987) and, more recently, Andreoni and Harbaugh (2008) and Beatty and Crawford (2010) -rather convincingly- argue that revealed preference test results (indicating pass or fail of the data for some behavioral condition) should be complemented with power measures to obtain a fair empirical assessment of the rationalizability conditions under evaluation. Here, power is measured as the probability of rejecting the revealed preference test given that the model does not hold. Favorable test results (i.e. a high pass rate for some given data), which prima facie suggest a good empirical fit, have little value if the test has little discriminatory power (i.e. the conditions are hard to reject for the data at hand).

For all revealed preference tests under evaluation, we compute a power measure for every individual household. This measure quantifies discriminatory power in terms of the probability to detect random behavior, and is constructed as follows. We simulate 1000 random series of eight consumption choices by drawing for each of the eight observed household budgets, a random quantity bundle from a uniform distribution on the given budget hyperplane for the corresponding prices and total expenditure. The power measure is then calculated as one minus the proportion of these randomly generated consumption series that are consistent with the rationalizability conditions under evaluation. The distribution of this power measure for the different models is given in Table 1. We see that the standard utility maximization model has a rather low power. On average only about 11% of all random data sets violate the revealed preference conditions of Theorem 1. By contrast, the power distribution for the homothetic separability test is entirely centered around 1, with almost no spread. In other words, nearly all random data sets fail this test, which confirms its stringency. Finally, the weak separability test has reasonably high power while the power of the indirect weak separability test is fairly low.

This last finding suggests that, from an empirical point of view, indirect separability is much less stringent than weak separability. In this respect, we recall that the weak separability model was associated with a higher pass rate for the sample at hand. Now, it seems that this better fit may simply be due to a lower discriminatory power rather than a better model per se. Our following exercise accounts for the possible trade-off between pass rate and power.

Predictive success. The above analysis compares the four behavioral models in terms of to their pass rates and discriminatory power. Beatty and Crawford (2010) recently suggested to combine these two (often inversely related) performance measures into a single metric. More specifically, building further on an original idea of Selten (1991), they suggest to assess the empirical performance of a model by a so-called predictive success measure which, for a given household, is computed as the difference between the pass rate (either 1 or 0) and 1 minus the power. By construction, this measure takes values between -1 and 1. Negative values then suggest that the model under study is rather inadequate to describe the household data at hand: the model provides a poor fit of the household behavior (pass rate is zero) even though the model's power is low (i.e. the model is difficult to reject empirically). Conversely, a high and positive predictive success value points to a potentially useful model: it is able to explain the observed consumption behavior (i.e. pass rate equals 1) while its power is high (i.e. the model would rapidly be rejected in case of random behavior).

Table 2 presents the mean and quartile values of the predictive success measures associated with the four models under study. We observe that the standard utility maximization model has the highest mean predictive success. However, the value of 0.013 is still very low. In general, the mean predictive success values do not provide a strong empirical case in favor of one or the other model.

Let us then consider the quartile values. Here, we get a more balanced picture. For the homothetic separability model, the predictive success measure is entirely centered around zero with (practically) no spread. This result directly follows from the fact that this model has, for each household, a zero pass rate combined with power (close to) unity. Next, the distributions of the predictive success measures are almost identical for the standard utility maximization model and the indirect weak separability model. In other words, indirect separability seems to add little value (if any) over and above basic utility maximization in terms of predictive success. Finally, the predictive success distribution of the weak separability model has fatter tails than the ones of both the standard utility maximization model and the indirect separability model: on the one hand, there are a lot of households with very negative predictive success values for weak separability but, on the other hand, there are also a lot of households with large and positive predictive success values. One interpretation is that the weak separability model performs rather well empirically for one subgroup of households while it does a fairly poor job for other households. Given this, it can be useful to investigate which household characteristics determine the good fit of the weak separability model. Because our empirical application is mainly meant to be illustrative, we will not explore this route here, but we do see this as a potentially interesting avenue for follow-up research.

A related point concerns the observation that the weak separability model dominates the indirect weak separability model in terms of predictive success for the median, third quartile and maximum values. This suggests that weak separability may effectively constitute an appropriate model to describe the consumption behavior of most households in our sample. While it provides a worse fit than indirect separability at the overall sample level, for those households that do pass the weak separability test the higher discriminatory power effectively makes this model more useful from an empirical point of view. That is, for many households we obtain a predictive success value that is substantially above zero.

Table 2: Predictive success

model	mean	min	1st quartile	median	3rd quartile	max
general ut. max.	0.013	-1	0	0.031	0.171	0.610
weak separability	-0.005	-0.950	-0.443	0.099	0.428	0.942
homothetic separability	0	-0.002	0	0	0	0
indirect separability	-0.056	-0.998	0	0.016	0.191	0.733

Table 3: Computational speed for different data set sizes

size of data set	number of datasets	pass rate	(mean) time (in seconds)	var time
8	152	84	0.023	0.00001
16	87	5	0.099	0.0002
24	49	1	0.256	0.001
32	38	0	0.559	0.009
40	48	0	1.125	0.021
48	22	0	1.911	0.067
56	22	0	3.223	0.098
64	14	0	5.206	0.144
72	7	0	7.74	1.256
80	9	0	12.042	1.835
88	6	0	15.319	1.223
96	5	0	21.644	5.367
104	1	0	39.267	N.A.
120	2	0	50.543	13.256

Large data sets and computational speed. The above empirical illustration considers data sets with (only) 8 observations. Not very surprisingly, for such small data sets the integer programming method we propose comes to a test result very rapidly. Here, it seems interesting to assess whether this 'speediness' also holds if we increase the size of the data sets. As is well known, integer programming problems might become increasingly hard to solve as the size of the problem gets larger. To assess whether our integer programming method also works well for substantially large data sets, we assume identical preferences for all households with the same age of the male and female household members. In practice, this means that we perform our separability tests on pooled data sets containing all households with equally aged household members. A similar homogeneity assumption is frequently used in econometric demand analysis, i.e. demand estimation is often conditioned on ages of the household members as demographic factors.

As can be seen from Table 3, the size of our newly constructed data sets varies from 8 to 120 observations, with the average number of observations equal to 27.44. Clearly, this implies relatively big data sets as compared to other data sets that have been considered in empirical revealed preference analysis.[12] To keep our discussion focused, we here only report on the revealed preference conditions for the weak separability model. The results for the other separability models are more or less similar in terms of computational speed.

The third column of Table 2 reports the pass rates for the data sets of different sizes. However, our main interest here is in the fourth column of the table, which gives the average computation time of our algorithm for the different data set sizes that we consider.[13] Generally,

[12] See, for example, Cherchye, De Rock, Sabbe, and Vermeulen (2008) for a discussion on the typical size of data sets considered in empirical revealed preference analysis.

[13] We performed all our computations on a laptop computer with 2.4GHz clock speed and 4GB RAM with a

these results provide a fairly strong case in favor of our integer programming approach. For example, checking the revealed preference conditions for weak separability takes (on average) less than a second for data sets with up to 32 observations. However, if we keep increasing the number of observations, the computational time increases rapidly. Nevertheless, even for the largest data sets with 120 observations we obtain an outcome in less than a minute (on average), which -in our opinion- is still reasonably fast.

5 Conclusion

We considered the revealed preference conditions for weak separability. From a theoretical perspective, we found that verifying these conditions is a difficult (= NP-complete) problem. Given this, we introduced an integer programming approach to test data consistency with the conditions. We illustrated the versatility of this approach by deriving formally similar integer programming tests for the case of homothetic separability and indirect separability.

Further, we showed the empirical viability of our integer programming approach by providing an application to Spanish household consumption data. In this application, we focused on separability between food expenditures and other expenditures (on nondurables). An interesting observation was that indirect weak separability was associated with a higher pass rate than weak separability for the sample of households at hand. However, we also found that the weak separability test had substantially more discriminatory power than the indirect separability test. As such, the weak separability model was associated with a rather favorable predictive success measure (indicating a high degree of empirical usefulness) for most households that we considered. Finally, we also investigated the computational tractability of our integer programming procedure if we increased the size of the data sets. Here, we demonstrated that the test procedure works rather quickly even in the case of substantially large data sets (with up to 120 observations).

We see multiple avenues for further research. First of all, at the theoretical level, we have concentrated on three most commonly used types of separability, which have been established in the literature for a long time: weak separability, homothetic separability and indirect weak separability. More recently, Blundell and Robin (2000) introduced the notion of latent separability, a generalization of weak separability that provides an attractive empirical and theoretical framework for investigating the grouping of goods and prices. Crawford (2004) has derived the revealed preference conditions for latent separability. Just like for weak separability, in their original formulation these conditions are nonlinear (quadratic) and thus hard to verify. We believe it interesting to explore whether and to what extent the integer programming approach set out in the current paper may help to derive necessary and/or sufficient testable (integer programming) formulations of Crawford's conditions for latent separability.

Next, at the methodological level, we focused our discussion by only considering revealed preference tests for alternative separability specifications. If observed behavior is consistent with a particular specification (i.e. can be rationalized), then a natural next question pertains to recovering/identifying the structural features of the model under consideration. For example, in the present context such recovery can focus on identifying group (price/quantity) indices

standard configuration. For solving the integer programming problem, we used the commercial solver CPLEX©.

that are consistent with a separable representation of the utility structure. Because the revealed preference approach does not require a prior specification for the utility functions, it addresses recovery questions by 'letting the data speak for themselves' (i.e. it only uses the information that is directly revealed by the data). See, for example, Afriat (1967) and Varian (1982) for detailed discussions of revealed preference recoverability. These authors consider the standard utility maximization model. By using the integer programming formulations developed in the current paper, one can address similar recovery questions under alternative separability assumptions.[14]

Finally, the rationalizability tests discussed above are 'sharp' tests: they only tell us whether observations are exactly consistent with the rationalizability conditions that are under evaluation. However, as argued by Varian (1990), exact consistency may not be a very interesting hypothesis. Rather, one may be interested whether the behavioral model under study provides a reasonable way to describe observed behavior; for most purposes, 'nearly optimizing behavior' is just as good as 'optimizing' behavior. Also, in empirical applications of revealed preference tests it is often useful to account for measurement errors in the price and quantity data. This pleads for refinements of our integer programming tests that account for optimization and/or measurement errors. In this respect, we indicate that our integer programming formulations can easily account for such issues. Specifically, it is fairly straightforward to include the methodological extensions that Varian (1985, 1990) originally proposed in the case of the standard utility maximization model to deal with (measurement/optimization) errors.[15]

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[14]Compare with Cherchye, De Rock, and Vermeulen (2011a),who address recovery questions for collective consumption models by using a closely similar integer programming approach.

[15]See Cherchye, De Rock, Sabbe, and Vermeulen (2008), who consider such extensions for formally similar integer programming tests that apply to collective consumption models. In a more recent paper, Hjerstrand (2011a) also proposed using integer programming to deal with optimization error in revealed preference tests.

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Appendix A: proof of Theorem 3

Proof. In order to show that the problem of rationalizability by a weakly separable utility function is in the class NP, we need to reduce a known NP-complete problem to this decision problem. For this we use the problem of Monotone 3SAT (M3SAT).

M3SAT

INSTANCE: A set of binary variables $b_1, \ldots, b_t$ and a set of clauses $C_1, \ldots, C_r$. Each clause C_ℓ, $\ell = 1, \ldots, r$, contains three literals $l_{1,\ell}, l_{2,\ell}$ and $l_{3,\ell}$ and each literal either equals a variable or its negation. The condition monotone refers to the fact that for every clause all literals within this clause are either negated or unnegated.

QUESTION: Does there exist an assignment to the variables $b_1, \ldots, b_t$ (either 1 or 0) such that each clause contains at least one literal with the values equal to 1?

Now, consider an instance of M3SAT. We first construct the set of observations T and the sets of goods T and S.:

- For every literal $l_{k,\ell}$ ($\ell = 1, \ldots, r$ and $k = 1, 2, 3$), we construct two observations $t(k, \ell)$ and $v(k, \ell)$. These observations are gathered in the set T'.
- For every literal $l_{k,\ell}$ ($\ell = 1, \ldots, r$ and $k = 1, 2, 3$), we create two goods $g(t, k, \ell)$ and $g(v, k, \ell)$.
- For every literal $l_{k,\ell}$ ($\ell = 1, \ldots, r$ and $k = 1, 2, 3$), we create two goods $h(t, k, \ell)$ and $h(v, k, \ell)$.

For two literals l and l', we say that they are opposites if l corresponds to a variable b_i and l' corresponds to $(1 - b_i)$ or l corresponds to $(1 - b_i)$ and l' corresponds to b_i (i.e. $l \equiv (1 - l')$). We consider some special subsets of the set of goods.

- $\mathcal{G}_t = \{g(t, k, \ell) | k = 1, 2, 3; \ell = 1, \ldots, r\}$.
- $\mathcal{G}_v = \{g(v, k, \ell) | k = 1, 2, 3; \ell = 1, \ldots, r\}$.
- $\mathcal{O}(t, k, \ell) = \left\{ g(v, k', \ell') \,\middle|\, \begin{array}{l} \text{the k-th literal in clause } \ell \text{ and the } k'\text{th literal} \\ \text{in clause } \ell' \text{ are opposites} \end{array} \right\}$.
- $\mathcal{H}_t = \{h(t, k, \ell) | k = 1, 2, 3; \ell = 1, \ldots, r\}$.
- $\mathcal{H}_v = \{h(v, k, \ell) | k = 1, 2, 3; \ell = 1, \ldots, r\}$.

The goods in the separable group (bundle $\mathbf{y}$) are the goods $g(t, k, \ell)$ and $g(v, k, \ell)$. For k and $l \in \mathbb{N}$ denote by $k \oplus l$ the number $(k + l) \mod 3$. The remaining goods are the goods for the nonseparable group (bundel $\mathbf{x}$). The prices and quantities for each observation and good are summarized in the following tables for all $k = 1, 2, 3$ and $\ell = 1, \ldots, r$ (prices are before the separator '|', quantities after).

Here, the numbers $\mathfrak{p}, \mathfrak{z}$ and $\mathfrak{y}$ are given by:

$$\mathfrak{p} = 14 + 35r, \qquad \mathfrak{z} = 16 + 42r, \qquad \mathfrak{y} = 11 + 29r,$$

Table 4: Prices and quantities for instance of weak separability

observation	$g(t,k,\ell)$	$\mathcal{G}_t - \{g(t,k,\ell)\}$	$\mathcal{G}_v - \mathcal{O}(t,k,\ell)$	$O(t,k,\ell)$
$t(k,\ell)$	$\mathfrak{p}\|1$	$1\|2$	$1\|2$	$1\|1-\frac{1}{\mathfrak{p}}$

observation	$h(t,k,\ell)$	$\mathcal{H}_t - \{h(t,k,\ell)\}$	$h(v,k,\ell)$	$\mathcal{H}_v - \{h(v,k,\ell)\}$
$t(k,\ell)$	$\mathfrak{z}\|2$	$1\|3$	$1\|1-\frac{1}{\mathfrak{y}}$	$1\|2$

observation	$g(v,k,\ell)$	$\mathcal{G}_t$	$\mathcal{G}_v - \{g(v,k,\ell)\}$	
$v(k,\ell)$	$\mathfrak{p}\|1$	$1\|2$	$1\|2$	

observation	$h(t,k\oplus 2,\ell)$	$\mathcal{H}_t - \{h(t,k\oplus 2,\ell)\}$	$h(v,k,\ell)$	$\mathcal{H}_v - \{h(v,k,\ell)\}$
$v(k,\ell)$	$1\|1$	$1\|3$	$\mathfrak{y}\|1$	$1\|2$

with r the number of clauses.

We have to show that M3SAT has a solution if and only if the data set constructed above is weakly separable rationalizable. First let us assume that the data set is weakly separable rationalizable. Let $S_{t(k,\ell)}$ and $S_{v(k,\ell)}$ and $U_{t(k,\ell)}, U_{v(k,\ell)}$ be the Afriat numbers for the observations $t(k,\ell)$ and $v(k,\ell)$ that correspond to this rationalization. The idea is to set the value of the variables in such a way as to guarantee that the kth literal in the ℓth clause is equal to one whenever $S_{t(k,\ell)} \geq S_{v(k,\ell)}$. We need to verify that this is possible and that this leads to a solution of M3SAT. The following facts will be helpful.

Fact 1. *For all $k, k' = 1,2,3$ and $\ell, \ell' = 1,\ldots,r$, if the kth literal in the ℓth clause and the k'th literal in the ℓ'th are opposites, then $S_{v(k,\ell)} > S_{t(k',\ell')}$.*

Proof. We have that:

$$\begin{aligned}
S_{t(k',\ell')} - S_{v(k,\ell)} &\leq \delta_{v(k,\ell)} \mathbf{q}_{v(k,l)} \left[\mathbf{y}_{t(k',\ell')} - \mathbf{y}_{v(k,\ell)}\right] \\
&= \delta_{v(k,\ell)} \left[-1 - 1 + (-2 + 1 - 1/\mathfrak{p}) \left|O(t,k',\ell') \cap (\mathcal{G}_v - \{g(v,k,\ell)\})\right|\right] \\
&< 0
\end{aligned}$$

□

Fact 2. *For all $\ell, \ell' = 1,\ldots,r$ and $k, k' = 1,2,3$ if the kth literal in the ℓth clause and the k'th literal in the ℓ'th clause are opposites then it is not the case that both $S_{t(k,\ell)} \geq S_{v(k,\ell)}$ and $S_{t(k',\ell')} \geq S_{v(k',\ell')}$.*

Proof. If, on the contrary, $S_{t(k,\ell)} \geq S_{v(k,\ell)}$ and $S_{t(k',\ell')} \geq S_{v(k',\ell')}$, we would have that (by fact 1):

$$S_{t(k,\ell)} \geq S_{v(k,\ell)} > S_{t(k',\ell')} \geq S_{v(k',\ell')} > S_{t(k,\ell)},$$

a contradiction. □

Facts 1 and 2 show that above construction above can be performed (i.e. it is never the case that two opposite literals have the value of one). The following fact demonstrates that it provides a solution to M3SAT.

Fact 3. *For all $\ell = 1, \dots, r$, there is at least one value $k = 1, 2, 3$ such that $S_{t(k,\ell)} \geq S_{v(k,\ell)}$.*

Proof. Let us first show that for all $k = 1, 2, 3$ and $\ell = 1, \dots, r$, $U_{t(k,\ell)} > U_{v(k\oplus 1,\ell)}$. Indeed,

$$\begin{aligned} U_{v(k\oplus 1,\ell)} - U_{t(k,\ell)} &\leq \lambda_{t(k,\ell)} \mathbf{p}_{t(k,\ell)} \left[\mathbf{x}_{v(k\oplus 1,\ell)} - \mathbf{x}_{t(k,\ell)}\right] + \lambda_{t(k,\ell)} \mathbf{q}_{t(k,\ell)} \left[\mathbf{y}_{v(k\oplus 1,\ell)} - \mathbf{y}_{t(k,\ell)}\right] \\ &= \lambda_{t(k,\ell)} \begin{bmatrix} \mathfrak{p} - 1 + (2 - 1 + 1/\mathfrak{p}) \left|O(t,k,\ell) \cap (\mathcal{G}_v - \{g(v,k,\ell\})\right| \\ -\mathfrak{z} + (2 - 1 + 1/\mathfrak{y}) + (1-2) \end{bmatrix} \\ &\leq \lambda_{t(k,\ell)} \left[\mathfrak{p} - 1 + 6r - \mathfrak{z} + 2 - 1\right] \\ &= \lambda_{t(k,\ell)} \left[(14 + 35r) + 6r - (16 + 42r)\right] < 0 \end{aligned}$$

Now, consider the identity

$$\begin{aligned} 0 = &\left[U_{v(k\oplus 1,\ell)} - U_{t(k,\ell)}\right] + \left[U_{v(k\oplus 2,\ell)} - U_{t(k\oplus 1,\ell)}\right] + \left[U_{v(k\oplus 3,\ell)} - U_{t(k\oplus 2,\ell)}\right] \\ &+ \left[U_{t(k,\ell)} - U_{v(k,\ell)}\right] + \left[U_{t(k\oplus 1,\ell)} - U_{v(k\oplus 1,\ell)}\right] + \left[U_{t(k\oplus 2,\ell)} - U_{v(k\oplus 2,\ell)}\right] \end{aligned}$$

The first three terms on the right hand side are negative, hence,

$$\begin{aligned} 0 &< \left[U_{t(k,\ell)} - U_{v(k,\ell)}\right] + \left[U_{t(k\oplus 1,\ell)} - U_{v(k\oplus 1,\ell)}\right] + \left[U_{t(k\oplus 2,\ell)} - U_{v(k\oplus 2,\ell)}\right] \\ &\leq \frac{\lambda_{v(1,\ell)}}{\delta_{v(1,\ell)}} \left[S_{t(1,\ell)} - S_{v(1,\ell)}\right] + \frac{\lambda_{v(2,\ell)}}{\delta_{v(k\oplus 1,\ell)}} \left[S_{t(2,\ell)} - S_{v(2,\ell)}\right] + \frac{\lambda_{v(3,\ell)}}{\delta_{v(3,\ell)}} \left[S_{t(3,\ell)} - S_{v(3,\ell)}\right] \end{aligned}$$

As such at least for one $k = 1, 2, 3$ it must be that $S_{t(k,\ell)} > S_{v(k,\ell)}$. □

Now, consider a 'yes' instance of M3SAT. We need to construct Afriat numbers S and δ for each observation that satisfy the the conditions for rationalizability by weak separability (see Theorem 2). Let us start by constructing a binary relation $\succ$. For $k, k' = 1, 2, 3$ and $\ell, \ell' = 1, \dots, r$ if the k-th literal in the ℓth clause and the k'th literal in the ℓ'th clause are opposites, we set $v(k,\ell) \succ t(k', \ell')$. Further, for all $k = 1, 2, 3$ and $\ell = 1, \dots, r$ if the kth literal in the ℓth clause has the value 1, we set $t(k,\ell) \succ v(k,\ell)$. These are the only comparisons in $\succ$. Observe that $\succ$ has no cycles and any path in $\succ$ contains no more than 4 observations.

Let M_1 be the set of $\succ$-maximal elements of T':

$$M_1 = \{a \in T \mid \nexists b \in T', b \succ a\}.$$

For all observations a in M_1, we set $S_a = 4$. Let M_2 be the set of $\succ$-maximal elements in $T' - M_1$. For all $a \in M_2$, set $S_a = 3$. Next, let M_3 be the set of $\succ$-maximal elements in $T' - (M_1 \cup M_2)$ and set $S_a = 2$ for all $a \in M_3$. Finally let M_4 be the set of $\succ$-maximal element in $T' - (M_1 \cup M_2 \cup M_3)$ and set for all $a \in M_4$, $S_a = 1$. It is easy to see that $M_1 \cup M_2 \cup M_3 \cup M_4 = T$, hence all observations are allocated a value. Observe that when the kth literal in the ℓth clause equals one, then $S_{t(k,\ell)} > S_{v(k,\ell)}$. Finally, for all $k = 1, 2, 3$ and $\ell = 1, \dots, r$, set $\delta_{t(k,\ell)} = 1$ and set $\delta_{v(k,\ell)} = \dfrac{1}{3 + 7r}$, where r is the number of clauses.

We need to proof two things. First we need to verify that all Afriat inequalities hold for every two observations in the set $\left\{t(k,\ell), v(k,\ell), t(k',\ell'), v(k',\ell')\right\}_{k,k'=1,2,3;\ell,\ell'=1,\ldots,r}$ (i.e. condition (ii.1) of Theorem 2). Second, we need to show that the data set $\{\mathbf{p}_w, 1/\delta_w, \mathbf{x}_w, S_w\}_{w\in T'}$ satisfies GARP (condition (ii.2)). For the first, it is a straightforward but cumbersome exercise to verify every possible combination of states. As such we refer to the appendix D. Now, let us verify the second claim. Consider the direct revealed preference relation R^D for the data set $\{\mathbf{p}_w, 1/\delta_w; \mathbf{x}_w, S_w\}_{w\in T'}$. We have following results.

Fact 4. *For all $k = 1,2,3$ and $\ell = 1,\ldots,r$, we have that the observation $t(k,\ell)$ is directly revealed preferred to the observation $v(k\oplus 1,\ell)$ (i.e. $(t(k,\ell), v(k\oplus 1,\ell)) \in R^D$).*

Proof. We have that:

$$
\begin{aligned}
&\mathbf{p}_{t(k,\ell)}\left[\mathbf{x}_{t(k,\ell)} - \mathbf{x}_{v(k\oplus 1,\ell)}\right] + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{v(k\oplus 1,\ell)}\right] \\
&= \mathfrak{z} + (1 - 1/\mathfrak{y} - 2) + (2-1) + \left[S_{t(k,\ell)} - S_{v(k\oplus 1,\ell)}\right] \\
&\geq \mathfrak{z} - 2 - 3 = 16 + 42r - 5 > 0
\end{aligned}
$$

□

Fact 5. *For all $k = 1,2,3$ and $\ell = 1,\ldots,r$, $(v(k,\ell), t(k,\ell)) \in R^D$ if and only if $S_{v(k,\ell)} \geq S_{t(k,\ell)}$ (which implies that the kth literal in the ℓth clause is equal to zero).*

Proof. We have that,

$$
\mathbf{p}_{v(k,\ell)}(\mathbf{x}_{v(k,\ell)} - \mathbf{x}_{t(k,\ell)}) + \frac{1}{\delta_{v(k,\ell)}}\left[S_{v(k,\ell)} - S_{t(k,\ell)}\right] = \frac{1}{\delta_{v(k,\ell)}}\left[S_{v(k,\ell)} - S_{t(k,\ell)}\right].
$$

This is positive or negative depending on the sign of $S_{v(k,\ell)} - S_{t(k,\ell)}$. □

Fact 6. *The relation R^D contains no comparisons except for the cases mentioned by Facts 4 and 5.*

Proof. See appendix E. □

Now, assume a violation of GARP. Above Facts show that this implies the following cycle for some $\ell = 1,\ldots,r$:

$$
\begin{aligned}
&(t(1,\ell), v(2,\ell)),\ (v(2,\ell), t(2,\ell)),\ (t(2,\ell), v(3,\ell)) \\
&\qquad\qquad (v(3,\ell), t(3,\ell)),\ (t(3,\ell), v(1,\ell)),\ (v(1,\ell), t(1,\ell)).
\end{aligned}
$$

Fact 5 shows that in this case $S_{v(1,\ell)} \geq S_{t(1,\ell)}$, $S_{v(2,\ell)} \geq S_{t(2,\ell)}$ and $S_{v(3,\ell)} \geq S_{t(3,\ell)}$. This can only be the case if all literals in the clause ℓ are zero, a contradiction. □

Appendix B: proof of Theorem 4

Proof. Assume that the data set $D = \{\delta_t \mathbf{p}_t, 1; \mathbf{x}_t, S_t\}_{t \in T}$ satisfies GARP. From Theorem 1, we know that this data set is rationalizable. As such, consider a utility function u that rationalizes this data set. It is always possible to rescale u such that, for all $t \in T$, $u(\mathbf{x}_t, S_t) < 1$. Then, for all $t \in T$, define $U_t = u(\mathbf{x}_t, S_t)$ and, for all $t, v \in T$, define $X_{t,v} = 1$ if and only if $U_t \geq U_v$ (i.e. $X_{t,v} = 0$ if $U_t < U_v$) . We must show that conditions (cs.2)-(cs.5) hold. By definition, conditions (cs.2) and (cs.3) are always satisfied. Let $\delta_t \mathbf{p}_t \mathbf{x}_t + S_t \geq \delta_t \mathbf{p}_t \mathbf{x}_v + S_v$. Then, as u rationalizes the data set D, it must be that $U_t \geq U_v$ and thus $X_{t,v} = 1$. This demonstrates that condition (cs.4) holds. Next, assume that $X_{t,v} = 1$, which implies $U_t \geq U_v$. As u rationalizes D, we have that it is impossible that $\delta_v \mathbf{p}_v \mathbf{x}_v + S_v > \delta_v \mathbf{p}_v \mathbf{x}_t + S_t$. Otherwise, we would have that $U_v > U_t$. This shows that (cs.5) is also satisfied.

For the reverse, assume that (cs.2)-(cs.5) has a solution. We need to show that D satisfies GARP. If $\delta_t \mathbf{p}_t \mathbf{x}_t + S_t \geq \delta_t \mathbf{p}_t \mathbf{x}_v + S_v$, we have, from (cs.4), that $X_{t,v} = 1$. Condition (cs.3) then requires that $U_t \geq U_V$. As such, by transitivity, if $(\mathbf{x}_t, S_t)$ is indirectly revealed preferred to $(\mathbf{x}_v, S_v)$, it must also be the case that $U_t \geq U_v$. Condition (cs.2) then shows that $X_{t,v} = 1$. Now, if $(\mathbf{x}_t, S_t)$ is indirectly revealed preferred to $(\mathbf{x}_v, S_v)$ (and hence $X_{t,v} = 1$), then condition (cs.5) requires that $\delta_v \mathbf{p}_v \mathbf{x}_v + S_v \leq \delta_v \mathbf{p}_v \mathbf{x}_t + S_t$. As such, the data set D satisfies GARP. $\square$

Appendix C: MIP formulation for rationalizability by indirect weak separability

Multiplying (iv.1) by minus one gives:

$$(-W_t) - (-W_v) \leq \delta_v \mathbf{y}_v (\mathbf{q}_t - \mathbf{q}_v)$$

Next, observe that (iv.2) can be rewritten as:

$$(-V_t) - (-V_v) \leq \lambda_v \left(\mathbf{x}_v (\mathbf{p}_t - \mathbf{p}_v) + \frac{1}{\delta_v} ((-W_t) - (-W_v)) \right).$$

Now, transform each variable V_t and W_t by adding a strict positive number A such that for each t, $A - V_t$ and $A - W_t$ become positive. This does not change the left and right hand side of above inequalities. Then, using $\tilde{W}_t = A - W_t$ and $\tilde{V}_t = A - V_t$, we obtain:

$$\begin{aligned} \tilde{W}_t - \tilde{W}_v &\leq \delta_v \mathbf{y}_v (\mathbf{q}_t - \mathbf{q}_v), \\ \tilde{V}_t - \tilde{V}_v &\leq \lambda_v \left(\mathbf{x}_v (\mathbf{p}_t - \mathbf{p}_v) + \frac{1}{\delta_v} (\tilde{W}_t - \tilde{W}_v) \right). \end{aligned}$$

From Theorem 2, we see that the last condition is equivalent to the condition that the data set $\{\mathbf{x}_t, 1/\delta_t, \mathbf{p}_t, \tilde{W}_t\}_{t \in T}$ satisfies GARP. Observe that the prices and quantity vector have exchanged places. The result follows then from the proof of Theorem 4.

Appendix D: supplement to Appendix A

Case 1: $(t(k,\ell), t(k',\ell'))$

$$\begin{aligned}\delta_{t(k,\ell)}\mathbf{q}_{t(k,\ell)}\left[\mathbf{y}_{t(k',\ell')}-\mathbf{y}_{t(k,\ell)}\right] = & \begin{array}{l}\mathfrak{p}+(1-2)+(1-2-1/\mathfrak{p})\left|\mathcal{G}-\mathcal{O}(t,k,\ell)\cap O(t,k',\ell')\right| \\ +(2-1+1/\mathfrak{p})\left|\mathcal{O}(t,k,\ell)\cap\left(\mathcal{G}_v-\mathcal{O}(t,k',\ell')\right)\right|\end{array} \\ & \geq \mathfrak{p}-1-6r = 14+35r-6r > 3 \geq S_{t(k',\ell')}-S_{t(k,\ell)}\end{aligned}$$

Case 2: $(v(k,\ell), v(k',\ell'))$

$$\begin{aligned}\delta_{v(k,\ell)}\mathbf{q}_{v(k,\ell)}\left[\mathbf{y}_{v(k',\ell')}-\mathbf{y}_{v(k,\ell)}\right] &= \delta_{v(k,\ell)}\left[\mathfrak{p}+(1-2)\right] \\ &= \frac{14+35r-1}{3+7r} > 3 \geq S_{v(k',\ell')}-S_{v(k,\ell)}\end{aligned}$$

Case 3: $(t(k,\ell), v(k',\ell'))$

$$\begin{aligned}\delta_{t(k,\ell)}\mathbf{q}_{t(k,\ell)}\left[\mathbf{y}_{v(k',\ell')}-\mathbf{y}_{t(k,\ell)}\right] = & \begin{array}{l}\mathfrak{p}+(1/\mathfrak{p})\left|\mathcal{O}(t,k,l)\cap\{g(v,k',\ell')\}\right| \\ +(1-2)\left|(\mathcal{G}_v-\mathcal{O}(t,k,\ell))\cap\{g(v,k',\ell'\}\right| \\ +(2-1+1/\mathfrak{p})\left|\mathcal{O}(t,k,l)\cap\left(\mathcal{G}_v-\{g(v,k',\ell')\}\right)\right|\end{array} \\ & \geq \mathfrak{p}-1 = 14+35r-1 > 3 \geq S_{v(k',\ell')}-S_{t(k,\ell)}\end{aligned}$$

Case 4: $(v(k,\ell), t(k',\ell'))$ and the kth literal in the ℓth clause and the k'th literal in the ℓ'th clause have opposite signature. First of all, observe that $-1 \geq S_{t(k',\ell')}-S_{v(k,\ell)}$.

Then:

$$\begin{aligned}\delta_{v(k,\ell)}\mathbf{q}_{v(k,\ell)}\left[\mathbf{y}_{t(k',\ell')}-\mathbf{y}_{v(k',\ell')}\right] &= \delta_{v(k,\ell)}\left[\begin{array}{l}\mathfrak{p}(1-1/\mathfrak{p}-1)+(1-2) \\ +(1-2-1/\mathfrak{p})\left|(\mathcal{G}_v-\{g(v,k,\ell)\})\cap\mathcal{O}(t,k',\ell')\right|\end{array}\right] \\ &\geq \frac{-1-1-6r}{3+7r} > -1 \geq S_{t(k',\ell')}-S_{v(k,\ell)}\end{aligned}$$

Case 5: $(v(k,\ell), t(k',\ell'))$ and the kth literal in the ℓth clause and the k'th literal in the ℓ'th clause do not have opposite signature.

$$\begin{aligned}\delta_{v(k,\ell)}\mathbf{q}_{v(k,\ell)}\left[\mathbf{y}_{t(k',\ell')}-\mathbf{y}_{v(k',\ell')}\right] &= \delta_{v(k,\ell)}\left[\begin{array}{l}\mathfrak{p}+(1-2) \\ +(1-1/\mathfrak{p}-2)\left|(\mathcal{G}_v-\{g(v,k,\ell)\})\cap\mathcal{O}(t,k',\ell')\right|\end{array}\right] \\ &\geq \frac{\mathfrak{p}-1-6r}{3+7r} = \frac{13+29r}{3+7r} > 3 \geq S_{t(k',\ell')}-S_{v(k,\ell)}\end{aligned}$$

Appendix E: supplement to Appendix A

Case 1: $(t(k,\ell), t(k',\ell'))$

$$\begin{aligned}
&\mathbf{p}_{t(k,\ell)}\left[\mathbf{x}_{t(k,\ell)} - \mathbf{x}_{t(k',\ell')}\right] + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{t(k',\ell')}\right] \\
&= -\mathfrak{z} + (3-2) + \left[S_{t(k,\ell)} - S_{t(k',\ell')}\right] \\
&\leq -16 - 42r + 1 + 3 < 0
\end{aligned}$$

Case 2: $(t(k,\ell), v(k',\ell'))$ with $\ell \neq \ell'$.

$$\begin{aligned}
&\mathbf{p}_{t(k,\ell)}\left[\mathbf{x}_{t(k,\ell)} - \mathbf{x}_{v(k',\ell')}\right] + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{v(k',\ell')}\right] \\
&= -\mathfrak{z} + (3-1) + (1 - 1/\mathfrak{y} - 2) + (2-1) + \left[S_{t(k,\ell)} - S_{v(k',\ell')}\right] \\
&\leq -16 - 42r + 2 + 1 + 3 < 0
\end{aligned}$$

Case 3: $(t(k,\ell), v(k,\ell))$

$$\begin{aligned}
&\mathbf{p}_{t(k,\ell)}\left[\mathbf{x}_{t(k,\ell)} - \mathbf{x}_{v(k,\ell)}\right] + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{v(k,\ell)}\right] \\
&= -\mathfrak{z} + (3-1) + (1 - 1/\mathfrak{y} - 1) + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{v(k,\ell)}\right] \\
&\leq -16 - 42r + 2 + 3 < 0
\end{aligned}$$

Case 4: $(t(k,\ell), v(k \oplus 2,\ell))$

$$\begin{aligned}
&\mathbf{p}_{t(k,\ell)}\left[\mathbf{x}_{t(k,\ell)} - \mathbf{x}_{v(k\oplus 2,\ell)}\right] + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{v(k\oplus 2,\ell)}\right] \\
&= -\mathfrak{z} + (3-1) + (1 - 1/\mathfrak{y} - 2) + 1 + \frac{1}{\delta_{t(k,\ell)}}\left[S_{t(k,\ell)} - S_{v(k\oplus 2,\ell)}\right] \\
&\leq -16 - 42r + 2 + 1 + 3 < 0
\end{aligned}$$

Case 5: $(v(k,\ell), v(k',\ell'))$

$$\begin{aligned}
&\mathbf{p}_{v(k,\ell)}\left[\mathbf{x}_{v(k,\ell)} - \mathbf{x}_{v(k',\ell')}\right] + \frac{1}{\delta_{v(k,\ell)}}\left[S_{v(k,\ell)} - S_{v(k',\ell')}\right] \\
&= -\mathfrak{y} + (2-1)\frac{1}{\delta_{v(k,\ell)}}\left[S_{v(k,\ell)} - S_{v(k',\ell')}\right] \\
&\leq -11 - 29r + 1 + (3+7r)3 < 0
\end{aligned}$$

Case 6: $(\nu(k,\ell), t(k',\ell'))$ with $\ell \neq \ell'$.

$$\begin{aligned}
&\mathbf{p}_{\nu(k,\ell)}\left[\mathbf{x}_{\nu(k,\ell)} - \mathbf{x}_{t(k',\ell')}\right] + \frac{1}{\delta_{\nu(k,\ell)}}\left[S_{\nu(k,\ell)} - S_{t(k',\ell')}\right] \\
&= (1-3) + (3-2) - \mathfrak{y} + (2-1+1/\mathfrak{y}) + \frac{1}{\delta_{\nu(k,\ell)}}\left[S_{\nu(k,\ell)} - S_{t(k',\ell')}\right] \\
&\leq -11 - 29r + 1 + (3+7r)3 < 0
\end{aligned}$$

Case 7: $(\nu(k,\ell), t(k\oplus 1,\ell))$

$$\begin{aligned}
&\mathbf{p}_{\nu(k,\ell)}\left[\mathbf{x}_{\nu(k,\ell)} - \mathbf{x}_{t(k\oplus 1,\ell)}\right] + \frac{1}{\delta_{\nu(k,\ell)}}\left[S_{\nu(k,\ell)} - S_{t(k\oplus 1,\ell)}\right] \\
&= (1-3) + (3-2) - \mathfrak{y} + (2-1+1/\mathfrak{y}) + \frac{1}{\delta_{\nu(k,\ell)}}\left[S_{\nu(k,\ell)} - S_{t(k\oplus 1,\ell)}\right] \\
&\leq -11 - 29r + 1 + (3+7r)3 < 0
\end{aligned}$$

Case 8: $(\nu(k,\ell), t(k\oplus 2,\ell))$

$$\begin{aligned}
&\mathbf{p}_{\nu(k,\ell)}\left[\mathbf{x}_{\nu(k,\ell)} - \mathbf{x}_{t(k\oplus 2,\ell)}\right] + \frac{1}{\delta_{\nu(k,\ell)}}\left[S_{\nu(k,\ell)} - S_{t(k\oplus 2,\ell)}\right] \\
&= (1-2) - \mathfrak{y} + (2-1+1/\mathfrak{y}) + \frac{1}{\delta_{\nu(k,\ell)}}\left[S_{\nu(k,\ell)} - S_{t(k\oplus 2,\ell)}\right] \\
&\leq -11 - 29r + 1 + (3+7r)3 < 0
\end{aligned}$$