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by

Luc Lauwers

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Strong anonymity and infinite streams

Luc Lauwers

E-mail: luc.lauwers@econ.kuleuven.be

Center for Economic Studies, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium

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Abstract. The extended rank-discounted utilitarian social welfare order introduced and axiomatized by Stéphane Zuber and Geir B. Asheim satisfies strong anonymity (J. Econ. Theory (2011), doi:10.1016/j.jet.2011.08.001). We question the appropriateness of strong anonymity in the context of a countably infinite sequence of subsequent generations. A modified criterion that is incomplete and satisfies finite anonymity is presented.

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1 The extended rank-discounted utilitarian social welfare order

Zuber and Asheim study a countably infinite sequence of subsequent generations and consider the collection X of all bounded infinite streams. For each x in X a related non-decreasing stream $x_{[]} = (x_{[1]}, x_{[2]}, \ldots, x_{[t]}, \ldots)$ is defined. Let $\ell(x)$ denote $\liminf_{t \to +\infty} x_t$ and consider the restriction y of x to $L(x) = \{t \text{ in } \mathbb{N} : x_t < \ell(x)\}$. If y is an infinite subsequence of x, then $x_{[]}$ rewrites y in non-decreasing order. If $y = (y_1, y_2, \ldots, y_n)$ is a finite vector, then $x_{[]}$ rewrites the sequence $(y_1, y_2, \ldots, y_n, \ell(x), \ell(x), \ldots, \ell(x), \ldots)$ in non-decreasing order.

An extended rank-discounted utilitarian social welfare order [1, Definition 2] is represented by a social welfare function $W : X \to \mathbb{R}$ defined by

$$W(x) \;=\; u(\ell(x)) + (1-\beta) \sum_{r=1}^{|L(x)|} \beta^{r-1} \Big(u(x_{[r]}) - u(\ell(x)) \Big),$$

with $0 < \beta < 1$ and u a continuous and increasing real valued function. The length of the discounted sum in $W(x)$ either is finite (if $|L(x)| < +\infty$) or infinite (if $|L(x)| = +\infty$).

1.1 Strong anonymity

The welfare order represented by W satisfies strong anonymity [1, Theorem 1]. Hence, for each stream x in X and for each permutation π on $\mathbb{N}$, the stream x and its permuted stream $x_\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(t)}, \ldots)$ are equally good.

Let us illustrate this axiom. Consider $y = (1/2, 2/3, \ldots, t/(t+1), \ldots)$ and let $z = 1\,000\,000$. Define infinite streams a and b as follows:

$$a_{10^k} \;=\; y_k \text{ for each } k \text{ in } \mathbb{N} \qquad \text{and} \qquad a_t = z \text{ elsewhere,}$$

and b is equal to y except that for each k in $\mathbb{N}$ at position 10^k the value $b_{10^k} = z$ is inserted in between two subsequent coordinates of y. The streams a and b both accommodate the infinite streams y and $(z, z, \ldots, z, \ldots)$. Therefore, there exists a permutation π on $\mathbb{N}$ such that $a = b_\pi$. Also, $\liminf a = \liminf b = 1$ and $a_{[]} = b_{[]} = y$. Hence, according to W the streams a and b are equally good. Now, change b into b' by changing $b_3 = 3/4$ into $b'_3 = 3/4 + 0.01$. The extended rank-discounted utilitarian social welfare order considers the resulting stream,

$$b' \;=\; \Big(1/2,\, 2/3,\, 0.76,\, 4/5,\, \ldots,\, 9/10,\, z,\, 10/11,\, \ldots,\, 98/99,\, z,\, 99/100,\, \ldots\Big),$$

as strictly better than b and, therefore, strictly better than a.

The set $\{10, 10^2, 10^3, \ldots, 10^k, \ldots\}$ has an asymptotic density (see below) equal to $\lim_{k\to+\infty} k/10^k = 0$. In this language, the consumption stream a allocates to almost each generation the high level z. In contrast, the consumption stream b allocates to almost each generation the low level 1. Nevertheless, the extended rank-discounted "*utilitarian*" rule is unable to see the difference and considers a and b as equally good. In addition, according to this criterion stream b' should be strictly preferred to a. I believe the indifference between the consumption streams a and b, and the ranking of stream b' over a is hard to defend.

2 Anonymity

Anonymity in a finite context is easy to understand and to model. Consider a population $N = \{1, 2, \ldots, n\}$. A welfare order on $\mathbb{R}^n$ is anonymous if for each permutation $\pi : N \to N$ and for each vector a in $\mathbb{R}^n$ we have indifference between a and a_π. A permutation only reshuffles the positions of the coordinates in a vector and has no effect upon the anonymous ranking. In addition, the counting measure (on a finite set) is invariant under each permutation: a subset S of individuals and a permuted set $\pi(S)$ have exactly the same weight, that is the relative numbers $|S|/|N|$ and $|\pi(S)|/|N|$ coincide. In sum, anonymity entails equal treatment of individuals and equal treatment of equally large groups of individuals.

In the framework of subsequent generations the weight of a subset S is indicated by its asymptotic density (when it exists[1]):

$$\delta(S) \;=\; \lim_{t\to\infty} \frac{|\,S \cap \{1, 2, \ldots, t\}|}{t}.$$

[1]If $|S \cap \{1, 2, \ldots, t\}|/t$ does not converge, the asymptotic density $\delta(S)$ is not defined.

For example, sets of the form $\{n, 2n, \ldots, kn, \ldots\}$ have asymptotic density equal to $1/n$. If anonymity is supposed to guarantee equal treatment of generations and of groups of generations, then the set of permutations used to operationalize this idea should be restricted to the set of density-preserving permutations. A finite permutation π on $\mathbb{N}$ satisfies $\pi(t) = t$ for all but a finite set of numbers t in $\mathbb{N}$ and preserves asymptotic densities. On the other hand, the previous section presented an infinite permutation that transforms a set of generations with density 0 into a set with density 1. Obviously, that kind of permutations do not preserve asymptotic densities. As illustrated, a social welfare order that satisfies strong anonymity is unable to take the weights of subsets into account.

We now argue that the frameworks of subsequent generations and of an infinite set of individuals are different and that asymptotic densities are relevant in the former framework. In case one considers an (unordered) set of individuals, the asymptotic density is not relevant. The countably infinite sets

$$\{1, 2, \ldots, n, \ldots\} \quad \text{and} \quad \{1, 2, 4, 3, 6, 8, \ldots, 2n-1, 4n-2, 4n, \ldots\}$$

should be considered as being the same. Each list presents the very same set of individuals. The queuing, however, is different.

In contrast, the countably infinite vectors

$$(1, 2, \ldots, n, \ldots) \quad \text{and} \quad (1, 2, 4, 3, 6, 8, \ldots, 2n-1, 4n-2, 4n, \ldots)$$

should be considered as being different. The asymptotic density of the set of odd numbers is equal to $1/2$ in the first and $1/3$ in the second vector. The permutation that maps the first enumeration into the second one, again, does not preserve the asymptotic densities. As soon the set is ordered, e.g. if one considers moments in time (with an infinite future), then the asymptotic density becomes relevant. For example, a situation in which a particular event occurs once in each week differs from a situation in which the same event occurs once in each year (although in both situations the event happens infinitely many times).

2.1 A modified criterion

Consider an infinite number of subsequent generations. Let x be a bounded infinite stream with only a finite number of cluster points (or accumulation points). For each cluster point c consider the set

$$C = \{t \mid c - \varepsilon \leq x_t \leq c + \varepsilon\},$$

with $\varepsilon > 0$ such that the interval $[c - \varepsilon\, ,\, c + \varepsilon]$ contains exactly one cluster point (to wit, the cluster point c). If the asymptotic density $\delta(C)$ is well defined, then $\delta(c) = \delta(C)$ is said to be the weight of the cluster point c. Note that $\delta(c)$ does not depend upon the particular value of ε. Let Y collect those infinite streams that have a finite number of cluster points and for which the cluster points have well defined weights.

The stream a in the previous section has two cluster points 1 and z with weights 0 and 1, the stream b has the same cluster points 1 and z with weights 1 and 0.

We now define a modified criterion. A stream x in Y can be evaluated on the basis of its cluster points and their weights, e.g.

$$V(x) \;=\; u_k(c_1, c_2, \ldots, c_k\,;\, \delta(c_1), \delta(c_2), \ldots, \delta(c_k)),$$

with $c_1 < c_2 < \cdots < c_k$ the finite number of different cluster points in x and with u_k a continuous real valued function (defined for each natural number k).[2] In case V is indifferent between two streams x and y, some strong Pareto criterion might be used for further investigation.

Let us apply this modified criterion V to the stream $y = (x_1, x_2, \ldots, x_n)_{\text{rep}}$, i.e. the infinite repetition of the finite vector $(x_1, x_2, \ldots, x_n)$. Each x_i is a cluster point of y with weight $1/n$ (assume that the x_i are all different). We obtain

$$V(y) \;=\; u_n(x_1, x_2, \ldots, x_n\,;\, 1/n, 1/n, \ldots, 1/n).$$

In contrast, according to the extended rank-discounted utilitarian social welfare function, we obtain

$$W(y) \;=\; u(\min\{x_1, x_2, \ldots, x_n\}).$$

The effect of strong anonymity is again revealed. The extended rank-discounted utilitarian social welfare order only takes one cluster point into account (while V considers each cluster point as relevant).

This note questions the appropriateness of the extended rank-discounted utilitarian social welfare order introduced and characterized by Zuber and Asheim. Nevertheless, I want to stress that this welfare order, when restricted to the set of non-decreasing streams does not suffer from the strong anonymity problem and contributes to the ongoing debate on intertemporal welfare.

Reference

[1] Zuber, S, and G B Asheim, 2011. Justifying social discounting: The rank-discounted utilitarian approach. J. Econ. Theory, doi:10.1016/j.jet.2011.08.001.

[2] For example, $V(x) = \delta(c_1)\, c_1 \,+\, \delta(c_2)\, c_2 \,+ \cdots +\, \delta(c_k)\, c_k$.