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Stochastic signaling: information substitutes and complements

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Abstract

In a model of stochastic costly signaling in the presence of exogenous imperfect information, I study whether equilibrium signaling decreases ('information substitutes') or increases ('information complements') if the accuracy of exogenous information increases. A unique threshold level of prior beliefs generically exists that separates the cases of information complements and substitutes. More accurate exogenous information can induce a less informative signaling equilibrium, and can result in a lower expected accuracy of the uninformed party's equilibrium beliefs.

Keywords: Asymmetric Information, Monotonic Costly Signaling, Stochastic Signaling, Noisy Signaling, Advertising, Job Market Signaling, Conspicuous Consumption

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1 Introduction

Costly signaling models explain ostentatious waste as a way of communicating private information that otherwise cannot be credibly communicated, and have found numerous applications in recent decades.[1] Policy

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[1]See e.g. Riley (2001) for a survey of the economic literature. Examples include labor economics (Spence, 1973), advertising (Milgrom and Roberts, 1986), finance

makers and economists have since long deplored the welfare losses due to conspicuous waste, and occasionally applauded the welfare gains associated with the resulting information transfer (e.g. by solving Akerlof's (1970) market for lemons problem).[2] But what happens to ostentatious waste if better information is exogenously provided to the uninformed parties ('Receiver')? A common (but false) intuition is that better information about the subject of the informed party's ('Sender') private information is generally an efficient way of reducing wasteful signaling.

Veblen (1899(1994), pp.53-55) observes that "*Conspicuous consumption claims a relatively larger portion of the income of the urban than of the rural population, and the claim is also more imperative.* [...] *So it comes, for instance, that the American farmer and his wife and daughters are notoriously less modish in their dress, as well as less urbane in their manners, than the city artisan's family with equal income.* [...] *And in the struggle to outdo one another the city population push their normal standard of conspicuous consumption to a higher point* [...]." Veblen suggests the availability of exogenous information as an explanation. "*The means of communication and the mobility of the population now expose the individual to the observation of many persons who have no other means of judging of his reputability than the display of goods [...]. One's neighbors, mechanically speaking, often are socially not one's neighbors, or even acquaintances; and still their transient good opinion has a high degree of utility.*" If the exogenous information is perfect, Veblen's intuition is trivially true: if exogenous information resolves the information asymmetry, one expects no costly signaling. But how does equilibrium signaling depend on the accuracy of exogenous information when both signaling and the exogenous information are imperfect? And what happens to the expected accuracy of Receiver's equilibrium beliefs, if exogenous information becomes more accurate?

Real world costly signals are usually imperfect information sources, and Receiver usually has other information (beyond Sender's control) about the subject of asymmetric information. In a job market example, an academic degree can imperfectly reflect a job candidate's productivity because of luck with examination questions, a bad day during the exams or an employers' hardship to judge a program's difficulty. Moreover, employers often observe additional information: they often use psychometric tests during recruitment or learn about the candidate

(Myers and Majluf 1984, John and Williams, 1985, Bhattacharya 1979), animal behavior and morphology (Zahavi, 1975, Grafen, 1990a,b) consumption (Frank, 1999; or Truyts (2010) for a recent survey).

[2] See Truyts (2012) and the references therein for a discussion of various policies proposed for reducing the welfare costs of signaling.

from social relations.[3] An important distinction is whether Sender knows the actual realization of exogenous imperfect information when choosing a signaling strategy. If she does (e.g. ethnic markers in job market signaling), more accurate exogenous information alters the prior beliefs in equilibrium. Such marginal changes in prior beliefs were studied in e.g. Matthews and Mirman (1983) and Jeitschko and Norman (2012). This article's focus is on cases in which Sender knows the accuracy of exogenous information, but not its realization (e.g. psychometric tests during recruitment).

Veblen's observations suggest that better exogenous information enables Receiver to distinguish more between different Sender types, and thus reduces Sender's need for costly signaling. Costly signaling and exogenous information can then be called 'information substitutes': more accurate exogenous information reduces equilibrium signaling. The natural counterpart is 'information complements': more accurate exogenous information increases equilibrium costly signaling.

The main difference to earlier work on signaling in the presence of exogenous information lies in the imperfectly observed signals, which induce a smooth dependence of equilibrium signaling on the accuracy of exogenous information. For non-stochastic signaling with exogenous imperfect information, Feltovitch et al. (2002) show the existence of a non-monotonic signaling equilibrium: middle types signal while the high and low types pool at zero signaling, if high types can sufficiently rely on exogenous information to separate them from the low types. Daley and Green (2012) show that separating equilibria do not survive the common stability-based equilibrium refinements (e.g. D1) in the presence of sufficiently informative exogenous imperfect information. Welch (1989) shows how high quality firms underprice at their initial public offering (IPO), in order to obtain a higher price in a later seasoned offering, and finds that equilibrium underpricing decreases if high quality firms are more likely to be revealed as such between the two offerings. Frank (1985) studies status consumption as an imperfect signal of ability in the presence of exogenous imperfect information, and concludes that if uninformed parties aggregate both information sources linearly by means of a minimum variance unbiased estimator, "*the ability-signaling rationale* [...] *suggests that incentives to distort consumption in favor of observable goods will be inversely related to the amount and reliability of independent information that exists concerning individual abilities*". Note that these findings all suggest that more accurate exogenous information induces lower equilibrium signaling (i.e. information substitutes).

[3] More examples are discussed at the end of section 2.

This article develops a stochastic signaling model with exogenous imperfect information, and thus relates to a small literature on stochastic costly signaling. Matthews and Mirman (1983) introduce noise in terms of demand shocks in a limit pricing model and demonstrate a number of advantages of stochastic signaling games: a limited number of equilibria, smooth comparative statics and a solution that depends on prior beliefs.[4] Carlsson and Dasgupta (1997) develop vanishing noise as an equilibrium selection criterion for non-stochastic signaling games. De Haan et al. (2011) and Jeitschko and Norman (2012) test the implications of stochastic signaling models experimentally.

In what follows, Sender has binary private information and sends a costly signal to Receiver. Receiver observes this signal distorted by random noise, and also sees a binary exogenous imperfect signal. Upon observing both information sources, Receiver chooses an action from a continuum. Under mild regularity conditions, a unique sequential equilibrium exists. First, equilibrium signaling is non-monotonic with respect to the accuracy of exogenous information. A threshold level of prior beliefs is shown to separate the cases of information complements and substitutes, such that an interval of sufficiently low prior beliefs generically exists for which more accurate exogenous information implies higher equilibrium signaling. Second, more accurate exogenous information can result in a lower expected accuracy of Receiver's equilibrium beliefs, due to changes in equilibrium signaling.

This paper is structured as follows: the second section introduces the formal setting and suggests some specific examples. The third section characterizes equilibrium signaling in the presence of exogenous imperfect information. The final section concludes. All proofs are collected in a mathematical appendix.

2 Setting

A player, Sender, has private information about a quality parameter θ ('her type'), which is either high θ^H or low θ^L. She cares about the beliefs of an uninformed player, Receiver, about θ. Receiver has prior belief $p \in (0,1)$ that θ is high, and deems θ low with probability $1-p$. Sender sends a costly signal $s \in \bar{\mathbb{R}}_+$. As in Carlsson and Dasgupta (1997),[5] Receiver observes this signal imperfectly as y, the sum of s and random

[4]Note that these three points are major problems of non-stochastic signaling models (e.g. Spence, 1973, Riley, 1979). See e.g. Mailath et al. (1993) for a critique of this last feature of non-stochastic costly signaling games.

[5]Carlsson and Dasgupta (1997) demonstrate how this additive technology encompasses a.o. the demand shocks model of Matthews and Mirman (1983).

noise ε:

$$y = s + \varepsilon. \tag{1}$$

Noise term ε is independently distributed according to a density function φ, with $E(\varepsilon) = 0$ and a variance which is finite and bounded away from zero. Assume that φ satisfies the following properties.

Condition 1 *Let φ be a C^2 probability density function which*

1. (symmetry) *is symmetric around the mean,*
2. (MLR) *satisfies the strict monotone likelihood ratio property,*[6]
3. (support) *has full support on $\mathbb{R}$.*

Prominent examples of distributions satisfying condition 1 are the normal and logistic distributions. Continuous differentiability, full support and MLR are in line with Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and de Haan et al. (2011). Full support on $\mathbb{R}$ implies that all y have an equilibrium interpretation, such that specifying out-of-equilibrium beliefs and the resulting multitude of equilibria is no cause of concern.

Receiver observes two pieces of imperfect information about θ: distorted signal y and exogenous imperfect information ω, the distribution of which is independent of Sender's signaling. Assume for simplicity binary exogenous information

$$\omega \in \{L, H\},$$

of which the accuracy is denoted $q \in \left(\frac{1}{2}, 1\right)$, such that $q \equiv \Pr\left(\omega = H|\theta^H\right) = \Pr\left(\omega = L|\theta^L\right)$.

Sender's preferences are represented by a utility function

$$u(s, y, \omega|\theta, \beta) = v(s|\theta) + \kappa\beta(y, \omega) \tag{2}$$

in which $\beta(y, \omega)$ represents Receiver's posterior 'believed' probability of Sender being a high type (her 'beliefs'), given the pair of imperfect signals (y, ω). Parameter $\kappa > 0$ represents Sender's constant marginal utility of β.

[6] For two means $\mu > \mu'$, a density function φ satisfies the strict monotone likelihood ratio property (MLR) if the ratio $\frac{\varphi(\varepsilon|\mu)}{\varphi(\varepsilon|\mu')}$ strictly increases with ε everywhere. Note that this is equivalent to log-supermodularity of φ w.r.t. ε and μ, i.e. that for $\varepsilon > \varepsilon'$ and $\mu > \mu'$: $\varphi(\varepsilon|\mu)\varphi(\varepsilon'|\mu') > \varphi(\varepsilon'|\mu)\varphi(\varepsilon|\mu')$. See a.o. Karlin and Rubin (1956) or Athey (2002).

Condition 2 *Let v be C^2 with $v_1(0|.) > 0$, $v_{12}(.) > 0$ and $v_{11}(.) < \eta$ for an $\eta < 0$.*

Condition 2 imposes a standard Spence-Mirrlees single crossing condition, and ensures that both Sender types have a unique utility maximizing choice of s in the absence of signaling concerns, denoted $\bar{s}^H$ and $\bar{s}^L$, such that $\bar{s}^H > \bar{s}^L > 0$.[7] The utility function in (2) departs from the utility function in Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and Jeitschko and Normann (2012) in two respects. First, it takes Receiver's beliefs directly as an argument. This either represents a problem in which Sender cares about Receiver's beliefs directly, or is shorthand notation by omitting an explicit analysis of Receiver's optimal choice of action in function of her beliefs. Receiver's choice is easily introduced explicitly, as illustrated at the end of this section. Second, (2) assumes that Sender's utility is strictly increasing with β, which reflects that Receiver's choice set is a continuum.[8] The fact that (2) is linear in β and additively separable in β and v may seem restrictive at first sight. But other than ensuring tractability, this formulation also aims to focus on the interaction between imperfect signaling and imperfect exogenous information by maximally separating the uncertainty associated with noisy information transmission from attitudes towards risk and other particularities in the utility function of Sender and Receiver.

Sender maximizes expected utility, considering all possible realizations of ε and ω, for a given interpretation β of distorted signals:

$$\mathrm{E}u(s, y, \omega|\theta, \beta) = v(s|\theta) + \kappa B(s|\theta), \tag{3}$$

with

$$B(s|\theta) \equiv \sum_{\omega' \in \{L,H\}} \int \Pr(\omega = \omega'|\theta) \beta(y, \omega') \varphi(y|s) dy.$$

We consider pure strategy sequential equilibria (S.E.) of the stochastic signaling game.[9] Let s^L and s^H denote respectively the (pure) sig-

[7]This assumption is not crucial for the results and intuitions developped below. A pure costly signaling model with linear signaling costs is analytically more involved, but produces similar results.

[8]In the stochastic signaling models listed above, Receiver has a binary choice, which results in combination with MLR in a cut-off strategy as best reply: Receiver chooses the action most preferred by Sender if $y \geq y^*$, with y^* an optimally chosen threshold.

[9]A Sequential Equilibrium (S.E.) is described by a pair of strategy profile and posterior beliefs $\left(\left(\hat{s}^L(q), \hat{s}^H(q)\right), \beta\right)$, such that:
1. $\left(\hat{s}^L(q), \hat{s}^H(q)\right)$ maximizes expected utility (3) of each type given β
2. Beliefs $\beta(y, \omega)$ are Bayesian consistent with equilibrium strategies $\left(\hat{s}^L(q), \hat{s}^H(q)\right)$ as in (4) and (5).

naling strategy of the low and high Sender type. Receiver's beliefs are consistent with pure strategy profile $\left(s^L, s^H\right)$ if they satisfy Bayes' rule for each (y, ω):

$$\beta\left(y, H\right) = \left(1 + \frac{1-q}{q}\frac{1-p}{p}\frac{\varphi(y|s^L)}{\varphi(y|s^H)}\right)^{-1} \tag{4}$$

$$\beta\left(y, L\right) = \left(1 + \frac{q}{1-q}\frac{1-p}{p}\frac{\varphi(y|s^L)}{\varphi(y|s^H)}\right)^{-1}. \tag{5}$$

The likelihood ratio formulations (4) and (5) illustrate that MLR imposes consistent posterior beliefs $\beta\left(y, \omega\right)$ to be strictly monotonic with y if $s^L \neq s^H$.

Note that if $q = \frac{1}{2}$ and $Var\left(\varepsilon\right) = 0$, this game reduces to a textbook Spence costly signaling game with quasilinear preferences. Therefore, a number of standard examples in the literature are easily adapted to this setting of stochastic signaling with imperfect exogenous information.

Example 1 (Status Signaling) *Sender wishes to signal her income θ to other consumers because she cares directly about their beliefs and esteem. Sender divides her income between invisible rest consumption and visible status consumption s, such that her utility is represented by $v^{SS}\left(\theta - s, s\right) + \kappa\beta\left(y, \omega\right)$. The 'intrinsic' utility of consumption, v^{SS}, is strictly increasing in both arguments and strictly concave. Status consumption is an imperfect signal because status goods can be bought at a discount price, second hand or can be cheap imitations, and because there are far too many visible consumption goods to keep track of prices. On the other hand, one can typically rely on gossip for additional information ω about a consumer's reputability.*

Example 2 (Job Market Signaling) *As in Spence (1973), Sender is a job candidate of high or low productivity θ, and invests in education s at cost $-\left(s-\theta\right)^2$. Hence, job candidates intrinsically enjoy some education up to θ for its own sake. Receiver is an employer in a competitive job market, who sees a noisy educational score y and an additional imperfect test result ω and offers in equilibrium a contract with wage $\theta^L + \beta\left(y\,, \omega\right)\left(\theta^H - \theta^L\right)$. The expected utility of a job candidate is then $\theta^L - \left(s-\theta\right)^2 + \left(\theta^H - \theta^L\right) B\left(s|\theta\right)$. Education is an imperfect signal because Sender may have been lucky with exam questions or have had a bad day during the exams, or an employer may have difficulty judging the difficulty of a degree. On the other hand, the employer typically has extra psychometric tests at her disposal during the recruitment stage, or* *can ask social relations whether they know more about the job candidate. Note that w never equals the true productivity of Sender in this stochastic job market signaling game. The return to education thus only concerns a period needed by employers to learn about Sender's true productivity and to alter a possibly rigid contract.*

Example 3 (Advertising) *As in Milgrom and Roberts (1986) and Hertzendorf (1993), Sender is a monopolist, selling a new product of high or low quality θ to a continuum of consumers, distributed uniformly on $[0,1]$. For simplicity, we take the commodity price as exogenously fixed. Before launching the new product, Sender can invest in advertising s at strictly convex costs $-v^{AD}(s|\theta)$ in a first period.*[10] *Advertising is an imperfect signal because consumers typically fail to observe the total number of advertisements bought, ignore their costs and have difficulty comparing the importance of these advertising costs to the size of the firm and market. They can often also rely on product tests in magazines or discussions on the internet. After observing both imperfect signals (y,ω), consumers decide whether or not to buy the product. Consumers buy the product if they deem the probability of a high quality product higher than their position on $[0,1]$.*[11] *Only consumers who buy the product observe the true quality θ, and can buy the product again in a second period (they all do if θ is high). If each consumer draws an independent y and ω, and profits per unit sold are π_θ (with $2\pi_H > \pi_L$), then profits of a high and low quality monopolist are respectively $-v^{AD}\left(s|\theta^H\right) + 2\pi_H B\left(s|\theta^H\right)$ and $-v^{AD}\left(s|\theta^L\right) + \pi_L B\left(s|\theta^L\right)$, such that Sender's preferences can be written as $B\left(s|\theta^H\right) - \frac{v^{AD}\left(s|\theta^H\right)}{2\pi_H}$ and $B\left(s|\theta^L\right) - \frac{v^{AD}\left(s|\theta^L\right)}{\pi_L}$.*

3 Information Substitutes and Complements

Before presenting the main results, this section first highlights a few simple features of the stochastic signaling game under consideration. First, by condition 1, Receiver's consistent beliefs are never degenerate for finite y and s, such that Receiver's best choice is generically suboptimal with respect to Sender's true type. In expectation, the weighted average

[10] Note that by condition 2, $v_1^{AD}(0|.) > 0$. This can reflect other advantages of advertising (informing consumers of the existence of the product, entry deterrence...) as summarized in Bagwell (2007).

[11] If a risk neutral consumer's willingness to pay for a high and low quality product is resp. $\lambda^H > \lambda^L > 0$, she buys at price γ if $\beta(y,\omega)\, 2\left(\lambda^H - \gamma\right) + (1-\beta(y))\left(\lambda^L - \gamma\right) \geq 0$, i.e. if $\beta(y,\omega) \geq \frac{\gamma - \lambda^L}{\left(\lambda^H - \gamma\right) + \left(\lambda^H - \lambda^L\right)} \equiv \zeta$. We thus assume ζ uniformly distributed on $[0,1]$. Consumers with ζ negative or greater than 1 never and always buy, respectively. See also Milgrom and Roberts (1986) and Hertzendorf (1993).

of Receiver's consistent beliefs equals the prior belief, as stated by the following lemma.

Lemma 1 *For Receiver's beliefs consistent with a strategy profile* $\left(s^L, s^H\right)$,

1. *the stochastic signaling game is zero sum in* B*:*
$$pB\left(s^H|\theta^H\right) + (1-p)\,B\left(s^L|\theta^L\right) = p, \tag{6}$$

2. B *depends only on* $\Delta \equiv s^H - s^L$ *and not on actual levels of* s.

Using lemma 1, it will be convenient to normalize $B\left(s^H|\theta^H\right)$ for consistent beliefs, such that it is written

$$B\left(\Delta|p,q\right) \equiv \int \left(q\tilde{\beta}\left(y,H|p,q\right) + (1-q)\,\tilde{\beta}\left(y,L|p,q\right)\right)\varphi(y|\Delta)dy,$$

with

$$\tilde{\beta}\left(y,H|p,q\right) = \left(1 + \frac{1-q}{q}\frac{1-p}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}\right)^{-1} \tag{7}$$

and

$$\tilde{\beta}\left(y,L|p,q\right) = \left(1 + \frac{q}{1-q}\frac{1-p}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}\right)^{-1}. \tag{8}$$

Let $B'\left(\Delta|p,q\right)$ denote the marginal effect of Δ on B for fixed consistent beliefs, i.e.[12]

$$B'\left(\Delta|p,q\right) \equiv \int \left(q\tilde{\beta}\left(y,H|p,q\right) + (1-q)\,\tilde{\beta}\left(y,L|p,q\right)\right)\varphi_2\left(y|\Delta\right)dy.$$

Unless potentially confusing, the two last arguments p and q are omitted from B, B' and $\tilde{\beta}$ to economize on notation. The next lemma shows that B increases with Δ.

Lemma 2 *If* φ *satisfies condition 1, then* $\Delta > 0$ *implies* $B'\left(\Delta\right) > 0$ *and* $B_1\left(\Delta\right) > 0$, *while* $B'\left(0\right) = 0$.

As such, the stochastic signaling game can be understood as an arms race in which both Sender types waste means to secure for themselves a larger share of a given resource of fixed size: Receiver's expected consistent beliefs. The division of this resource depends only on the difference in signaling efforts Δ. The high Sender type can create more distinction in expectation by increasing s^H, while the low Sender type can increase

[12] The notation B' is used to distinguish from the first order derivative to all Δ in B (i.e. in $\tilde{\beta}$ and φ), denoted B_1.

signaling s^L to confuse Receiver more and undo the expected distinction established by the high type. If both Sender types signal, then an amount min $\{s^L - \bar{s}^L, s^H - \bar{s}^H\}$ is wasted, in the sense that exogenously reducing the signaling efforts of both Sender types by this much (and adapting beliefs accordingly) improves the welfare of both Sender types without affecting the information transferred to Receiver in expectation. Note also that $B(\Delta)$, the expectation of Receiver's believed probability that the high Sender type is a high type, measures the expected accuracy of Receiver's consistent beliefs, such that Receiver is by assumption *ceteris paribus* better off with higher Δ.

Pure strategy sequential equilibria are characterized by consistent beliefs, (7) and (8), and a strategy profile $(\hat{s}^L(q), \hat{s}^H(q))$ which satisfies the following first order conditions for expected utility maximization for given consistent beliefs:

$$v_1\left(\hat{s}^H(q)\,|\theta^H\right) + \kappa B'\left(\hat{\Delta}(q)\right) = 0 \tag{9}$$

and

$$v_1\left(\hat{s}^L(q)\,|\theta^L\right) + \kappa\frac{p}{1-p}B'\left(\hat{\Delta}(q)\right) = 0. \tag{10}$$

If Sender's problem is strictly concave for all strategy profiles,[13] one can construct for each Sender type a function similar to best response functions in e.g. Cournot games. For the high Sender type, such a function indicates for each level of s^L the unique level of s^H which satisfies (9) for consistent beliefs (7) and (8), under the restriction that $\Delta \geq 0$ (as $\Delta < 0$ cannot be an equilibrium, cfr. infra). After constructing a similar function for the low Sender type, any crossing of both functions constitutes an S.E. The following proposition shows that such an S.E. is unique.

Proposition 1 *If φ and v satisfy respectively conditions 1 and 2 and if Sender's problem is strictly concave, then a unique S.E. in pure strategies exists, in which equilibrium strategies $(\hat{s}^H(q), \hat{s}^L(q))$ are such that $\hat{\Delta}(q) > 0$, $\hat{s}^H(q) > \bar{s}^H$ and $\hat{s}^L(q) > \bar{s}^L$.*

The existence and uniqueness of such an S.E. for stochastic signaling games was shown by Matthews and Mirman (1983)), and is in this framework shown by an elementary application of the Poincaré-Hopf index theorem. As in Jeitschko and Normann (2012) and unlike in nonstochastic costly signaling games in line with Spence (1973), signaling causes distortion at the top and bottom. The low Sender type wastes

[13] An extensive characterization of strict concavity of Sender's problem in terms of the fundamentals is provided in section A.3 of the appendix.

means in equilibrium to confuse Receiver and undo distinction with the high Sender type.

How does more accurate exogenous information affect equilibrium signaling? I impose an additional technical condition, which bounds the accuracy of exogenous information ω from above.[14]

Condition 3 *Let* $q < \frac{2+\sqrt{3}}{4} \cong 0.933$.

The following theorem then shows that imperfect signaling and exogenous information are information complements for prior beliefs below a threshold, and information substitutes for prior beliefs above the same threshold.

Theorem 1 *If φ, v and q satisfy, respectively, conditions 1, 2 and 3 and if Sender's problem is strictly concave, then a unique threshold $\bar{p}(q)$ exists such that:*
if $p < \bar{p}(q)$, then $\hat{s}_1^H(q) > 0$ and $\hat{s}_1^L(q) > 0$ (information complements),
if $p > \bar{p}(q)$, then $\hat{s}_1^H(q) < 0$ and $\hat{s}_1^L(q) < 0$ (information substitutes).
Moreover, $\bar{p}(q)$ is a continuous function of q.

Hence, an interval of sufficiently low prior beliefs generically exists for which both imperfect information channels are information complements. Figure 1 displays a numerical solution of threshold $\bar{p}(q)$, for φ the normal density function at $\sigma = 2$ and for three values of Δ.

A marginal increase in accuracy q affects $B'(\Delta)$, and thus the marginal rewards to signaling, by changing the probability density of exogenous signal ω and by changing Receiver's consistent beliefs:

$$B_3'(\Delta|p,q) = \int \left[\tilde{\beta}(y,H|p,q) - \tilde{\beta}(y,L|p,q)\right] \varphi_2(y|\Delta)\,dy$$
$$+ \int \left[q\tilde{\beta}_4(y,H|p,q) + (1-q)\,\tilde{\beta}_4(y,L|p,q)\right] \varphi_2(y|\Delta)\,dy.$$

First, the effect of high Sender types drawing $\omega = H$ more often depends on the magnitude of p and Δ. If p or Δ are so high that, for almost all y with a nontrivial probability mass under $\varphi(y|\Delta)$, Receiver deems Sender almost certainly a high type if $\omega = H$, while she is less certain for part of the same y if $\omega = L$, then the marginal rewards to signaling are greater if $\omega = L$. Drawing $\omega = H$ more often then decreases

[14] This restriction reflects a limitation of my method of proof. Extensive numerical simulations failed to generate a counterexample to theorem 1 beyond condition 3, i.e. for $q > \frac{2+\sqrt{3}}{4}$.

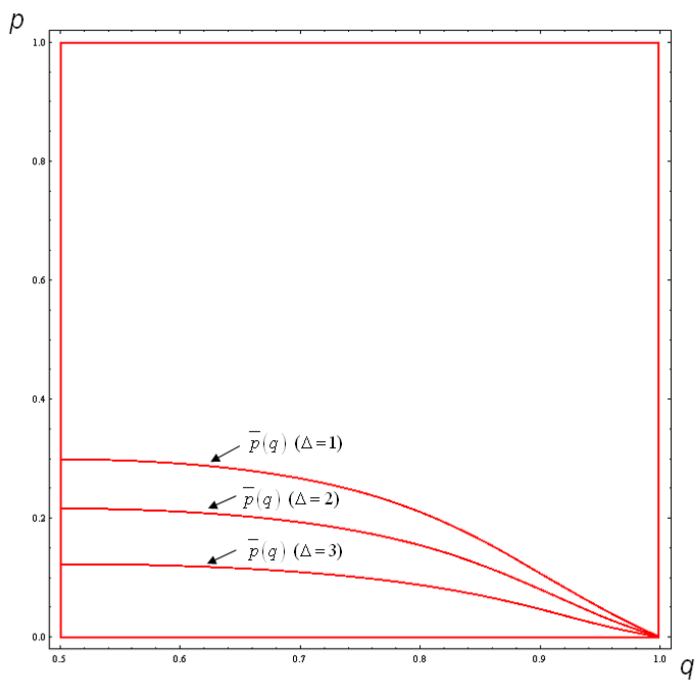

Figure 1: $\bar{p}(q)$ for φ the normal density function with $\sigma = 2$ at Δ equal to 1, 2 and 3.

the high Sender type's marginal rewards to signaling. By a similar logic, the opposite is true if Δ and p are sufficiently low. One can show the existence of a unique threshold p such that a greater occurrence of $\omega = H$ increases or rather decreases the marginal rewards to signaling for p, respectively, below and above this threshold.

Second, conditional on ω being H or L, an increased accuracy of ω affects Receiver's consistent beliefs, respectively, as a increase or decrease in prior beliefs. A marginally more accurate exogenous signal $\omega = H$ increases or rather decreases the marginal rewards to signaling if prior beliefs are respectively below or above a threshold level p. For low prior beliefs, Receiver is convinced only after very high y that Sender is a high type. This is achieved at lower y, more within reach of a high Sender type, if Receiver observes more accurate exogenous information $\omega = H$, such that the latter enhances the marginal rewards to signaling. For similar reasons, more accurate exogenous information $\omega = L$ decreases or rather increases the conditional marginal benefits to signaling for prior beliefs respectively below and above another (lower) threshold level.

Bringing these partial effects together, theorem 1 states that a threshold $\bar{p}(q)$ exists, such that y and ω are information complements for prior beliefs below $\bar{p}(q)$, and information substitutes for prior beliefs above $\bar{p}(q)$. Or, an open interval of sufficiently low prior beliefs generically exists for which signaling and exogenous information are information complements. If Receiver is sufficiently pessimistic about Sender being a high type and exogenous information is inaccurate, then imperfect signals (y, ω) that Receiver attributes to a high Sender type with an intermediate or high probability are only drawn by Sender if she is very lucky. More accurate exogenous information brings such imperfect signals more likely within reach of Sender, thus inciting greater rewards for increased signaling efforts. By the zero sum nature of the stochastic signaling game, the high Sender type's enhanced opportunities for distinction also raise the stakes for the low type, who equally increases equilibrium signaling. Note that higher Δ implies lower $\bar{p}(q)$, as illustrated in figure 1: on average more informative y implies that only for the most pessimistic Receivers more accurate exogenous information is needed to bring imperfect signals (y, ω) which Receiver attributes to a high Sender type with an intermediate or high probability more within reach of Sender. By the same logic, $\bar{p}(q)$ can be seen to decrease with q.

In example 3, a monopolist is selling a new product, about which specialized media will publish product tests to distinguish between a true innovation and a marketing scam. If customers deem the chance of a true innovation sufficiently low, then an improved reliability of tests increases advertising by both true innovators and imitators selling junk. More reliable tests more often convince customers that a truly good product might indeed be a true innovation, whereas without these tests, the advertising needed to convince enough customers is prohibitively high. By distinguishing better between true innovators imitators, more reliable tests also raise the stakes for the latter, who accordingly increase their equilibrium advertising to restore confusion with true innovators.

How does more accurate exogenous information help Receiver? A greater accuracy of exogenous information ω affects the expected accuracy of Receiver's equilibrium beliefs, as measured by B, in two ways: directly by providing more accurate ω, and indirectly by changing the average informativeness of equilibrium signaling:

$$B_q\left(\hat{\Delta}(q)\,|p,q\right) = B_3\left(\hat{\Delta}(q)\,|p,q\right) + B_1\left(\hat{\Delta}(q)\,|p,q\right)\hat{\Delta}_1(q).$$

The direct effect $B_3\left(\hat{\Delta}(q)\,|p,q\right)$ is always positive: given Δ, more accurate exogenous information improves the expected accuracy of Receiver's equilibrium beliefs. For the indirect effect, $B_1\left(\hat{\Delta}(q)\,|p,q\right) > 0$

by lemma 2: more separation in signaling helps Receiver in expectation to distinguish between Sender types. Note then in (9) and (10) that in the S.E.

$$\frac{p}{1-p} v_1 \left(\hat{s}^H (q) | \theta^H \right) = v_1 \left(\hat{s}^L (q) | \theta^L \right) = -\kappa \frac{p}{1-p} B' \left(\hat{\Delta} (q) \right),$$

and define

$$h \left(s^L, s^H \right) \equiv (1-p) v_{11} \left(s^L | \theta^L \right) - p v_{11} \left(s^H | \theta^H \right)$$

as the weighted difference in the rate at which the marginal utility costs of signaling increase for either Sender type. The next result shows that an open interval of intermediate prior beliefs generically exists for which a marginal increase in q induces a decrease in the average informativeness of costly signaling, and that this decrease can come to dominate the direct information benefits of more accurate exogenous information. Better exogenous information can thus make Receiver worse off in expectation.

Proposition 2 *If the conditions of theorem 1 apply, then:*

1. $\hat{\Delta}_1 (q)$ *takes the opposite sign of* $\hat{s}_1^H (q) h \left(\hat{s}^L (q), \hat{s}^H (q) \right)$,

2. *for* φ *the normal distribution,* $B_q \left(\hat{\Delta} (q) | p, q \right) < 0$ *for a non-empty part of parameter space.*

First, if $\hat{p}$ denotes the prior beliefs at which $h \left(s^L, s^H \right) = 0$,[15] then the first part of proposition 2 shows that $\hat{\Delta}_1 (q) < 0$ only in an open interval between $\bar{p}$ and $\hat{p}$. Second, in this case the negative indirect effect can outweigh the positive direct effect on the expected accuracy of Receiver's equilibrium beliefs. For information substitutes, this occurs if the low Sender type's marginal signaling costs increase very quickly, such that a decrease in the marginal rewards to signaling induces a negligible reduction in s^L, while the high Sender type's reduction in signaling is sufficiently large to outweigh the direct effect.

4 Conclusions

The ostentatious waste associated with costly signaling is generally understood as a necessary cost for a transfer of information which otherwise cannot be credibly communicated. This paper developed a simple model

[15] That is: $\hat{p} = \frac{v_{11} \left(s^L | \theta^L \right)}{v_{11} \left(s^L | \theta^L \right) + v_{11} \left(s^H | \theta^H \right)}$.

of stochastic costly signaling in the presence of exogenous imperfect information, and studied how more accurate exogenous information affects the equilibrium signaling costs as well as the information eventually held in expectation by Receiver. Previous literature, mostly focussing on non-stochastic signaling with imperfect exogenous information, has found that better exogenous information can reduce equilibrium signaling, by offering Receiver more means to distinguish between Sender types. The present analysis demonstrates that more accurate exogenous information can generically both decrease and increase equilibrium costly signaling, depending on Receiver's prior beliefs. The intuition for the latter result is generic: for sufficiently pessimistic prior beliefs, the signaling levels required to generate with non-negligible likelihood noisy signals which Receiver attributes to a high Sender type with intermediate or high probability are prohibitively high. More accurate exogenous information brings these noisy signals more likely within reach of high Sender types, thus increasing their marginal benefits of signaling. More accurate exogenous information can also cause Receiver to be less well informed in equilibrium. More accurate exogenous information, although improving Receiver's information as a direct effect (i.e. for fixed signaling strategies), can by changing equilibrium signaling induce a decrease in the average informativeness of the distorted equilibrium signals. The latter effect of more accurate exogenous information can dominate the former, thus decreasing the expected accuracy of Receiver's equilibrium beliefs.

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A Mathematical Appendix: Proofs

It will be convenient to write

$$c\left(y,q\right)\equiv q\tilde{\beta}\left(y,H\right)+\left(1-q\right)\tilde{\beta}\left(y,L\right),$$

and to denote

$$\begin{aligned} F^{H}\left(s^{H},s^{L}|q\right)&\equiv v_{1}\left(s^{H}|\theta^{H}\right)+\kappa B'\left(\Delta\right)\\ F^{L}\left(s^{L},s^{H}|q\right)&\equiv v_{1}\left(s^{L}|\theta^{L}\right)+\kappa\frac{p}{1-p}B'\left(\Delta\right). \end{aligned}$$

A.1 Proof of lemma 1

To see the first part, write

$$\begin{aligned} &pB\left(s^{H}|\theta^{H}\right)+\left(1-p\right)B\left(s^{L}|\theta^{L}\right)\\ =&\int\beta\left(y,H\right)\left[pq\varphi\left(y|s^{H}\right)+\left(1-q\right)\left(1-p\right)\varphi\left(y|s^{L}\right)\right]dy\\ &+\int\beta\left(y,L\right)\left[q\left(1-p\right)\varphi\left(y|s^{L}\right)+\left(1-q\right)p\varphi\left(y|s^{H}\right)\right]dy\\ =&pq+\left(1-q\right)p=p, \end{aligned}$$

while the second part follows directly from the assumption that the distribution of ε is independent of s.

A.2 Proof of lemma 2

By condition 1, $\tilde{\beta}$ is strictly increasing with y if $\Delta > 0$ and constant if $\Delta = 0$, such that

$$\begin{aligned} B'(\Delta) &= q \int_{\Delta}^{+\infty} \left(\tilde{\beta}(y, H) - \tilde{\beta}(2\Delta - y, H)\right) |\varphi'(y|\Delta)| \, dy \qquad (11) \\ &+ (1-q) \int_{\Delta}^{+\infty} \left(\tilde{\beta}(y, L) - \tilde{\beta}(2\Delta - y, L)\right) |\varphi'(y|\Delta)| \, dy > 0 \end{aligned}$$

if $\Delta > 0$ and $B'(0) = 0$. Next, use lemma 1 to write

$$B(\Delta) = 1 - \frac{1-p}{p} \int c(y, q)\, \varphi(y|0) dy,$$

such that

$$B_1(\Delta) = - \int \left(\frac{q^2}{1-q} \left(1 - \tilde{\beta}(y, H)\right)^2 + \frac{(1-q)^2}{q} \left(1 - \tilde{\beta}(y, L)\right)^2 \right) \varphi_2(y|\Delta) dy,$$

in which, by condition 1, $-\left(1 - \tilde{\beta}(y, \omega)\right)^2$ is strictly increasing with y if $\Delta > 0$ and constant if $\Delta = 0$. Use condition 1 to write $B_1(\Delta)$ as an integral over $[\Delta, \infty)$, as in (11), to obtain $B_1(\Delta) > 0$ for $\Delta > 0$ and $B_1(0) = 0$.

A.3 Proof of proposition 1

Define the second order derivative of B, taking $\tilde{\beta}$ as given,

$$B''(\Delta) \equiv \int c(y, q)\, \varphi_{22}(y|\Delta) dy,$$

while differentiating $B'(\Delta)$ to Δ (including the Δ in $\tilde{\beta}$) gives

$$B_1'(\Delta) = B''(\Delta) + \frac{p}{1-p} \int \left(\begin{array}{c} \frac{q^2}{1-q} \left(1 - \tilde{\beta}(y, H)\right)^2 \\ + \frac{(1-q)^2}{q} \left(1 - \tilde{\beta}(y, L)\right)^2 \end{array} \right) \frac{(\varphi_2(y|\Delta))^2}{\varphi(y|0)} dy, \qquad (12)$$

in which the second term is always positive. In general, $B_1'(\Delta)$ can be both positive and negative, such that condition 2 must be strenghtened with an additional strict concavity condition.

Condition 4 *Let u and φ be such that for all $\left(s^L, s^H\right)$ with $\Delta \geq 0$:*

$$\begin{aligned} v_{11}\left(s^H | \theta^H\right) + \kappa B_1'(\Delta) &< 0 \\ v_{11}\left(s^L | \theta^L\right) - \kappa \frac{p}{1-p} B''(\Delta) &< 0. \end{aligned}$$

This condition encompasses two sets of second order conditions. First, a solution to (9) and (10) is a maximum for given beliefs (7) and (8) if for all $\left(s^L, s^H\right)$ with $\Delta \geq 0$:

$$v_{11}\left(s^H|\theta^H\right) + \kappa B''(\Delta) < 0 \tag{13}$$

$$v_{11}\left(s^L|\theta^L\right) - \kappa\frac{p}{1-p}B''(\Delta) < 0. \tag{14}$$

On the other hand, for a given level of signaling of the other type, an interior solution to (9) and (10) defines a unique interior level of signaling consistent with an S.E. if for all $\left(s^L, s^H\right)$ with $\Delta \geq 0$:

$$S^H\left(s^H\right) \equiv v_{11}\left(s^H|\theta^H\right) + \kappa B_1'(\Delta) < 0 \tag{15}$$

$$S^L\left(s^L\right) \equiv v_{11}\left(s^L|\theta^L\right) - \kappa\frac{p}{1-p}B_1'(\Delta) < 0. \tag{16}$$

Because the second term in (12) is always nonnegative, (13) is implied by (15) and (16) is implied by (14).

Proof. i) *Any S.E. strategy profile* $\left(\hat{s}^L(q), \hat{s}^H(q)\right)$ *must be above* $\left(\bar{s}^L, \bar{s}^H\right)$. Assume otherwise. First, if $\hat{s}^L(q) \leq \hat{s}^H(q)$, then $B'\left(\hat{\Delta}(q)\right) \geq 0$ and either $v_1\left(\hat{s}^H(q)|\theta^H\right) > 0$ or $v_1\left(\hat{s}^L(q)|\theta^L\right) > 0$, such that (9) and (10) cannot both be satisfied. Second, if $\hat{s}^L(q) > \hat{s}^H(q)$ and $\hat{s}^L(q) \leq \bar{s}^H$, then at $\hat{s}^H(q) = \hat{s}^L(q)$ we have $v_1\left(\hat{s}^H(q)|\theta^H\right) \geq 0$ and $B'(0) = 0$, which implies in combination with condition 4 that (9) cannot be satisfied. If $\hat{s}^L(q) > \hat{s}^H(q)$ and $\hat{s}^L(q) > \bar{s}^H$, then $v_1\left(\hat{s}^L(q)|\theta^L\right) < 0$ and $B'\left(\hat{\Delta}(q)\right) < 0$, such that (10) cannot be satisfied.

ii) *In any equilibrium,* $\hat{\Delta}(q) \geq 0$. If $\hat{\Delta}(q) < 0$ (and $\hat{s}^L(q) \geq \bar{s}^H$ by the previous point), then the low Sender type can strictly improve herself by signaling less, because $v_1\left(\hat{s}^L(q)|\theta^L\right) < 0$ and $B'\left(\hat{\Delta}(q)\Delta\right) < 0$.

iii) *Existence of an S.E.* Let $b^L\left(s^H\right)$ represent for each value of $s^H \geq \bar{s}^H$ the s^L for which (10) is satisfied. By condition 4, this value is unique, and by conditions 1 and 2, b^L is continuously differentiable. Let $b^H\left(s^L\right)$ represent for each value of s^L the unique value of s^H for which $F^H\left(s^H, s^L|q\right)\Delta = 0$, such that $b^H\left(s^L\right) \geq s^L$ satisfies (9) or $b^H\left(s^L\right) = s^L$ if the constraint $\Delta \geq 0$ is binding. By conditions 1 and 2, b^H is continuously differentiable.

A crossing of $b^H\left(s^L\right)$ and $b^L\left(s^H\right)$ constitutes an S.E. Note then that $b^L\left(s^H\right) \in \left(\bar{s}^L, s^H\right)$, because for $s^H \geq \bar{s}^H$ by construction $v_1\left(\bar{s}^L|\theta^L\right) = 0$ and $B'\left(s^H - \bar{s}^L\right) > 0$ while $v_1\left(s^H|\theta^L\right) < 0$ and $B'(0) = 0$.at $s^L = s^H$. On the other hand, $b^H\left(\bar{s}^L\right) > \bar{s}^H$ because $v_1\left(\bar{s}^H|\theta^H\right) = 0$ and

$B'\left(\bar{s}^H - \bar{s}^L\right) > 0$ by construction. Moreover, a threshold $\zeta \geq \bar{s}^H$ exists such that $b^H\left(s^L\right) = s^L$ for all $s^L \geq \zeta$, because in this case $F_1^H\left(s, s^L|q\right) < 0$ for all $s > s^L$. This implies that $b^H\left(s^L\right)$ and $b^L\left(s^H\right)$ cross at least once, and at such crossing $\Delta > 0$, $b^L\left(s^H\right) > \bar{s}^L$ and $b^H\left(s^L\right) > \bar{s}^H$.

iv) *Uniqueness.* This is shown by an elementary instance of the Poincaré-Hopf index theorem (Guillemin and Pollack, , p. 134), as exemplified in a.o. Chenault (1986). Construct the auxiliary function $d\left(s^L\right) \equiv b^H\left(s^L\right) - \left(b^L\right)^{-1}\left(s^L\right)$, measuring the distance between $b^H\left(s^L\right)$ and $b^L\left(s^H\right)$. Then

$$\begin{aligned} d_1\left(s^L\right) &= b_1^H\left(s^L\right) - \frac{1}{b_1^L\left(s^H\right)} = -\frac{F_2^H\left(s^H, s^L|q\right)}{S^H\left(s^H\right)} + \frac{S^L\left(s^L\right)}{F_2^L\left(s^L, s^H|q\right)} > 0 \\ &\Leftrightarrow F_2^H\left(s^H, s^L|q\right) F_2^L\left(s^L, s^H|q\right) - S^L\left(s^L\right) S^H\left(s^H\right) \\ &= -\frac{p}{1-p}\left(\kappa B_1'\left(\Delta\right)\right)^2 - S^L\left(s^L\right) S^H\left(s^H\right) < 0, \end{aligned}$$

which is always satisfied under condition 4. Because $d\left(s^L\right)$ crosses 0 at most once, the S.E. is unique. ■

A.4 Proof of theorem 1

This proof proceeds in 3 steps.

Claim 1 *$\hat{s}_1^H(q)$ and $\hat{s}_1^L(q)$ have the same sign as $B_3'\left(\hat{\Delta}(q)|p, q\right)$.*

Claim 2 *$B'(\Delta|p, q)$ is continuously differentiable w.r.t. q for $q \in \left(\frac{1}{2}, \frac{2+\sqrt{3}}{4}\right)$. $B_3'(\Delta|p, q)$ is strictly positive for p sufficiently close to 0, and strictly negative for p sufficiently close to 1.*

Claim 3 *$B_3'(\Delta|p, q)$ is continuous w.r.t. p, and at the $\bar{p}(q)$ (where $B_3'(\Delta|\bar{p}(q), q) = 0$), it must be that $B_{23}'(\Delta|\bar{p}(q), q) < 0$.*

Claims 1, 2 and 3 together imply theorem 1.

A.4.1 Proof of claim 1

Proof. Write $F_q^H\left(\hat{s}^H(q), \hat{s}^L(q)|q\right) = 0$ and $F_q^L\left(\hat{s}^L(q), \hat{s}^H(q)|q\right) = 0$ as a system

$$A \cdot \begin{pmatrix} \hat{s}_1^L(q) \\ \hat{s}_1^H(q) \end{pmatrix} = \begin{pmatrix} -\kappa B_3'\left(\hat{\Delta}(q)|p, q\right) \\ -\kappa \frac{p}{(1-p)} B_3'\left(\hat{\Delta}(q)|p, q\right) \end{pmatrix}, \tag{17}$$

with

$$A = \begin{pmatrix} -\kappa B_1'\left(\hat{\Delta}(q)\right) & v_{11}\left(\hat{s}^H(q)\,|\theta^H\right) + \kappa B_1'\left(\hat{\Delta}(q)\right) \\ v_{11}\left(\hat{s}^L(q)\,|\theta^L\right) - \frac{\kappa p}{(1-p)} B_1'\left(\hat{\Delta}(q)\right) & \frac{\kappa p}{(1-p)} B_1'\left(\hat{\Delta}(q)\right) \end{pmatrix}.$$

System (17) has a unique solution if $|A| \neq 0$ everywhere, which is satisfied under condition 4 as:

$$|A| = -\frac{p}{(1-p)}\left(\kappa B_1'\left(\hat{\Delta}(q)\right)\right)^2 - S^L\left(\hat{s}^L(q)\right) S^H\left(\hat{s}^H(q)\right) < 0.$$

The system is solved for $\hat{s}_1^L(q)$ and $\hat{s}_1^H(q)$ by Cramer's rule, such that $\hat{s}_1^L(q) = \frac{\kappa B_3'\left(\hat{\Delta}(q)|p,q\right) v_{11}\left(\hat{s}^L(q)|\theta^L\right)}{|A|}$ and $\hat{s}_1^H(q) = \frac{\kappa \frac{p}{(1-p)} B_3'\left(\hat{\Delta}(q)|p,q\right) v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{|A|}$. This implies that $\hat{s}_1^L(q)$ and $\hat{s}_1^H(q)$ take the same sign as $B_3'\left(\hat{\Delta}(q)\,|p,q\right)$.
■

A.4.2 Proof of claim 2

It will be convenient to define $z(y,p) \equiv \frac{(1-p)}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}$, such that $z(y,p) \in \mathbb{R}^+$ and $z_1(y,p) < 0$. Whenever obvious, the arguments of z are omitted. Further, I denote $P^1(y,\omega) \equiv \left(1 - \tilde{\beta}(y,\omega)\right)\tilde{\beta}(y,\omega)$ and $P^2(y,\omega) \equiv \left(1 - \tilde{\beta}(y,\omega)\right)\left(\tilde{\beta}(y,\omega)\right)^2$.

Proof. Write $B_3'(\Delta|p,q) = \int f(z|p,q)\,\varphi_2(y|\Delta)\,dy$, with

$$\begin{aligned} f(z|p,q) \equiv c_2(y,q) &= \tilde{\beta}(y,H) + \frac{P^1(y,H)}{1-q} - \tilde{\beta}(y,L) - \frac{P^1(y,L)}{q} \\ &= \frac{z^2(2q-1)(z+1)}{\left((q+(1-q)z)((1-q)+qz)\right)^2}. \end{aligned}$$

Note that $f(z|p,q) > 0$ for all $z \in \mathbb{R}^+$. By condition 1, $f(z|p,q)$ is continuous and bounded, such that $B'(\Delta)$ is differentiable w.r.t. q. Moreover, it is easily verified that $f(0,q) = 0$ and $\lim_{z\to+\infty} f(z|p,q) = 0$. Furthermore, $f(z|p,q)$ has a unique extremum in terms of z, a maximum, because

$$f_1(z|p,q) = z(2q-1)\frac{\left(-z^3 a + z^2(1-4a) + 3za + 2a\right)}{\left((q+(1-q)z)((1-q)+qz)\right)^3},$$

with $a \equiv q(1-q)$, has for $q \in \left(\frac{1}{2},1\right)$ and $z > 0$ a strictly positive denominator which is finite for finite z. Then $f_1(z|p,q) = 0$ only where $-z^3a + z^2(1-4a) + 3za + 2a = 0$, which has a unique real root because its discriminant is $\delta = -a\left(-1088a^3 + 564a^2 - 105a + 8\right) < 0$ for $q \in \left(\frac{1}{2},1\right)$.

This root, denoted ξ, is strictly positive and finite for q bounded away from 1 (while only $q < \frac{2+\sqrt{3}}{4}$ is considered):

$$\xi = \frac{(1-4a)}{3a} + \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(X + \sqrt{-27a^2\delta}\right)} + \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left(X - \sqrt{-27a^2\delta}\right)} > 0,$$

with

$$X \equiv \left(2(1-4a)^3 + 27a^2(1-4a) + 54a^3\right).$$

Hence, $c_2(y,q)$ is unimodal with a unique maximum at $y_\xi(p)$, which solves $z(y_\xi(p), p) = \xi$, such that $c_2(y,q)$ is strictly increasing with y for $y < y_\xi(p)$ and strictly decreasing with y for $y > y_\xi(p)$.
Note that ξ is independent of p, and that by taking p sufficiently close to 0, $f(z|p,q)$ is strictly increasing with y for almost all mass under $|\varphi_2(y|\Delta)|$ such that $B_3'(\Delta|p,q) > 0$. Similarly, for p sufficiently close to 1, $f(z|p,q)$ is strictly decreasing with y for almost all mass under $|\varphi_2(y|\Delta)|$ such that $B_3'(\Delta|p,q) < 0$. ■

A.4.3 Proof of claim 3

Proof. First, $B_{23}'(\Delta|p,q) = -\frac{1}{p(1-p)}\int (zf_1(z|p,q))\varphi_2(y|\Delta)\,dy$ exists everywhere because $zf_1(z|p,q)$ is continuous w.r.t. y and bounded for $q \in \left(\frac{1}{2}, \frac{2+\sqrt{3}}{4}\right)$. Note also that $\lim_{z\to 0} zf_1(z|p,q) = 0$ and $\lim_{z\to+\infty} zf_1(z|p,q) = 0$.
Consider then $B_{23}'(\Delta|p,q) = \int f_2(z|p,q)\varphi_2(y|\Delta)\,dy$ with $\tilde{\beta}_3(y,\omega|p,q) = \frac{P^1(y,\omega)}{p(1-p)}$ such that

$$\begin{aligned} p(1-p) f_2(z|p,q) &= P^1(y,H) - P^1(y,L) \\ &\qquad + \frac{P^1(y,H) - 2P^2(y,H)}{1-q} - \frac{P^1(y,L) - 2P^2(y,L)}{q} \\ &= f(z|p,q) - g(z|p,q), \end{aligned}$$

with

$$\begin{aligned} g(z|p,q) &\equiv \left(\tilde{\beta}(y,H)\right)^2 - \left(\tilde{\beta}(y,L)\right)^2 + 2\left(\frac{P^2(y,H)}{1-q} - \frac{P^2(y,L)}{q}\right) \\ &= \frac{z^2(2q-1)(5q^2z^2 - 2q^2z - 3q^2 - 5qz^2 + 2qz + 3q + 2z^2 + z)}{\left((q+(1-q)z)((1-q)+qz)\right)^3}, \end{aligned}$$

such that

$$p(1-p) B_{23}'(\Delta|p,q) = B_3'(\Delta|p,q) - \int g(z|p,q)\varphi_2(y|\Delta)\,dy$$

Define

$$r(z) \equiv \frac{g(z|p,q)}{f(z|p,q)} = \frac{(5q^2z^2 - 2q^2z - 3q^2 - 5qz^2 + 2qz + 3q + 2z^2 + z)}{((q + (1-q)z)((1-q) + qz))(z+1)}.$$

One can verify that

$$r_1(z) = -2(1-q)q\left((q + (1-q)z)^{-2} + (q(z-1)+1)^{-2}\right) + (z+1)^{-2} < 0 \tag{18}$$

for all $z \in \mathbb{R}^+$ if $q < \frac{2+\sqrt{3}}{4} \cong 0.93301$. At $\bar{p}$, we have by definition $B_3'(\Delta|\bar{p},q) = \int_{\Delta}^{+\infty} [-f(z(2\Delta - y,\bar{p})|\bar{p},q) + f(z(y,\bar{p})|\bar{p},q)] |\varphi_2(y|\Delta)| dy = 0$, which implies for all $q < \frac{2+\sqrt{3}}{4}$ that

$$-\bar{p}(1-\bar{p}) B_{23}'(\Delta|\bar{p},q) = \int_{\Delta}^{+\infty} \begin{bmatrix} -g(z(2\Delta - y,\bar{p})|\bar{p},q) \\ +g(z(y,\bar{p})|\bar{p},q) \end{bmatrix} |\varphi_2(y|\Delta)| dy > 0,$$

because by (18) we have

$$-g(z(2\Delta - y,p)|p,q) + g(z(y,p)|p,q) > -f(z(2\Delta - y,p)|p,q) + f(z(y,p)|p,q)$$

for all $y \in (\Delta, \infty)$. Hence, $B_{23}'(\Delta|p,q) < 0$ at $\bar{p}$, and this implies that $\bar{p}(q)$ is unique for all $q \in \left(\frac{1}{2}, \frac{2+\sqrt{3}}{4}\right)$. The continuity of $\bar{p}(q)$ follows from the differentiability of $B_3'(\Delta|p,q)$ w.r.t. p. ■

A.5 Proof of proposition 2

First, from the proof of claim 1 we obtain $\hat{\Delta}_1(q) = \hat{s}_1^H(q) - \hat{s}_1^L(q) = \frac{\kappa B_3'(\Delta|p,q) h\left(\hat{s}^L(q),\hat{s}^H(q)\right)}{|A|(1-p)}$, which establishes the first part of proposition 2.

The second part of proposition 2 is shown by constructing a numerical example for which $B_3\left(\hat{\Delta}(q)|p,q\right) + B_1\left(\hat{\Delta}(q)|p,q\right)\hat{\Delta}_1(q) < 0$. Consider $\hat{\Delta}(q) = \frac{3}{2}$, $p = 0.9$, $q = 0.91$ and φ the normal density function with $\sigma = 2$. In this case, we seek to construct an S.E. where $\frac{\kappa h\left(\hat{s}^L(q),\hat{s}^H(q)\right)}{|A|(1-p)} < \frac{B_3\left(\hat{\Delta}(q)|p,q\right)}{B_1\left(\hat{\Delta}(q)|p,q\right)B_3'\left(\hat{\Delta}(q)|p,q\right)} \simeq -395.095$. For these parameter values we also have $\max_{\Delta}\{|B_1'(\Delta)|\} < 0.008052 \equiv C$, $\max_{\Delta}\{|B''(\Delta)|\} < 0.0104 \equiv D$ and $B''\left(\frac{3}{2}\right) \simeq 0.00612$. Choose a utility function for which $-v_{11}\left(.|\theta^H\right)$ is minimal at $\hat{s}^H(q)$ and for which $v\left(.|\theta^L\right)$ is sufficiently concave to guarantee $\frac{-v_{11}\left(.|\theta^L\right)}{-v_{11}\left(\hat{s}^H(q)|\theta^H\right)} > \frac{9D}{C} = 15.289$. Note that this implies $h\left(\hat{s}^L(q), \hat{s}^H(q)\right) < 0$ and $\hat{\Delta}_1(q) < 0$ and that condition 4 is satisfied if we choose

$$\kappa = \frac{-v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{C}.$$

Thus, we obtain

$$\frac{\kappa h\left(\hat{s}_1^L(q),\hat{s}_1^H(q)\right)}{|A|(1-p)} = \left(\frac{-C}{\left(1-\frac{p}{(1-p)}\frac{v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^L(q)|\theta^L\right)}\right)} + B_1'(\Delta)\right)^{-1}$$

$$= \left(\frac{-0.008052}{\left(1-9\frac{v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^L(q)|\theta^L\right)}\right)} + 0.00612\right)^{-1}$$

which is smaller than -395.095 if $\frac{v_{11}\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^L(q)|\theta^L\right)} \lessapprox 7.720\times 10^{-3}$, i.e. if $-v_{11}\left(\hat{s}^L(q)|\theta^L\right) > 129.53\left(-v_{11}\left(\hat{s}^H(q)|\theta^H\right)\right)$.

Finally, an S.E. is constructed which satisfies the above restrictions. First, $B'\left(\hat{\Delta}(q)\right) \simeq 0.0111$ for the given parameter values, such that (9) can be written $\frac{v_1\left(\hat{s}^H(q)|\theta^H\right)}{v_{11}\left(\hat{s}^H(q)|\theta^H\right)} = \frac{0.0111}{0.008052} = 1.3748$ and likewise (10) becomes $\frac{v_1\left(\hat{s}^L(q)|\theta^L\right)}{v_{11}\left(\hat{s}^H(q)|\theta^H\right)} = 9\frac{(0.0110702)}{0.008052} = 12.374$. No other restrictions impede the construction of such a function v.