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Revealed preference theory for finite choice sets

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Abstract

The theory of revealed preferences offers an elegant way to test the neoclassical model of utility maximization subject to a linear budget constraint. In many settings, however, the set of available consumption bundles does not take the form of a linear budget set. In this paper, we adjust the theory of revealed preferences to handle situations where the set of feasible bundles is finite. Such situations occur frequently in many real life and experimental settings. We derive the revealed preference conditions for consistency with utility maximization in this finite choice-set setting. Interestingly, we find that it is necessary to make a distinction between the cases where the underlying utility function is weakly monotone, strongly monotone and/or concave. Next, we provide conditions on the structure of the finite choice sets for which the usual revealed preference condition (i.e. GARP) is still valid. We illustrate the relevance of our results by means of an application based on two experimental data sets that contain choice behavior from children.
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1 Introduction

The theory of revealed preferences provides an attractive methodology to verify whether a finite data set of chosen consumption bundles from a collection of linear budget sets is consistent with the neo-classical model of utility maximization. Revealed preference theory provides a number of advantages in comparison to other (econometric) methods. First of all, revealed preference theory is nonparametric in the sense that it abstains from imposing a functional specification of the utility function prior to the analysis. As such, the revealed preference approach avoids that a specific behavioral model is rejected because of an erroneous functional specification (while the actual consumption behavior is consistent with the model of utility maximization). Second, revealed preference methods are particularly attractive from a practical point of view: for a given data set, it allows for testing data consistency with the utility maximization model in a very straightforward and quick way. A third important advantage of the revealed preference approach is that it can be meaningfully applied to small data sets. As such, by restricting the revealed preference test to repeated observations from the same individual, it can avoid the (often debatable) preference homogeneity assumptions across individuals. As a consequence, revealed preference theory provides an attractive methodology for many real life applications and experimental settings.

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Finite choice sets Revealed preference theory was initially developed to deal with situations where choices are made from linear budget sets.[1] On the other hand, in many settings choice sets are inherently finite.

A first and obvious case where finite choice sets occur naturally is when the goods under consideration can only be bought in discrete amounts. In such cases, the choice set can be represented as the intersection of a linear budget set and the space $\mathbb{Z}_+^n$. This particular setting is studied in the recent paper by Polisson and Quah (2013). Polisson and Quah show that for such choice sets, the standard revealed preference condition (i.e. GARP) characterizes the data sets which are rationalizable by a utility function which is separable in an unobserved good which can be consumed in continuous quantities. Their analysis differs from ours in two ways. First of all, our focus is on more general discrete choice settings, i.e. we do not necessarily require that the choice sets are constructed as the intersection of a discrete set and a linear budget set. Second, and more important, we have a different focus. Our main goal is to obtain the set of revealed preference conditions that characterizes the standard model of utility maximization when choices are made from finite choice sets. On the other hand, Polisson and Quah (2013) look for the specific utility maximization model that underlies the standard revealed preference condition (i.e. GARP).

A second setting where discrete choice sets in combination with revealed preferences are pertinent is for experimental data. Indeed, revealed preference theory is remarkably well suited to analyze the rationality of subjects in experimental settings. Its main advantage lies in the fact that experiments can be specifically designed to allow for very powerful tests (e.g. by letting prices vary and keeping budgets constant across different choice problems). The usual procedure for such revealed preference experiments is to let the subjects of the experiment solve a number of different exercises. For each exercise, the subject is endowed with a budget (expressed as a number of tokens) and is informed on a vector of prices. Next, the subjects are instructed to allocate their budget over a set of goods subject to the budget constraint defined by the income and the prices.[2] However, this experimental design may pose two potential problems. First of all, it requires that the subjects understand the concepts of money, prices and income. Furthermore, he/she must also be able to compute the total expenditure and compare it with the total available budget. This requirement is not always satisfied, especially when the subjects under consideration are children.[3] Second, revealed preference theory usually requires that the entire available budget must be exhausted (i.e. the total expenditure should be equal to the available budget). In settings where there are only two goods, this requirement can be met by representing choice problems graphically as a 2–dimensional budget line.[4] Then, budget exhaustion can easily be imposed by restricting the choices to lie on the budget line. However, if there are more than two goods, graphical illustrations are no longer feasible (or much more difficult to represent). As such, the requirement that the entire budget must be exhausted might require a lot of fine–tuning on the part of the subject. In these settings, subjects are usually given calculators (or other computing devices) to check whether there are any tokens left to spend. Nevertheless, this fine–tuning might still impose a considerable burden on the subjects. In fact, this burden might become large enough such that it actually interferes with the optimality of the choices.[5]

[1] Although the theory has been extended to deal with nonlinear budgets (See, for instance, Yatchew, 1985; Matzkin, 1991; Forges and Minelli, 2009; Cherchye, Demuynck, and De Rock, 2012), none of these papers looks at the situation where choice sets are finite. Recently Forges and Iehlé (2012) also discuss the revealed preference conditions when the available data only consist of a so called "essential experiment" given by observed consumption bundles and a feasibility matrix. In such setting the experimental observer only knows to which extent a bundle that has been chosen at some date is also available at another date.

[2] See, for example, Cox (1997), Andreoni and Miller (2002), Février and Visser (2004), Fishman, Kariv, and Markovits (2007), Huck and Rasul (2008), Bruyneel, Cherchye, and De Rock (2012b) and Cherchye, Demuynck, and De Rock (2013) for similar experimental designs.

[3] This problem is also present in our empirical application which is based on experimental data obtained from choices by children.

[4] This setting can even be extended to include choices under uncertainty (Choi, Fisman, Gale, and Kariv, 2007).

[5] Indeed, if there are many goods, then the opportunity cost (i.e. additional time) that is needed to fine–tune the choices may become quite large. This, in turn, might lead to situations where the subjects choose to lower the time spent on fine-tuning at the expense of choosing a less–optimal bundle. The difficulty is clearly illustrated in an article by Mattei (2000) which reports on three experiments. The second experiment involved choices from 100 business students who each had to solve 20 exercises. Each exercise

An elegant solution to the two aforementioned problems is to design the experiment in such a way that the subjects choose from a finite set of distinct bundles. Restricting the choice sets to be finite makes the choices of the respondents much easier: they only have to pick one bundle from a finite collection of feasible options. The first experimental study that uses this option is by Harbaugh, Krause, and Berry (2001) who investigated the rationality of choice behavior by children. Their experimental design has been replicated by several others (see for instance Burghart, Glimcher, and Lazzaro (2012) and List and Millimet (2008)). We will also use their data set in our empirical application.

A third relevant case where choice sets are finite is when choices are made by picking a single item from a finite set of distinct alternatives (e.g. the choice between different cars from a catalogue). In these settings, it is useful to think of the different alternatives as representing different bundles of characteristics (e.g. the price, the top speed, the fuel efficiency, etc.). Usually, these kind of discrete choice models are analyzed by econometric methods which are based on limited dependent variables models (see for instance Train (2009) for a thorough overview). Given this, our results can actually be seen as a first step towards an analysis of such discrete choice characteristics models by nonparametric revealed preference techniques.[6] Related to this, we also like to point to the large choice theoretic (and behavioral) literature that models the choice behavior over arbitrary (discrete) sets of alternatives, which need not necessarily be representable as bundles of goods or characteristics. The main rationalizability concept in this setting is due to Richter (1966), who provided a choice theoretic analogue to the 'consumption based revealed preference literature' founded by Samuelson (1938, 1948) and Houthakker (1950). By developing a 'consumption based' revealed preference theory for finite choice sets, we are in a certain sense building a bridge between these two largely separate literatures, thereby opening the door for empirical applications of various other choice theoretic models.

Contributions Our paper has several contributions. First of all, from a theoretical perspective, we derive a number of revealed preference conditions that can be applied to settings where choices are made from finite sets of distinct consumption bundles. Towards this end, we distinguish between four cases: rationalizability by a weakly monotone utility function, rationalizabililty by a weakly monotone and concave utility function, rationalizability by a strongly monotone utility function and rationalizability by a strongly monotone and concave utility function. For each of these rationalizability concepts we obtain a different set of revealed preference conditions. Interestingly, these different revealed preference conditions do not coincide. As such, it is for example possible to find a data set that is rationalizable by a weakly monotone utility function but not by a concave and weakly monotone utility function. This result is interesting, because such distinctions can not be made when choices are obtained from linear budget sets. Indeed, a well known result in revealed preference theory (Afriat's theorem) tells us that in such cases, all four rationalizability concepts coincide (see section 2 for more details). Next, we adapt an idea of Houtman and Maks (1985) and develop a goodness-of-fit measure for our different revealed preference tests. The measure coincides with the size of the largest subset of the data set that still satisfies the relevant revealed preference conditions. In this way, it gives an indication of the severity of violation in cases where the data set under consideration is not rationalizable. We also show that our measure can be computed by solving a linear programming problem with binary variables.

Second, we provide a number of conditions on the finite choice sets for which it is allowed to neglect the fact that choices are made from finite choice sets. In other words, we present a collection of assumptions such that the standard revealed preference condition (i.e. GARP) is still valid for consistency with utility maxi-

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consisted of allocating a given budget over 8 different commodities. He found that (despite the computer warning signal) 4 out of 100 students exceeded the budget by more than 1 percent, 47 subjects spent less than 99 percent of the budget and 6 subjects even spent less than 90 percent of the budget. The third experiment was done using questionnaires by mail. For this experiment, Mattei (2000) found that out of 320 respondents, 4 percent spent beyond 10 percent of the available budget and 12 percent spend less than 90 percent of the budget.

[6] The revealed preference conditions of the characteristics model in a continuous choice space were analyzed by Blow, Browning, and Crawford (2008).

mizing behavior. These conditions can be used to design experimental settings for which the results can still be analyzed using the standard revealed preference conditions. For example, we show that the experimental design of Harbaugh, Krause, and Berry (2001) satisfies the conditions such that GARP characterizes the data sets that are consistent with utility maximization by a strongly monotone (and concave) utility function. However, we also show that is is not possible to strengthen this to utility maximization with a weakly monotone (and concave) utility function. In other words, it possible that the data set violates GARP, but the behavior was nevertheless generated by some weakly monotone utility function (e.g. a Leontief utility function).

Finally, we show the relevance of our results by analyzing two experimental data sets that contain choices made by children. We compare the empirical fit of the different rationalizability concepts and we compute the goodness-of-fit measure. The first data set is from the previously mentioned experiment of Harbaugh, Krause, and Berry (2001). The second is from Bruyneel, Cherchye, Cosaert, De Rock, and Dewitte (2012a). For both data sets, we find that imposing weak monotonicity instead of strong monotonicity on the utility functions improves the fit in terms of higher predictive success.

Outline In section 2, we give a brief summary of the most important results in revealed preference theory. This discussion will be useful to position our results within the revealed preference literature. Section 3 contains the main theoretical results of this paper. It develops the revealed preference conditions for the finite choice set framework. In section 4, we present some conditions for which the usual revealed preference conditions coincide with our revealed preference conditions. Section 4 contains an empirical application of our results to two experimental data sets. Finally, section 5 concludes. All proofs are in the appendix.

2 Revealed preference theory for linear budget sets

In this section, we present the basic revealed preference theory. This will be useful for comparison with the results that will be established in sections 3 and 4.

To start, consider a finite collection of sets $\{B_t\}_{t\in T}$, where T is a finite set of observations, $T = \{1, 2, \dots, |T|\}$. In this section, we assume that the choice sets take on the form of a linear budget set,

$$B_t = \{\mathbf{q} \in \mathbb{R}^n_+ | \mathbf{p}_t\mathbf{q} \leq m_t\},$$

In words, the choice set B_t contains all bundles $\mathbf{q} \in \mathbb{R}^n_+$ that can be bought with a certain income $m_t > 0$ at prices $\mathbf{p}_t \in \mathbb{R}^n_{++}$. In the next sections, we will consider the setting where each budget set B_t consists of a finite number of distinct bundles.

A data set $S = \{B_t, \mathbf{q}_t\}_{t\in T}$ then consists of a finite number of budget sets and for each budget set B_t a bundle $\mathbf{q}_t$ from this set, i.e. $\mathbf{q}_t \in B_t$. The idea is that $\mathbf{q}_t$ is the bundle which is chosen from the budget set B_t. Usually, it is assumed that $\mathbf{q}_t$ lies on the boundary of B_t, i.e. $\mathbf{p}_t\mathbf{q}_t = m_t$.[7] Given this, data sets are also frequently written as $\{\mathbf{p}_t, \mathbf{q}_t\}_{t\in T}$ in the understanding that the underlying budget set is implicitly given by:

$$B_t = \{\mathbf{q} \in \mathbb{R}^n_+ | \mathbf{p}_t\mathbf{q} \leq \mathbf{p}_t\mathbf{q}_t\}.$$

The following defines the standard rationality concept in revealed preference theory.

Definition 1 (Rationalizability). *A data set $S = \{B_t, \mathbf{q}_t\}_{t\in T}$ is* **rationalizable** *by the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ if for all $t \in T$,*

$$\mathbf{q}_t \in \arg\max_{\mathbf{q}\in B_t} u(\mathbf{q}).$$

[7] In most experimental settings, this condition is additionally imposed.

In words, a data set S is rationalizable by the utility function u if for each observation $t \in T$, the chosen bundle $\mathbf{q}_t$ maximizes the utility function $u(.)$ over the budget set B_t.

A utility function $u(.) : \mathbb{R}^n_+ \to \mathbb{R}$ is **weakly monotone** if $\mathbf{q} \geq \mathbf{q}'$ implies $u(\mathbf{q}) \geq u(\mathbf{q}')$ and $\mathbf{q} \gg \mathbf{q}'$ implies $u(\mathbf{q}) > u(\mathbf{q}')$.[8] A utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is **strongly monotone** if $\mathbf{q} \geq \mathbf{q}'$ implies $u(\mathbf{q}) \geq u(\mathbf{q}')$ and $\mathbf{q} > \mathbf{q}'$ implies $u(\mathbf{q}) > u(\mathbf{q}')$. The utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is **locally non-satiated** if for every open neighborhood N of $\mathbf{q}$ there is a bundle $\mathbf{q}' \in N \cap \mathbb{R}^n_+$ such that $u(\mathbf{q}') > u(\mathbf{q})$. Strong monotonicity is stronger than weak monotonicity (as it rules out situations like Leontief utility functions) which, in turn, is stronger than local non–satiation. The utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is **concave** if for all $\mathbf{q}$ and $\mathbf{q}' \in \mathbb{R}^n_+$ and all $\alpha \in [0, 1]$, $u(\alpha\mathbf{q} + (1 - \alpha)\mathbf{q}') \geq \alpha u(\mathbf{q}) + (1 - \alpha)u(\mathbf{q}')$.

Given these properties on the utility function, it is possible to define different rationalizability concepts, e.g. rationalizability by a strongly monotone and concave utility function, or rationalizability by a weakly monotone utility function. As will be demonstrated in Theorem 1 below, if budget sets are linear, then all these rationalizability concepts coincide. However, as we will demonstrate in the following sections, this equivalence breaks down when choice sets are finite.

Before we characterize the data sets that are rationalizable, we first define the Generalized Axiom of Revealed Preference (GARP) (see Varian (1982)).

Definition 2 (GARP). *The data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies the* **Generalized Axiom of Revealed Preference** *if there exists a binary relation R such that for all observations t, v and $s \in T$:*

1. *If $\mathbf{p}_t\mathbf{q}_t \geq \mathbf{p}_t\mathbf{q}_v$, then $\mathbf{q}_t R \mathbf{q}_v$.*
2. *If $\mathbf{q}_t R \mathbf{q}_v$ and $\mathbf{q}_v R \mathbf{q}_s$, then $\mathbf{q}_t R \mathbf{q}_s$.*
3. *If $\mathbf{q}_t R \mathbf{q}_v$, then it is not the case that $\mathbf{p}_v\mathbf{q}_v > \mathbf{p}_v\mathbf{q}_t$.*

The relation R is called the revealed preference relation. Condition 1 in Definition 2 states that $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$ whenever $\mathbf{p}_t\mathbf{q}_t \geq \mathbf{p}_t\mathbf{q}_v$, i.e. when $\mathbf{q}_v$ belongs to the budget set B_t. Indeed, in this case, $\mathbf{q}_t$ was chosen while $\mathbf{q}_v$ was also available. Condition 2 imposes transitivity on the revealed preference relation. The closing condition 3 says that if $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$, then it is not the case that $\mathbf{q}_v$ was chosen while $\mathbf{q}_t$ was in the interior of the budget B_v. GARP can be easily verified using Warshall (1962)'s algorithm to compute the transitive closure (see Varian (1982) for an algorithm).

It turns out that GARP is equivalent to rationalizability by a well behaved utility function. The proof of the theorem can be found in Varian (1982) and is based on previous results from Afriat (1967).

Theorem 1 (Afriat's theorem). *Consider a data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ where each set B_t ($t \in T$) is a linear budget set. Then the following conditions are equivalent:*

1. *The data set S is rationalizable by a locally non–satiated utility function.*
2. *The data set S satisfies GARP,*
3. *For all $t \in T$, there exist numbers φ_t and $\lambda_t > 0$ such that for all $t, v \in T$,*

$$\varphi_t - \varphi_v \leq \lambda_v \mathbf{p}_v(\mathbf{q}_t - \mathbf{q}_v).$$

[8] For two vectors $\mathbf{q}$ and $\mathbf{q}'$, we write $\mathbf{q} \geq \mathbf{q}'$ if every element of the vector $\mathbf{q}$ is as least as large as the corresponding element of the vector $\mathbf{q}'$. We denote $\mathbf{q} > \mathbf{q}'$ if $\mathbf{q} \geq \mathbf{q}'$ and $\mathbf{q} \neq \mathbf{q}'$. Finally, we have that $\mathbf{q} \gg \mathbf{q}'$ if every element of $\mathbf{q}$ is strictly larger than the corresponding element of $\mathbf{q}'$.

4. *The data set S is rationalizable by a concave, (continuous) and strongly monotone utility function.*

The above theorem shows that rationalizability by a locally non–satiated utility function is equivalent to GARP. Next, the equivalence between the second and fourth condition states that GARP is also equivalent to rationalizability by a concave, strongly monotone and continuous utility function. The equivalence between the first and fourth condition actually shows that it is impossible to reject concavity and strong monotonicity of the utility function without rejecting utility maximization by a locally non–satiated (and, hence, weakly monotone) utility function. In other words, if budget sets are linear, then all the different rationalizability concepts which are nested between the properties of 'local non–satiation' and 'strict monotonicity and concavity' coincide. In the next section, we will show that this property no longer holds if the choice sets B_t are finite. In other words, the equivalence between the first and fourth condition turns out to be a consequence from the fact that choice sets take the shape of linear budget sets.

The linear inequalities in the third condition are the so called Afriat inequalities. These have a nice interpretation when the underlying rationalization is concave. Indeed, if, for example, $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ is rationalizable by a concave and strongly monotone utility function $u(.)$ and if we take the simplifying assumption that $u(.)$ is differentiable, then from concavity of $u(.)$ we have that for all t and v,

$$u(\mathbf{q}_t) - u(\mathbf{q}_v) \leq \nabla_{\mathbf{q}} u(\mathbf{q}_v)(\mathbf{q}_t - \mathbf{q}_v). \tag{1}$$

Here $\nabla_{\mathbf{q}} u(\mathbf{q}_v)$ is the gradient of $u(.)$ at the bundle $\mathbf{q}_v$. Strict monotonicity requires that $\nabla_{\mathbf{q}} u(\mathbf{q}_v) \gg \mathbf{0}$. Next, the first order conditions for the utility maximization problem imply that,

$$\nabla_{\mathbf{q}} u(\mathbf{q}_v) = \lambda_v \mathbf{p}_v, \tag{2}$$

where λ_v is the strictly positive Lagrange multiplier corresponding to the budget constraint. Then, if we substitute equality (2) into inequality (1) and if we set $\varphi_t = u(\mathbf{q}_t)$ and $\varphi_v = u(\mathbf{q}_v)$, we effectively obtain the Afriat inequalities.

The Afriat inequalities form a set of linear inequalities. As such, they provide a second set of conditions by which it can be verified whether a data set is rationalizable.

3 Revealed preference theory for finite choice sets

In the previous section we considered the case where each choice set B_t takes on the form of a linear budget set. However, as explained in the introduction, in many contexts, individuals choose by picking one out of a finite number of distinct consumption bundles. To model this setting, we assume from now on that each choice set B_t consists of a finite number of distinct bundles $B_t = \{\mathbf{b}_t^1, \ldots, \mathbf{b}_t^{K_t}\} \in \prod_{i=1}^{K_t} \mathbb{R}_+^n$. Here K_t is the size of the choice set B_t, which may depend on the observation $t \in T$.

As in the previous section, we denote by $\mathbf{q}_t$ the observed choice from the set B_t, i.e. $\mathbf{q}_t \in \{\mathbf{b}_t^1, \ldots, \mathbf{b}_t^{K_t}\}$ and we denote a data set S as $\{B_t, \mathbf{q}_t\}_{t \in T}$. The concept of rationalizability in this setting is identical to the definition of the previous section: the data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ is rationalizable by the utility function $u(.)$ if for all $t \in T$,

$$\mathbf{q}_t \in \arg\max_{\mathbf{q} \in B_t} u(\mathbf{q}).$$

In contrast to the linear budget case, rationalizability by a locally non-satiated utility function is no longer a useful concept when choice sets are finite. The reason is that choice sets do not contain open subsets. For this reason, we will restrict ourselves to the two most natural strengthenings of local non–satiation, namely weak and strong monotonicity.

We will make a distinction between four different notions of rationalizability: (i) rationalizability by a weakly monotone utility function, (ii) rationalizability by a strongly monotone utility function, (iii) rationalizability by a weakly monotone and concave utility function and (iv) rationalizability by a strongly monotone and concave utility function. In principle, it is possible to obtain for all four rationalizability concepts, revealed preference restrictions both in terms of GARP-like conditions and in terms of Afriat-type inequalities.[9] However, we will restrict ourselves to GARP-like conditions for the first two rationalizability concepts (the cases without concavity) and we will restrict ourselves to the Afriat-type conditions for the rationalizability concepts with concavity.[10] The reason for doing this is twofold. First, from a theoretical viewpoint, it turns out that the conditions that we present are the most intuitive for the rationalizability concept under consideration. The other revealed preference conditions are much more difficult to interpret. Second, from an empirical viewpoint, it turns out that the omitted revealed preference conditions are computationally much more difficult to verify and therefore less useful in practice.

Weakly monotone rationalizability Let us start with rationalizability by a weakly monotone utility function. First, we introduce the Weakly Monotone Axiom of Revealed Preference (WMARP).

Definition 3 (WMARP)**.** *A data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies the* **Weakly Monotone Axiom of Revealed Preference** *if there exists a binary relation R such that for all t, v and $s \in T$,*

1. *If there exists a bundle $\mathbf{b}_t^k \in B_t$ such that $\mathbf{b}_t^k \geq \mathbf{q}_v$ then $\mathbf{q}_t R \mathbf{q}_v$.*
2. *If $\mathbf{q}_t R \mathbf{q}_v$ and $\mathbf{q}_v R \mathbf{q}_s$, then $\mathbf{q}_t R \mathbf{q}_s$.*
3. *If $\mathbf{q}_t R \mathbf{q}_v$ then for all $\mathbf{b}_v^k \in B_v$ it is not the case that $\mathbf{b}_v^k \gg \mathbf{q}_t$.*

Similar to GARP, we can interpret R as representing the revealed preference relation. The first condition states that if $\mathbf{q}_t$ was chosen from B_t and if there is another bundle in B_t, say $\mathbf{b}_t^k$ such that $\mathbf{b}_t^k \geq \mathbf{q}_v$, then $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$. This step combines the usual revealed preference idea (if a bundle $\mathbf{q}_t$ is chosen while another bundle $\mathbf{b}_t^k$ was also available, then it is at least as good as this second bundle) with the property of monotonicity (as $\mathbf{b}_t^k \geq \mathbf{q}_v$, $\mathbf{b}_t^k$ is at least as good as $\mathbf{q}_v$). The second condition imposes transitivity on the revealed preference relation. Finally, the third condition requires that if $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$, it is not the case that there is an option in B_v, say $\mathbf{b}_v^k$, such that $\mathbf{b}_v^k \gg \mathbf{q}_t$. The intuition behind this is straightforward: if the third condition is not satisfied, then it would have been better to choose $\mathbf{b}_v^k$ instead of $\mathbf{q}_v$, as it contains strictly more than the option $\mathbf{q}_t$ which is at least as good as $\mathbf{q}_v$.

Given the definition of WMARP, we can state our first result. All the proofs of the theorems can be found in the appendix.

Theorem 2. *Consider a data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$. Then the following conditions are equivalent:*

- *The data set S is rationalizable by a weakly monotone (and continuous) utility function.*
- *The data set S satisfies WMARP.*

Strongly monotone rationalizability The characterization of rationalizability by a strongly monotone utility function is similar to the case of weak monotonicity. Consider the following Strongly Monotone Axiom of Revealed preference (SMARP).

Definition 4 (SMARP)**.** *A data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies the* **Strongly Monotone Axiom of Revealed Preference** *if there exists a binary relation R such that for all t, v and $s \in T$,*

1. *If there exists a bundle $\mathbf{b}_t^k \in B_t$ such that $\mathbf{b}_t^k \geq \mathbf{q}_v$ then $\mathbf{q}_t R \mathbf{q}_v$.*

[9] As an example of Afriat-type restrictions for the cases of rationalizability by a weakly or strongly monotone utility function, we refer to the proofs of Theorems 2 and 3.

[10] The complementary revealed preference conditions can be obtained from the authors upon request.

2. *If* $\mathbf{q}_t R \mathbf{q}_v$ *and* $\mathbf{q}_v R \mathbf{q}_s$*, then* $\mathbf{q}_t R \mathbf{q}_s$*.*

3. *If* $\mathbf{q}_t R \mathbf{q}_v$ *then for all* $\mathbf{b}_v^k \in B_v$ *it is not the case that* $\mathbf{b}_v^k > \mathbf{q}_t$*.*

The only difference between WMARP and SMARP lies in the closing condition 3: for SMARP, the inequality $\mathbf{b}_v^k \gg \mathbf{q}_t$ is weakened to $\mathbf{b}_v^k > \mathbf{q}_t$.

As the following theorem shows, SMARP characterizes the data sets that are rationalizable by a strongly monotone utility function.

Theorem 3. *Consider a data set* $S = \{\mathbf{q}_t, B_t\}_{t \in T}$*. Then the following conditions are equivalent:*

- *The data set* S *is rationalizable by a strongly monotone (and continuous) utility function.*
- *The data set* S *satisfies SMARP.*

Both conditions WMARP and SMARP can easily be verified by using a simple adaptation of the algorithm presented by Varian (1982) which is the standard method to verify GARP.

Weakly monotone and concave rationalizability The characterizations in Theorems 2 and 3 were obtained by using variations on GARP (i.e. WMARP and SMARP). For rationalizations with a concave utility function, it will be easier to use conditions which are based on the Afriat inequalities.

In order to grasp the intuition behind these conditions, let us assume that $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ is rationalizable by a weakly monotone and concave utility function. Further, assume for convenience that u is also differentiable.[11] Then, by concavity of the function u, we have that for all observations t and $v \in T$ and all bundles $\mathbf{b}_t^k \in B_t$ and $\mathbf{b}_v^j \in B_v$:

$$u(\mathbf{b}_t^k) - u(\mathbf{b}_v^j) \leq \nabla_{\mathbf{q}} u(\mathbf{b}_v^j)\,(\mathbf{b}_t^k - \mathbf{b}_v^j). \tag{3}$$

Now, define the numbers $\varphi_t^k = u(\mathbf{b}_t^k), \varphi_v^j = u(\mathbf{b}_v^j)$ and vectors $\mathfrak{p}_v^j = \nabla_{\mathbf{q}} u(\mathbf{b}_v^j)$. For the latter, we have that $\mathfrak{p}_v^j > 0$ because u is weakly monotone. Then, substituting these values in (3) gives:

$$\varphi_t^k - \varphi_v^j \leq \mathfrak{p}_v^j(\mathbf{b}_t^k - \mathbf{b}_v^j).$$

This gives a set of Afriat–type inequalities. We augment these Afriat inequalities with a second set of conditions: given that $\mathbf{q}_t = \mathbf{b}_t^k$ was chosen while another bundle $\mathbf{b}_t^j$ was also feasible at $t \in T$, it must be that $\varphi_t^k = u(\mathbf{b}_t^k) = u(\mathbf{q}_t) \geq u(\mathbf{b}_t^j) = \varphi_t^j$. Hence, we also require that for all observations $t \in T$ and all $j \leq K_t$, if $\mathbf{q}_t = \mathbf{b}_t^k$, then $\varphi_t^k \geq \varphi_t^j$.

Definition 5 (WMCARP). *A data set* $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ *satisfies the* **Weakly Monotone and Concave Axiom of Revealed Preference** *if for all* $t \in T$ *and* $\mathbf{b}_t^k \in B_t$ *there exist numbers* φ_t^k *and vectors* $\mathfrak{p}_v^k > 0$ *such that for all* $t, v \in T$*:*

$$\varphi_t^k - \varphi_v^j \leq \mathfrak{p}_v^j(\mathbf{b}_t^k - \mathbf{b}_v^j) \text{ and,}$$
$$\text{if } \mathbf{q}_t = \mathbf{b}_t^k \text{ then, } \varphi_t^k \geq \varphi_t^j \text{ for all } j \leq K_t.$$

Given this definition, we can give our characterization for rationalizability by a weakly monotone and concave utility function.

Theorem 4. *Consider a data set* $S = \{B_t, \mathbf{q}_t\}_{t \in T}$*. Then the following conditions are equivalent:*

- *The data set* S *is rationalizable by a weakly monotone (continuous) and concave utility function.*
- *The data set* S *satisfies WMCARP.*

[11] If u is not differentiable, it suffices to replace the gradients $\nabla_{\mathbf{q}} u(\mathbf{b}_t^k)$ with a suitable subdifferential (see for instance Rockafellar (1970)).

Stongly monotone and concave rationalizability Now, in order to characterize the case where u is strongly monotone and concave, the only thing that we need to modify is the condition $\mathfrak{p}_v^j > 0$, which now becomes $\mathfrak{p}_v^j \gg 0$.

Definition 6 (SMCARP). *A data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies the* **Strongly Monotone and Concave Axiom of Revealed Preference** *if for all $t \in T$ and $\mathbf{b}_t^k \in B_t$ there exist numbers φ_t^k and vectors $\mathfrak{p}_v^k \gg 0$ such that for all $t, v \in T$:*

$$\varphi_t^k - \varphi_v^j \leq \mathfrak{p}_v^j (\mathbf{b}_t^k - \mathbf{b}_v^j),$$
$$\textit{if } \mathbf{b}_t^k = \mathbf{q}_t \textit{ then, } \varphi_t^k \geq \varphi_t^j \textit{ for all } j \leq K_t.$$

This gives our last characterization.

Theorem 5. *Consider a data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$. Then the following conditions are equivalent:*

- *The data set S is rationalizable by a strongly monotone (continuous) and concave utility function.*
- *The data set S satisfies SMCARP.*

Both SMCARP and WMCARP are expressed as a set of linear inequalities.[12] As such, they can easily be verified using suitable linear programming methods.

Goodness-of-fit Above revealed preference tests (WMARP, SMARP, WMCARP and SMCARP) tell us whether or not a data set is consistent with utility maximizing behavior for various conditions on the underlying utility function. However, as convincingly argued by Varian (1990), in many cases, nearly optimizing behavior is just as good as optimizing behavior. As such, it would be useful to have some indication that says how close a given data set is to being rationalizable if it violates the revealed preference conditions. Usually, nearly optimizing behavior is measured by using a goodness–of–fit measure. The most popular goodness–of–fit measure in the revealed preference literature is without doubt Afriat (1973)'s critical cost efficiency index. Intuitively, the critical cost efficiency index measures the amount by which each budget must be minimally adjusted in order to remove all GARP violations. Although this concept makes sense when choice sets are linear budgets, it is less suited when choice sets are finite.

A more interesting goodness–of–fit measure in our setting is the Houtman and Maks measure (HM–measure) (see Houtman and Maks, 1985)). The HM–measure gives the size of the largest subset of observations (i.e. the largest subset of T) which is still consistent with the revealed preference conditions under consideration. For example, the HM-measure for GARP looks at the largest subset of T, say A, such that $\{B_t, \mathbf{q}_t\}_{t \in A}$ still satisfies GARP.

A difficulty with the HM–index is that it is in general difficult to compute. In particular, the problem is known to be NP–hard (Houtman and Maks, 1985; Dean and Martin, 2008). Despite this difficulty, we will show that it is possible to compute the HM–index for our revealed preference tests by using binary programming methods. Although binary programming methods are known to have exponential worst time complexity, they can be solved relatively fast for small to moderately sized problems.

Let us first focus on WMARP. In order to present the optimization program that will compute the HM-measure, we first introduce for every two observations t and $v \in T$ two numbers $x_{t,v}$ and $y_{t,v}$ in $\{0, 1\}$. More specifically, we define for all $t, v \in T$:

- $x_{t,v} = 1$ if there is a $\mathbf{b}_t^k \in B_t$ such that $\mathbf{b}_t^k \geq \mathbf{q}_v$; and $x_{t,v} = 0$ else.

[12] The condition that $\mathfrak{p}_t^j > \mathbf{0}$ can be met by requiring that the sum of the elements in $\mathfrak{p}_t^j$ should be strictly positive.

- $y_{t,v} = 1$ if there is a $\mathbf{b}_t^k \in B_t$ such that $\mathbf{b}_t^k \gg \mathbf{q}_v$; and $y_{t,v} = 0$ else.

Now, given these numbers we consider the following linear binary program:

OP.HM-WMARP

$$\begin{aligned}
&\max_{A_t, Z_{t,v}} \sum_t A_t \\
\text{s.t. } & A_t \; x_{t,v} \leq Z_{t,v} && \forall t, v \in T \\
& Z_{t,v} + Z_{v,w} \leq 1 + Z_{t,w} && \forall t, v, w \in T \\
& 1 - Z_{t,v} \geq A_v \; y_{v,t}, && \forall t, v \in T \\
& A_t, Z_{t,v} \in \{0, 1\} && \forall t, v \in T.
\end{aligned}$$

In order to grasp the idea behind the program, let A be the largest subset of T that still satisfies WMARP. Let the variable A_t be equal to 1 if $t \in A$ and let A_t be equal to 0 if $t \notin A$. Observe that the objective function effectively maximizes the size of A.

The three restrictions of the program then guarantee that the data set $S = \{B_t, \mathbf{q}_t\}_{t \in A}$ satisfies WMARP. The intuition behind the variables $Z_{t,v}$ is that they capture the revealed preference relation for the observations $t, v \in A$. In particular, if $t, v \in A$ and $\mathbf{q}_t R \mathbf{q}_v$, then $Z_{t,v} = 1$. Indeed, the first condition imposes that when $t \in A$ (i.e. $A_t = 1$) and $\mathbf{q}_t R \mathbf{q}_v$ (i.e. $x_{t,v} = 1$), then $Z_{t,v} = 1$. The second condition imposes transitivity on the revealed preference relation (i.e. if $Z_{t,v} = 1$ and $Z_{v,w} = 1$, then $Z_{t,w} = 1$). Finally, the last condition requires that when $Z_{t,v} = 1$ (i.e. $\mathbf{q}_t R \mathbf{q}_v$) then either $v \notin A$ or $y_{v,t} = 0$ (i.e. there is no $\mathbf{b}_v^j \in B_v$ such that $\mathbf{b}_v^j \gg \mathbf{q}_t$).

Theorem 6. *If n is the solution to the program* **OP.HM-WMARP***, then the largest subset of T, say A, for which the data set $\{B_t, \mathbf{q}_t\}_{t \in A}$ satisfies WMARP is of size n.*

We can construct a similar program to compute the largest consistent subset of T that satisfies SMARP. In order to do so, we only need to modify the values of $y_{t,v}$ in the following way:

- $y_{t,v} = 1$ if there is a $\mathbf{b}_t^k \in B_t$ such that $\mathbf{b}_t^k > \mathbf{q}_v$, and $y_{t,v} = 0$ else.

Let us now look at the HM-index for the WMCARP and SMCARP tests. For WMCARP, we use the following program:

OP.HM-WMCARP

$$\begin{aligned}
&\max_{\varphi_t^k, A_t, \mathfrak{p}_t^k} \sum_t A_t \\
\text{s.t. } & \varphi_t^k - \varphi_v^j \leq \mathfrak{p}_v^j (\mathbf{b}_t^k - \mathbf{b}_v^j) + (1 - A_t) && \forall t, v \in T \\
& \text{if } \mathbf{q}_t = \mathbf{b}_t^k \text{ then } \varphi_t^k \geq \varphi_t^j && \forall t \in T \\
& A_t \in \{0, 1\} && \forall t \in T \\
& \mathfrak{p}_t^k > 0 && \forall t \in T
\end{aligned}$$

Again, the intuition is to set A_t equal to one if and only if $t \in A$. Then, if $A_t = 1$, we see that the conditions reduce to the usual WMCARP conditions. However, if $A_t = 0$, the first set of conditions are trivially satisfied (observe that we can always rescale the variables φ_t^k and $\mathfrak{p}_t^k$). As such, WMCARP is only imposed on the subset A of observations.

Theorem 7. *If n is the solution to the program* **OP.HM-WMCARP**, *then the largest subset of T, say A, such that the data set $S = \{B_t, \mathbf{q}_t\}_{t \in A}$ satisfies WMCARP is of size n.*

In order to compute the HM index for SMCARP, we can use the program **OP.HM-WMCARP** except that now we need to strengthen the final condition $\mathfrak{p}_t^k > 0$ to $\mathfrak{p}_t^k \gg 0$.

4 GARP for finite choice sets

In this section, we present a number of conditions on the finite budget sets $\{B_t\}_{t \in T}$ such that GARP is still a necessary and sufficient condition for rationalizability. These conditions could be used to design an experiment with finite choice sets for which the usual GARP condition can still be used to analyze the choice behavior of the subjects.

The GARP condition is expressed in terms of price vectors. In order to apply this, we need to introduce suitable price vectors. The easiest way to do this is to assume that all alternatives from the set B_t are situated on the same budget hyperplane. This will be our first assumption.

Assumption 1. *For all $t \in T$ there exists a price vector $\mathbf{p}_t \in \mathbb{R}^n_{++}$ and an income $m_t \in \mathbb{R}_{++}$ such that for all $\mathbf{b}_t^k \in B_t$:*

$$\mathbf{p}_t \mathbf{b}_t^k = m_t.$$

Next, given that Assumption 1 is satisfied, we consider the following additional assumption.

Assumption 2. *For all $t, v \in T$ and all $\mathbf{b}_v^k \in B_v$:*

- *If $m_t \geq \mathbf{p}_t \mathbf{b}_v^k$, then there exists a bundle $\mathbf{b}_t^j \in B_t$ such that,*

$$\mathbf{b}_t^j \geq \mathbf{b}_v^k.$$

- *If $m_t > \mathbf{p}_t \mathbf{b}_v^k$, then there exists a bundle $\mathbf{b}_t^j \in B_t$ such that,*

$$\mathbf{b}_t^j > \mathbf{b}_v^k.$$

Assumption 2 requires that if a bundle from a choice set B_v satisfies $\mathbf{p}_t \mathbf{b}_v^k \leq m_t$, then there is always a bundle in B_t which is (strictly) larger.

The following shows that under Assumptions 1 and 2, GARP and SMARP coincide. A similar result was also presented by Harbaugh, Krause, and Berry (2001) but was not proven as such.

Theorem 8. *If Assumptions 1 and 2 are satisfied, then a data set S satisfies SMARP if and only if it satisfies GARP.*

If we take into account Theorem 1, we also see that Theorem 8 implies that under Assumptions 1 and 2, SMARP is a necessary and sufficient condition for rationalizability by a concave (continuous) and strongly monotone utility function. As such, it indirectly follows that a data set will satisfy SMARP (or GARP) if and only it also satisfies SMCARP.

Now, in order to encompass the setting of a weakly monotone utility function, we must impose a third assumption.

Assumption 3. *For all $t, v \in T$ and all $\mathbf{b}_v^k \in B_v$, if $m_t > \mathbf{p}_t \mathbf{b}_v^k$, then there is a bundle $\mathbf{b}_t^j \in B_t$ such that*

$$\mathbf{b}_t^j \gg \mathbf{b}_v^k.$$

The significance of Assumption 3 in relation to Assumption 2 will become clear later on. Assumptions 1, 2 and 3 together imply the equivalence between GARP and WMARP.

Theorem 9. *If Assumptions 1, 2 and 3 are satisfied, then a data set S satisfies WMARP if and only if it satisfies GARP.*

Again taking into account Theorem 1, we also see that Theorem 9 implies that under Assumptions 1, 2 and 3, WMARP is a necessary and sufficient for rationalizability by a concave (continuous) and strongly monotone utility function, and consequentially, GARP, WMARP, SMARP, WMCARP and SMCARP will all be equivalent.

Assumption 3 cannot be left out in Theorem 9 because it is possible to find a collection of choice sets that satisfies Assumptions 1 and 2 but not Assumption 3 and which leads to a data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ that violates GARP but is nevertheless rationalizable by a weak monotone (and concave) utility function.

Figure 1: Illustration of Theorem 9

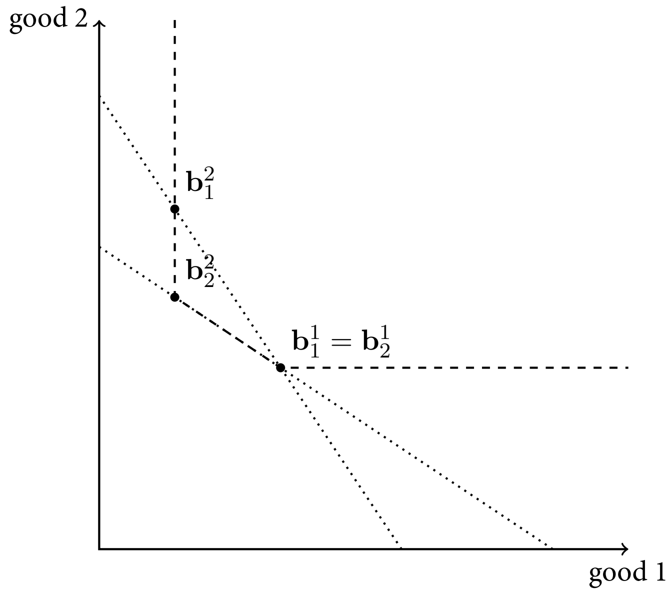
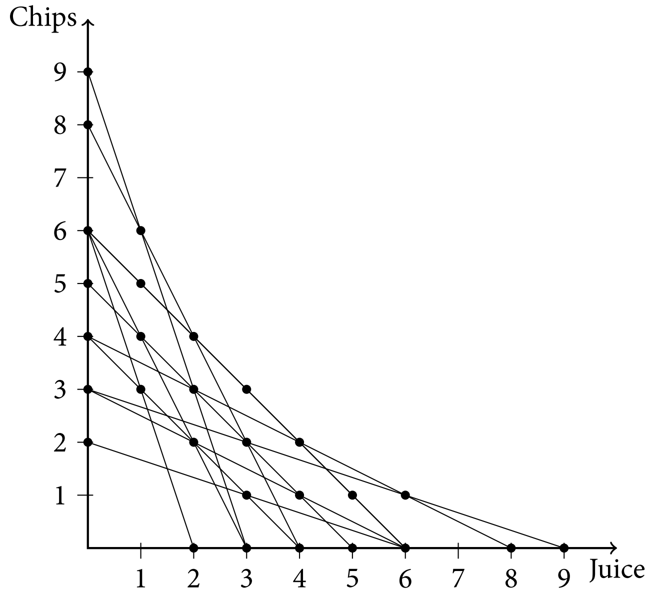


Figure 1 provides an example. There are two choice sets $B_1 = \{\mathbf{b}_1^1, \mathbf{b}_1^2\}$ and $B_2 = \{\mathbf{b}_2^1, \mathbf{b}_2^2\}$. These choice sets satisfy Assumptions 1 and 2 but violate Assumption 3. We assume that $\mathbf{b}_1^1$ is chosen from B_1 ($\mathbf{q}_1 = \mathbf{b}_1^1$) and $\mathbf{b}_2^2$ is chosen from B_2 ($\mathbf{q}_2 = \mathbf{b}_2^2$). It is easy to see that these choices violate GARP. As such, Theorem 8 says that these choices are not rationalizable by a strongly monotone (and concave) utility function (because they violate SMARP). However, it is easy to show that the choices are rationalizable by a weakly monotone (and concave) utility function. For example, the dashed curve provides an indifference curve which rationalizes all choices.

In order to see the relevance of the different assumptions, we take a closer look at the experimental design of Harbaugh, Krause, and Berry (2001) which will also be further analyzed in the next section. Figure 2 presents their experimental design. There are 11 choice sets consisting of the different points on the distinct budget lines. It is easy to verify that the choice sets satisfy both Assumptions 1 and 2. As such, GARP will be equivalent to SMARP and SMCARP. This also means that it is impossible to distinguish between rationalizability by a strongly monotone utility function and rationalizability by a strongly monotone and

Figure 2: Experimental design of Harbaugh, Krause, and Berry (2001)



concave utility function. On the other hand, the choice sets do not satisfy Assumption 3. As such, GARP will (in general) not be identical to WMARP and therefore, it might give us the opportunity to distinguish between rationalizability by a weakly monotone utility function and rationalizability by a weakly monotone and concave utility function. This feature will be demonstrated in the next section.

5 Application

We illustrate the usefulness of our results on the basis of two experimental data sets which use finite choice sets. The first is the data set from Harbaugh, Krause, and Berry (2001). The second is the data set from Bruyneel, Cherchye, Cosaert, De Rock, and Dewitte (2012a). Both data sets deal with choices made by children. As mentioned in the introduction, letting children choose from infinite budget sets is indeed problematic, because of difficulty that they may have in dealing with concepts like budgets and prices. We first give a brief description of the two data sets (more information can be found in the respective papers). Next, we present the different measures by which we will compare the different tests: pass rate, power and predictive success. Finally, we present and discuss our findings.

Brief dataset description The experiment from Harbaugh, Krause, and Berry (2001) contains information on 128 children (31 second grade students, 42 sixth grade students and 55 college undergraduates). Each child had to choose from 11 different choice sets ($|T| = 11$). Each choice set contains different bundles of chips and juice (the choice sets are illustrated in Figure 2). As already discussed in section 4, these choice sets satisfy Assumptions 1 and 2. As such, we have that GARP, SMARP and SMCARP will give identical results. However, it can still be useful to compare the outcome of these tests with WMARP and WMCARP.

The second experiment, from Bruyneel, Cherchye, Cosaert, De Rock, and Dewitte (2012a), contains information about choice behavior from 100 children (39 kindergarten respondents of about 5 years old, 31 third graders of about 8 years old and 30 sixth graders of about 11 years old). Each child was invited to solve nine successive choice problems ($|T| = 9$). Each choice set contained 7 distinct consumption bundles of three goods: grapes, mandarins and letter biscuits. The budget sets where chosen such that each bundle within the same budget lay, approximately, on the same hyperplane. The structure of these different hyperplanes can be found in Bruyneel, Cherchye, Cosaert, De Rock, and Dewitte (2012a).

Pass rate, power and predictive success In order to assess the empirical fit of the different revealed preference tests, we rely on three measures: the pass rate, the power and the predictive success. Next, we also look at the HM–measure.

The pass rate gives the percentage of all children that pass a certain revealed preference test. Of course, a higher pass rate implies a better fit as more children have made choices that can be rationalized. However, it is important to take into account the nestedness of the different tests. In particular, every data set that satisfies SMCARP wil also satisfy SMARP, WMCARP and WMARP, every data set that satisfies SMARP will also satisfy WMARP and every data set that satisfies WMCARP will also satisfy WMARP. The reason for this is simply that every data set which is rationalizable by a certain utility function, will also be rationalizable by a utility function with weaker properties. For example, if my choices are rationalizable by a strongly monotone and concave utility function, then they will also be rationalizable by a weakly monotone utility function. This nestedness implies, for example, that WMARP will have the highest pass rate of all revealed preference tests. However, this is only because it imposes the weakest set of restrictions on the underlying utility function.

On the other hand, since the main aim of revealed preference theory is to provide an accurate description of real consumer behavior, it is often favorable to look at more restrictive models. Otherwise stated, if a characterization is overly permissive, then it is unable to discriminate between real consumption data on the one hand and other non-rational behavior on the other hand, i.e. it has no discriminatory power. The strictness of a revealed preference test is usually measured by the power of the revealed preference test. The higher the power, the better the test is able to discriminate between irrational and rational behavior. Basically, the power of a revealed preference test measures the probability that non-rational choice behavior passes a certain revealed preference test.

Usually, power is computed using the procedure set out in Bronars (1987) which is based on Becker (1962)'s notion of non-rational, random behavior. For linear budget sets, the Bronars power measure is computed by constructing a high number of random data sets. Each random data set is constructed by drawing for each observation, a vector of random budget shares from a uniform distribution on the budget hyperplane. These budget shares then define the consumption bundle for the given observation. The power of the revealed preference test is then given by the percentage of these random data sets that violate the revealed preference condition. For example, if the power of GARP is equal to 0.9, then 90% of the random data sets violate GARP.

In our finite choice setting, we follow a modified version of the Bronars procedure. First, we generate 1000 random data sets. Each random data set is constructed by drawing, from each choice set B_t ($t \in T$) a bundle $\mathbf{q}_t$ at random (using a uniform distribution on $\{\mathbf{b}_t^1, \ldots, \mathbf{b}_t^K\}$).[13] This gives us 1000 random data sets $\{B_t, \mathbf{q}_t\}_{t \in T}$. The power of a revealed preference test is then computed as the percentage of these random data sets that fail the revealed preference test under consideration.

Using a similar reasoning as for the pass rates, it is easy to see that for two nested models, the power of the more restrictive model will be higher than the power for the more general model: if a random data set violates the weaker test, then it will also violate the more restrictive test. This implies, for example, that the power of WMARP will be lower than the power of all other revealed preference tests. In order to avoid these conflicting findings (high pass rates together with low power or low pass rates together with high power), we also use a measure that combines both pass rate and power into a single metric: the predictive success. The measure of predictive success was recently introduced and axiomatized by Beatty and Crawford (2011) and

[13] As an alternative, one could test the revealed preference conditions on all possible data sets and compute from this the fraction of all possible data sets that lead to a violation of the revealed preference conditions. However, assuming a relatively quick computation of 2 seconds per generated data set, this would lead us to a total computation time of about 400 days.

is based on an original proposal of Selten (1991). It is easily computed as the difference between the pass rate and one minus the power:

$$\text{predictive success} = \text{ pass rate } - [1 - \text{ power}].$$

The predictive success measure increases in both pass rate and power. As such, higher predictive success implies a better empirical fit. The interpretation of the predictive success measure is quite intuitive. In the best case scenario, both pass rate and power are equal to one, which gives us a predictive success of one. In such case, all observed data sets pass the revealed preference test while all random data sets are rejected by the test. In this sense, the revealed preference test is perfectly able to distinguish between actual and random behavior. In the worst case scenario, both pass rate and power are equal to zero, which gives us a predictive success of minus one. In this case, all observed data sets are rejected by the revealed preference test while all random data sets pass the test. In other words, the model explains random behavior perfectly but not the actual behavior. In intermediate cases, the measure of predictive success is found somewhere between minus one and plus one. A predictive success above zero points to a test which describes the observed data sets better than a model based on pure random behavior. A negative predictive success indicates a setting where the revealed preference test under consideration explains better random behavior than the actual observed behavior. A predictive success of zero implies that the test explains random behavior as good as the actual observed behavior (i.e. the test is unable to discriminate between random and observed behavior).

Results The pass rate, power and predictive success for the two experiments for the different revealed preference tests are given in Table 1. Let us first have a look at the results from the experiment of Harbaugh, Krause, and Berry (2001). As mentioned in Section 3 above, the experimental design is particular in the sense that the choice sets satisfy both Assumptions 1 and 2. By Theorem 8 this implies that GARP, SMARP and SMCARP are all equivalent. As the table shows, this is indeed the case. Interestingly, the pass rates of both WMARP (82%) and WMCARP (71%) are significantly higher than the pass rates of these other tests (54%). Furthermore, this increase in pass rates is not offset by an equal decrease in power. As a consequence, the highest predictive success is for WMARP (0.726) and WMCARP (0.708). The predictive success of GARP, SMARP and SMCARP is considerably lower (0.542).

Let us now look at the results for the experiment of Bruyneel, Cherchye, Cosaert, De Rock, and Dewitte (2012a). These choice sets do not satisfy Assumptions 2 and 3. We already argued that these cases are extremely interesting. Beside discriminating between rationalizability by a weakly and strongly monotone utility function, we are also able to discriminate between situations where the data set is rationalizable by a (strongly or weakly) monotone utility function on the one hand and situations where the data set is rationalizable by a (strongly or weakly) monotone and concave utility function on the other hand. Also notice that in this setting, GARP does not necessarily correspond to any 'nice' rationalizability concept (although it is still a sufficient condition for rationalizability by a strictly monotone and concave utility function). Nevertheless, we also give the results for GARP as it is frequently used in the literature. As for the previous experiment, we see that the pass rates for WMARP (71%) and WMCARP (55%) are higher than for SMARP (43%) and SMCARP (40%), which is the lowest. Then, if we also take into account the power, we see that WMCARP has the highest predictive success (0.429) of all models and is closely followed by WMARP (0.415). These numbers are higher than the predictive success of the other two models (0.358 for SMARP and 0.333 for SMCARP). As such, similar to the experiment of Harbaugh, Krause, and Berry (2001), we see that the revealed preference tests which only impose weak monotonicity instead of strong monotonicity, give a better fit of the observed behavior in terms of higher predictive success.

Let us now have a look at the goodness–of–fit of the different tests in terms of the HM–measure. Figures 3 and 4 give the distribution of the HM–index for the different revealed preference tests for the two experiments. The black histogram gives the distribution of the HM–measure for the randomly generated data sets which

Table 1: Results for pass rate, power and predictive success

Data set		GARP	WMARP	SMARP	WMCARP	SMCARP
Harbaugh et al.	Pass rate	0.547	0.828	0.547	0.719	0.547
	Power	0.995	0.898	0.995	0.989	0.995
	Pred succ	0.542	0.726	0.542	0.708	0.542
Bruyneel et al.	Pass rate	0.40	0.71	0.43	0.55	0.40
	Power	0.969	0.705	0.928	0.879	0.933
	Predictive success	0.369	0.415	0.358	0.429	0.333

were also used for computing the power. As such, it actually gives the distribution of the HM–measure under the hypothesis of random behavior. The gray histogram, on the other hand, gives the distribution of the HM–measure for the actual data sets.

As can be seen, the distributions of the real data sets have much more mass at higher values of the HM–measure compared to the distribution generated by the random data sets. In this sense, it can be safely stated that the hypothesis of random behavior is rejected for all models under consideration. As expected, the distribution of the HM–measure for the weakest test (WMARP) is most skewed to the right, but so is the distribution of the HM-measure when WMARP is applied to the randomly generated data sets.

Figure 3: Distribution of HM–index for random and actual data for the data sets of Harbaugh et al.

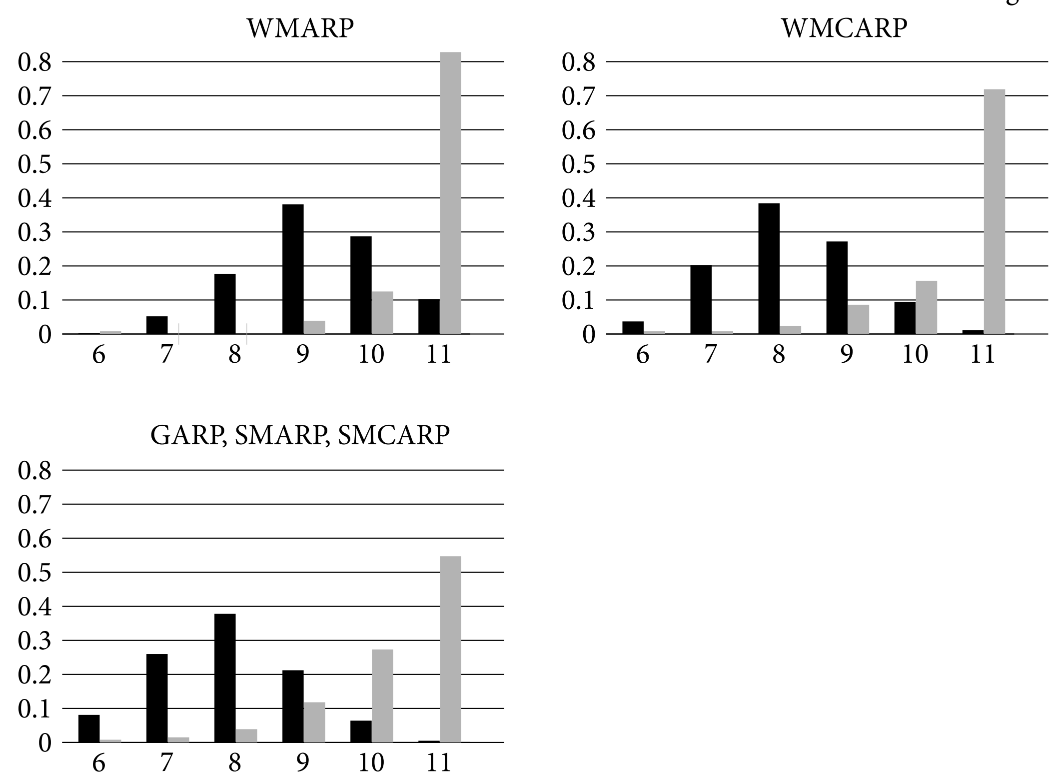
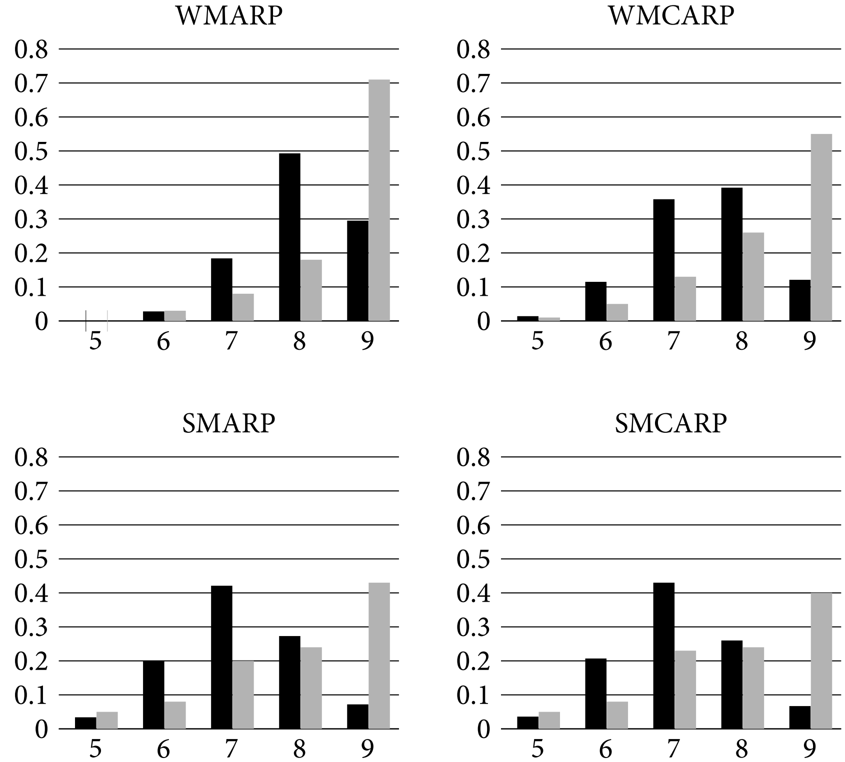


Figure 4: Distribution of HM–index for random and actual data for the data sets of Bruyneel et al.



What do we learn from all this? Our application has both a methodological and an empirical contribution. From a methodological point of view, we believe that our application demonstrates the usefulness of our revealed preference characterizations to deal with choice models when choice sets are finite. It also shows how to use pass rates, power, predictive success and goodness–of–fit (HM–measure) to compare the empirical fit of the different revealed preference tests.

Next at the empirical level, we have shown that rationalizability by a weakly monotone utility function may provide a better fit to actual observed behavior. In other words, the assumption that utility is increasing in the quantity of all goods is not supported by our findings. On the other hand, we found that all models perform considerably better than the model which is based on pure random behavior.

6 Conclusion

We developed a revealed preference analysis for situations where choices are made from a finite collection of bundles. This setting occurs in many real life and experimental settings.

First of all, we have shown that when choices are made from finite choice sets, then different rationalizability concepts will have different revealed preference restrictions. This makes it possible to test for various conditions on the utility function, like strong monotonicity or concavity. We have also shown how an existing goodness–of–fit measure (the Houtman and Maks measure) can easily be computed for our revealed preference tests.

Next, we have put forward a number of conditions for which our revealed preference conditions still coincide with the usual GARP condition. This result may be relevant for experimental researchers who do not wish to let their results depend on the specific conditions that are imposed on the utility function.

Finally, we applied our results using two experimental data sets that collect choices by children. We have shown that strong monotonicity may not be the best assumption to describe the actual choice behavior of children.

We see several avenues for follow up research. First of all, to focus our discussion, we have concentrated on testing rationalizability for basic conditions on the utility function, i.e. monotonicity and concavity. However, it is possible to obtain revealed preference conditions for even more stringent conditions on the utility function, like homotheticity or additivity (see Varian (1983) for such revealed preference conditions in the case of linear budget sets). A natural follow up research would be to derive the revealed preference conditions for such kind of utility functions when the choice sets are finite.

A second interesting subject for follow up research is the recovery or identification of the underlying preferences (or utility function) and to forecast behavior in new choice situations (see Varian (1982) for recovery in the linear budget set setting). As for the setting considered in the paper, recovery could proceed using the revealed preference relation as obtained from the definitions of WMARP and SMARP or the 'utility' values of φ_t^j as obtained from the definitions of WMCARP and SMCARP.

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A Proofs

Proof of Theorem 2

Assume that $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ is rationalizable by a weakly monotone utility function. Let us show that S satisfies WMARP.

We proceed by first defining the relation R as in the definition of WMARP and then we verify that this relation effectively satisfies the three conditions in the definition. Define $\mathbf{q}_t R \mathbf{q}_v$ if $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$. In order to see that R satisfies the first part in the definition of WMARP, assume that $b_t^k \geq \mathbf{q}_v$. Then obviously, by weak monotonicity, $u(\mathbf{b}_t^k) \geq u(\mathbf{q}_v)$. Next, as $\mathbf{q}_t$ was chosen from B_t, we also have that $u(\mathbf{q}_t) \geq u(\mathbf{b}_t^k)$. Therefore, $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$ and therefore $\mathbf{q}_t R \mathbf{q}_v$. The second part of the definition follows from the fact that if $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$ and $u(\mathbf{q}_v) \geq u(\mathbf{q}_s)$, then also $u(\mathbf{q}_t) \geq u(\mathbf{q}_s)$. Now, for the third part of the definition, let $\mathbf{q}_t R \mathbf{q}_v$ (i.e. $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$) and assume, towards a contradiction, that $\mathbf{b}_v^k \gg \mathbf{q}_t$. Then as $\mathbf{q}_v$ was chosen from B_v, we have that $u(\mathbf{q}_v) \geq u(\mathbf{b}_v^k)$. Next, from $\mathbf{b}_v^k \gg \mathbf{q}_t$ and weak monotonicity of the utility function, we have that $u(\mathbf{b}_v^k) > u(\mathbf{q}_t)$. A such, $u(\mathbf{q}_v) > u(\mathbf{q}_t)$. This contradicts with the assumption that $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$.

Now, assume that S satisfies WMARP. We need to show that it is also rationalizable by a weakly monotone utility function.

Consider the n-dimensional unit vectors,

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \ldots, 0); \\ \mathbf{e}_2 &= (0, 1, \ldots, 0); \\ &\ldots \\ \mathbf{e}_n &= (0, 0, \ldots, 1); \end{aligned}$$

Next, define the functions $a_t : \mathbb{R}_+^n \to \mathbb{R}$ in the following way:

$$a_t(\mathbf{q}) = \min_{k \leq K_t} \left(\max_i \mathbf{e}_i (\mathbf{q} - \mathbf{b}_t^k) \right)$$

This function satisfies the property that $a_t(\mathbf{q}) \leq 0$ if and only if there is a $k \leq K_t$ such that $\mathbf{q} \leq b_t^k$ and $a_t(\mathbf{q}) < 0$ if and only if there is a $k \leq K_t$ such that $\mathbf{q} \ll b_t^k$.

The function a_t is weakly monotone. Indeed if $\mathbf{q}' \geq (\gg) \mathbf{q}$, then $\max_i \mathbf{e}_i(\mathbf{q}' - \mathbf{b}_t^k) \geq (>) \max_i \mathbf{e}_i(\mathbf{q} - \mathbf{b}_t^k)$ for all $i = 1, \ldots, n$ and therefore, $a_t(\mathbf{q}') \geq (>) a_t(\mathbf{q})$. Next, $a_t(\mathbf{q})$ is also continuous as it is given by the the maximum of the minimum of continuous functions. We also have that for all $t \in T$, $a_t(\mathbf{q}_t) = 0$. Indeed, $a_t(\mathbf{q}_t) \leq 0$ because $\mathbf{q}_t \leq \mathbf{q}_t$. Now, if on the contrary $a_t(\mathbf{q}_t) \ll 0$ this would mean that there is a $k \leq K_t$ such that $\mathbf{q}_t \ll \mathbf{b}_t^k$, which would contradict WMARP.

Let $a_{t,v} = a_t(\mathbf{q}_v)$. We use the following definition of Cyclical Consistency.

Definition 7 (CC). *Consider a set of numbers $S = \{a_{t,v}\}_{t,v \in T}$. The set S is said to be cyclically consistent (CC) if there exist a binary relation W such that:*

1. *if $a_{t,v} \leq 0$, then tWv,*
2. *if tWv and vWw, then tWw,*
3. *if tWv then it is not the case that $a_{v,t} < 0$.*

Lemma 1. *The data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies WMARP if and only if $\{a_{t,v}\}_{t,v \in T}$ satisfies CC.*

Proof. Assume that S satisfies WMARP and let R be its revealed preference relation as defined in its definition. Assume that W is the relation as in the definition of CC. Let us first show that all three conditions of CC are satisfied if we take tWv if and only if $\mathbf{q}_t R \mathbf{q}_v$.

For the first, let $a_{t,v} \leq 0$. This means that $a_t(\mathbf{q}_v) \leq 0$ or equivalently $\mathbf{q}_v \leq \mathbf{b}_t^k$ for some $k \leq K_t$. However, by the first condition in the definition of WMARP, this implies that $\mathbf{q}_t R \mathbf{q}_v$ and, therefore, tWv. Hence, condition 1 in CC is satisfied.

The second condition in the definition of CC follows immediately from the second condition in the definition of WMARP.

For the third condition let tWv which implies $\mathbf{q}_t R \mathbf{q}_v$. This implies that for no $k \leq K_v$, $b_v^k \gg \mathbf{q}_t$. Assume on the contrary that $a_{v,t} < 0$. This implies that there is a $k \leq K_v$ such that $\mathbf{e}_i(\mathbf{q}_t - \mathbf{b}_t^k) < 0$ for all $i = 1, \ldots, n$. However, this implies that $\mathbf{q}_t \ll \mathbf{b}_t^k$ which contradicts with the requirement of WMARP.

The proof that CC implies WMARP can be shown along the same lines. □

Now, by a theorem of Fostel, Scarf, and Todd (2004) we have that CC is equivalent to the existence of numbers φ_t such that,

$$\varphi_t - \varphi_v \leq \lambda_v a_{v,t}.$$

Consider the function

$$u(\mathbf{q}) = \min_t \varphi_t + \lambda_t a_t(\mathbf{q})$$

This function is continuous (as it is the minimum of continuous functions), it is weakly monotone (because $a_t(.)$ is weakly monotone for all $t \in T$) and we have that for all $t \in T$, $u(\mathbf{q}_t) = \varphi_t$. In order to see this, notice that $u(\mathbf{q}_t) \leq \min_t \varphi_t + \lambda a_t(\mathbf{q}_t) = \varphi_t$. Now, if on the contrary $u(\mathbf{q}_t) > \varphi_t$, then there must be an observation $v \in T$ such that $\varphi_v + \lambda_v a_{v,t} > \varphi_t$, a contradiction.

Now, let us show that u rationalizes the data set. Assume, towards a contradiction that $u(\mathbf{b}_t^k) > u(\mathbf{q}_t)$ for some $k \leq K_t$. Then

$$\begin{aligned} u(\mathbf{b}_t^j) &= \min_v u_v + \lambda_v a_v(\mathbf{b}_t^j), \\ &\leq u_t + \lambda_v a_t(\mathbf{b}_t^j), \\ &\leq u_t. \end{aligned}$$

The last inequality comes from the fact that $\mathbf{b}_t^j \leq \mathbf{b}_t^j$, hence, $a_t(\mathbf{b}_t^k) \leq 0$.

A.1 Proof of Theorem 3

Assume that the data set $S = \{B_t, \mathbf{q}_t\}_{t\in T}$ is rationalizable by a strongly monotone utility function u. Let us show that S satisfies SMARP.

Define the binary relation R as $\mathbf{q}_t R \mathbf{q}_v$ if and only if $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$. To verify the first condition of the definition of SMARP, assume that $\mathbf{b}_t^k \geq \mathbf{q}_v$. Then from rationalizability $u(\mathbf{q}_t) \geq u(\mathbf{b}_t^k)$. Next, from monotonicity of the utility function, $u(\mathbf{b}_t^k) \geq u(\mathbf{q}_v)$. As such, we have that $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$ and therefore $\mathbf{q}_t R \mathbf{q}_v$.

To verify the second condition, assume that $\mathbf{q}_t R \mathbf{q}_v$ and $\mathbf{q}_v R \mathbf{q}_s$. Then we have that $u(\mathbf{q}_t) \geq u(\mathbf{q}_v) \geq u(\mathbf{q}_s)$, which gives $\mathbf{q}_t R \mathbf{q}_s$.

For the third condition, assume that $\mathbf{q}_t R \mathbf{q}_v$, which implies $u(\mathbf{q}_t) \geq u(\mathbf{q}_v)$. Now, if on the contrary $\mathbf{b}_v^j > \mathbf{q}_t$ for some $j \leq K_v$, then by strong monotonicity, $u(\mathbf{b}_v^j) > u(\mathbf{q}_t)$ and as such, $u(\mathbf{q}_v) \geq u(\mathbf{b}_v^j) > u(\mathbf{q}_t)$, which gives us a contradiction.

To see the reverse, let $S = \{B_t, \mathbf{q}_t\}_{t\in T}$ satisfy SMARP. Consider the vectors $\mathbf{e}_i$ such that

$$\begin{aligned}
\mathbf{e}_1 &= (1, \varepsilon, \dots, \varepsilon);\\
\mathbf{e}_2 &= (\varepsilon, 1, \dots, \varepsilon);\\
&\dots\\
\mathbf{e}_n &= (\varepsilon, \varepsilon, \dots, 1);
\end{aligned}$$

Here ε is a small but positive number.

Define the function a_t such that:

$$a_t(\mathbf{q}) = \min_{k \leq K_t} \left(\max_i \mathbf{e}_i(\mathbf{q} - \mathbf{b}_t^k) \right).$$

Now, it is easy to see that if there is a $k \leq K_t$ such that $\mathbf{q} \leq \mathbf{b}_t^k$, then $a_t(\mathbf{q}) \leq 0$. Also if there is a $v \in T$ such that $\mathbf{q}_v \not\leq \mathbf{b}_t^k$ for all $k \leq K_t$, we can set ε small enough such that $a_t(\mathbf{q}_v) > 0$. In other words, we can make ε small enough such that for all $v \in T$, $a_t(\mathbf{q}_v) \leq 0$ if and only if $\mathbf{q}_v \leq b_t^k$ for some $k \leq K_t$. Also, notice that $a_t(\mathbf{q}_t) = 0$ as otherwise $\mathbf{q}_t < \mathbf{b}_t^k$ for some $k \leq K_t$, which contradicts SMARP. Also, the function a_t is easily seen to be strongly monotone and continuous.

Lemma 2. *The set $\{a_{t,v}\}_{t,v}$ is cyclically consistent if and only if $\{B_t, \mathbf{q}_t\}_{t\in T}$ satisfies SMARP.*

Proof. Let $\{B_t, \mathbf{q}_t\}_{t\in T}$ satisfy SMARP and let R be the revealed preference relation. Define the relation W such that tWv if and only if $\mathbf{q}_t R \mathbf{q}_v$. Let us show that W satisfies the definition of CC.

First, let $a_{t,v} \leq 0$. This means that there is a $k \leq K_t$ such that $\mathbf{q}_v \leq \mathbf{b}_t^k$. However, this implies that $\mathbf{q}_t R \mathbf{q}_v$ and therefore tWv as was to be shown. The second condition follows from transitivity of the relation R.

For the third condition, let tWv which implies $\mathbf{q}_t R \mathbf{q}_v$. Now, if on the contrary $a_{v,t} < 0$, we know that there is a $k \leq K_t$ such that $\mathbf{q}_t \leq \mathbf{b}_v^k$. Now, if $\mathbf{q}_t = \mathbf{b}_v^k$, we have that $a_v(\mathbf{q}_t) = 0$, which is a contradiction. As such, it follows that $\mathbf{q}_t < \mathbf{b}_v^k$. However, this contradicts with SMARP. □

The remaining part of the proof is similar to that of Theorem 2.

Proof of Theorems 4 and 5

We only prove Theorem 4. The proof of Theorem 5 is very similar. The first part of the proof is established in the text. It is shown that the first condition implies the second. For the reverse, let us assume that the data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies WMCARP. Next, define the function

$$u(\mathbf{q}) = \min_{t \in T, k \leq K_t} \varphi_t^k + \mathfrak{p}_t^k(\mathbf{q} - \mathbf{b}_t^k).$$

This function is continuous, concave and weakly monotone. Let us show that it rationalizes the data. First of all, we show that $u(\mathbf{b}_t^k) = \varphi_t^k$. The inequality $u(\mathbf{b}_t^k) \leq \varphi_t^k$ follows simply from the definition of u. Now, if on the contrary $u(\mathbf{b}_t^k) < \varphi_t^k$, then there must exist an observation $v \in T$ and $j \leq K_v$ such that $\varphi_v^j + \mathfrak{p}_v^j(\mathbf{b}_t^k - \mathbf{b}_v^j) < \varphi_t^k$. This contradicts WMCARP.

Now in order to show that $u(.)$ rationalizes the data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ assume, towards a contradiction, that there is a $t \in T$ and $j \leq K_t$ such that $u(\mathbf{b}_t^k) > u(\mathbf{q}_t)$. Then if $\mathbf{q}_t = \mathbf{b}_t^j$ it follows that $\varphi_t^k > \varphi_t^j$. However, this contradicts with the second condition of WMCARP.

Proof of Theorem 6

Assume that n is the solution to **OP.HM-WMARP**. Consider the set $A = \{t | A_t = 1\}$, where A_t solves the program. Clearly, A has size n. We need to show that $\{B_t, \mathbf{q}_t\}_{t \in A}$ satisfies WMARP. Define $\mathbf{q}_t R \mathbf{q}_v$ if and only if $Z_{t,v} = 1$ and $t, v \in A$.

We proceed by verifying all three conditions of WMARP for the data set $\{B_t, \mathbf{q}_t\}_{t \in A}$. For the first, assume that $\mathbf{b}_t^k \geq \mathbf{q}_v$ for some $k \leq K_t$. Then by definition $x_{t,v} = 1$. If $A_t = 1$, the first constraint gives us that $Z_{t,v} = 1$, hence, $\mathbf{q}_t R \mathbf{q}_v$. For the second condition, assume that $\mathbf{q}_t R \mathbf{q}_v$ and $\mathbf{q}_v R \mathbf{q}_w$. Then the second constraint gives us that $Z_{t,v}$ and $Z_{v,w} = 1$. As such, $Z_{t,w} = 1$, which shows that $\mathbf{q}_t R \mathbf{q}_w$. Finally, for the third condition, if $\mathbf{q}_t R \mathbf{q}_v$, then $Z_{t,v} = 1$. The third constraint of the optimization problem then gives that we cannot have that both $A_v = 1$ and $y_{v,t} = 1$. As such, it is not the case that $v \in A$ and $b_v^j > \mathbf{q}_t$ for some $b_v^j \in B_v$. Therefore the third condition of WMARP is also satisfied.

For the other implication, assume that A is the (a) largest consistent subset of T that satisfies WMARP. It is easy to verify that setting $A_t = 1$ and $Z_{t,v} = 1$ if $\mathbf{q}_t R \mathbf{q}_v$ and $t, v \in A$ is a feasible solution to the optimization problem. Now, towards a contradiction, if the optimal value of the program is larger than $|A|$, then by the first part of the proof, there should exist a set W with $|W| > |A|$ such that $\{B_t, \mathbf{q}_t\}_{t \in W}$ satisfies WMARP. However, this violates the assumption that A is the largest of such sets.

Proof of Theorem 7

Assume that n is the solution to the above program and let A be the largest subset of T that still satisfies WMCARP. Let $A_t = 1$ if and only if $t \in A$. Then we see that all restrictions are satisfied, hence, we have that A_t is a feasible solution, hence $\sum_t A(t) \leq n$. On the other hand, assume towards a contradiction that $n > \sum_t A_t$ and let $\sum_t A_t^*$ be the optimal value of the program. Then define $Z = \{t | A_t^* = 1\}$. Then, we see that for all $t, v \in Z$

$$\mathbf{q}_t = \mathbf{b}_t^k \text{ then } \varphi_t^k \geq \varphi_t^j$$
$$\varphi_t^k - \varphi_v^j \leq \mathbf{p}_v^j(\mathbf{b}_t^k - \mathbf{b}_v^j)$$

As such, we see that Z satisfies WMCARP. This contradicts with the maximality of A.

Proof of Theorems 8 and 9

We only prove Theorem 8. The proof of Theorem 9 is very similar.

Assume that the data set $S = \{B_t, \mathbf{q}_t\}_{t \in T}$ satisfies Assumptions 1 and 2. Let us show that if S satisfies GARP, then S also satisfies SMARP. Let R be the revealed preference relation as given in the definition of GARP. We show that R also satisfies all conditions in the definition of SMARP. First, if $\mathbf{b}_t^k \geq \mathbf{q}_v$ for some

$t \in T$ and $k \leq K_t$, we have that $m_t = \mathbf{p}_t \mathbf{b}_t^k \geq \mathbf{p}_t \mathbf{q}_v$ and therefore $\mathbf{q}_t R \mathbf{q}_v$. The second condition of SMARP follows immediately from the second condition of GARP. Now, for the third condition, assume on the contrary that $\mathbf{q}_t R \mathbf{q}_v$ and $\mathbf{b}_v^j > \mathbf{q}_t$ for some $v \in T$ and $j \leq K_v$. But then, $\mathbf{p}_v \mathbf{q}_v = \mathbf{p}_v \mathbf{b}_v^j > \mathbf{p}_v \mathbf{q}_t$ which violates GARP, a contradiction. Conclude that S satisfies SMARP.

For the other implication, assume that S satisfies SMARP and let R be a revealed preference relation that satisfies the definition of SMARP. We show that R also satisfies the definition of GARP. For the first condition, assume that $\mathbf{p}_t \mathbf{q}_t \geq \mathbf{p}_t \mathbf{q}_v$. However, by Assumption 2, this implies that $\mathbf{b}_t^k \geq \mathbf{q}_v$ for some $k \leq K_t$. As such, $\mathbf{q}_t R \mathbf{q}_v$ as was to be shown. The second condition of GARP follows immediately from the second condition of GARP. For the third condition, let $\mathbf{q}_t R \mathbf{q}_v$ and assume on the contrary that $\mathbf{p}_v \mathbf{q}_v > \mathbf{p}_v \mathbf{q}_t$. From Assumption 2 this implies that $\mathbf{b}_v^j > \mathbf{q}_t$ for some $j \leq K_v$. However, this contradicts SMARP. As such, GARP must be satisfied.