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DISCUSSION PAPER SERIES
DPS13.10

MAY 2013

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Econometrics and Public Economics

Faculty of Economics
And Business

Prioritarian poverty comparisons with cardinal and ordinal attributes*

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May 25, 2013

Abstract. The ethical view of prioritarianism holds the following: if an extra bundle of attributes is to be allocated to either of two individuals, then priority should be given to the worse off among the two. We consider multidimensional poverty comparisons with cardinal and ordinal attributes and propose three axioms that operationalize the prioritarian view. Each priority axiom, in combination with a handful of standard properties, characterizes a class of poverty measures. We provide an empirical application to European Union Statistics on Income and Living Conditions data. For this application, we develop a unanimity criterion within the setting of a single cardinal attribute (income) augmented by several binary ordinal attributes.

Keywords. Multidimensional poverty measurement · Priority · Ordinal variables · Dominance

JEL classification. D31 · D63 · I31

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*We thank Koen Decancq, Jean-Yves Duclos, Tony Shorrocks, and the participants of the OPHI workshop in Oxford (University of Oxford) and the ProRESP conference in Wavre (CORE, Université catholique de Louvain) for fruitful discussion. The authors are responsible for remaining shortcomings.

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1 Introduction

"Benefiting people matters more the worse off these people are." This quote of Parfit (1997, p. 213) summarizes the ethical view of prioritarianism.[1] The view is straightforward to operationalize in the unidimensional setting of income distributions. Standard properties in unidimensional welfare and poverty measurement—with a central role for the Pigou-Dalton transfer principle—do the job (e.g., Fleurbaey, 2001, Tungodden, 2003, and Esposito and Lambert, 2011).[2] The implementation of prioritarianism is considerably more challenging in the multidimensional setting. In particular, the absence of a unique well-being indicator (such as income) complicates the identification of the worse off individuals to be prioritized. We consider the setting of multidimensional poverty comparisons and discuss three alternative axioms that operationalize the prioritarian view.

The weakest priority axiom is based on attribute dominance. Suppose a benefit—an extra bundle of attributes—can be given to either of two poor individuals. If one of the two individuals is worse off in each attribute, then she should receive the extra bundle according to the axiom. If not, then the axiom remains silent. We refer to this axiom as *dominance priority*. Dominance priority is in the spirit of the 'Pigou-Dalton bundle dominance' principle of Fleurbaey and Trannoy (2003).

The strongest priority axiom is based on the ranking of bundles by the poverty measure itself. Because comparing one-person distributions boils down to comparing single bundles, a poverty measure generates also a poverty ranking of individual bundles. Suppose again an extra bundle of attributes can be given to either of two poor individuals. This version of priority requires that the extra bundle goes to the poorer among the two individuals as judged by the poverty measure itself. We refer to this second axiom as *poverty priority*. Poverty priority is related to the 'consistent Pigou-Dalton principle' of Bosmans, Lauwers, and Ooghe (2009). Provided that the poverty measure is monotone in the attributes—an assumption maintained throughout the paper—poverty priority is stronger than (i.e., implies) dominance priority.

Figure 1 illustrates the dominance priority and poverty priority axioms. Individual 1's bundle dominates individual 2's bundle. Hence, dominance priority prescribes giving priority to individual 2 over individual 1. Given monotonicity, so does poverty priority. The depicted curve represents a level set of the poverty measure. Clearly, individual 4 is poorer than individual 3. Nonetheless, dominance priority does not recommend giving priority to individual 4 over individual 3, but remains silent. This disregard for the poverty measure's

[1] Parfit (1997, p. 214) presents prioritarianism as an alternative to egalitarianism. On the prioritarian view, the worse off should be prioritized "but that is only because these people are at a lower *absolute* level. It is irrelevant that these people are worse off *than others*. ... Egalitarians are concerned with *relativities*: with how each person's level compares with the level of other people."

[2] Esposito and Lambert (2011) stress that the distributional concern in unidimensional poverty measurement originates from a prioritarian rather than an egalitarian view. In his pioneering contribution, Watts (1968, p. 326) justifies this concern as follows: "poverty becomes more severe at an increasing rate as successive decrements of income are considered; in other words, ... poverty is reduced more by adding $500 to a family's command over goods and services if the family is at 50 percent of the poverty line than if it is at 75 percent." This justification is clearly prioritarian.

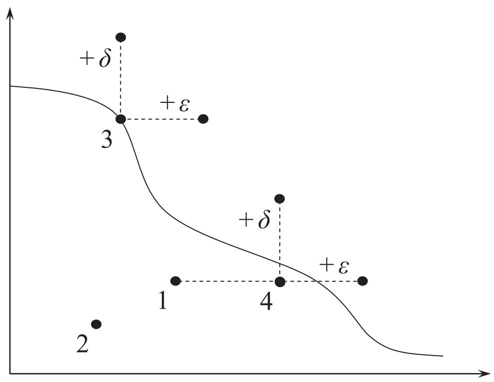

Figure 1. Who should receive the extra bundle?

own ranking of the individual bundles may be considered as a shortcoming. In contrast, poverty priority does respect this ranking and prescribes giving priority to individual 4 over individual 3.

However, it may be argued that the implications of the poverty priority axiom are too strong in some cases. Consider again the case where an extra bundle has to be allocated to either individual 3 or 4. Poverty priority recommends to allocate the extra bundle, δ or ε, to individual 4. However, individual 4 is already better endowed in terms of bundle ε than individual 3. If one takes into account possible diminishing returns to well-being—i.e., the possibly greater benefits for individual 3 of obtaining the bundle ε—then it is not clear that individual 4 should receive the bundle. As Parfit (1997, p. 213) puts it, "Benefits to the worse off should be given more weight. This priority is not, however, absolute. ... benefits to the worse off could be morally outweighted by sufficiently great benefits to the better off."

This motivates a third priority axiom that strengthens dominance priority, but, contrary to poverty priority, respects diminishing returns to well-being. If the poorer individual is better endowed in terms of the extra bundle, then this third priority axiom—in contrast to poverty priority—remains silent. If the poorer individual is not better endowed in terms of the extra bundle, then this third axiom follows poverty priority. Applied to individuals 3 and 4 in Figure 1, this axiom recommends to allocate the bundle δ to individual 4 and remains silent about the allocation of bundle ε. We refer to this third axiom as *bundle-dependent priority*. Bundle-dependent priority is intermediate in strength between dominance priority and poverty priority.

In the above, we have implicitly assumed attributes to be cardinal. Ordinal attributes require special treatment. Priority axioms express the idea that the *same* increase in attributes is more valuable if the worse off experiences it. But if, say, the bundle δ in Figure 1 concerns an ordinal attribute, then it is meaningless to state that the potential increase is the same for individuals 3 and 4. This statement is meaningful only if the initial value of the ordinal attribute is equal for the two individuals, such as in the allocation of bundle δ to individual 1 or 4. Hence, in each of the three priority axioms, we will impose this additional condition for bundles containing ordinal attributes.

Our main result characterizes a class of poverty measures using the bundle-dependent priority axiom in addition to a handful of standard axioms (see Section 5, which also discusses two further characterizations based on dominance priority and poverty priority). Let (c^i, o^i) be the attribute bundle of individual i, with c^i the k-vector listing the values of the cardinal attributes and o^i the ℓ-vector listing the values of the ordinal attributes. Poverty in a population of size n is measured by the average poverty level

$$\frac{1}{n}\sum_i \pi(c^i, o^i),$$

where the poverty level of individual i is

$$\pi(c^i, o^i) \;=\; f\big(\; \underbrace{g_1(c^i_1) + g_2(c^i_2) + \cdots + g_k(c^i_k)}_{\text{cardinal}} \;+\; \underbrace{h_1(o^i_1) + h_2(o^i_2) + \cdots + h_\ell(o^i_\ell)}_{\text{ordinal}} \;\big)$$

with f a decreasing and convex function, g_j increasing and concave, and h_j increasing. The different properties of the functions g_j and h_j reflect the different treatment of cardinal versus ordinal attributes. The separability of the individual poverty measure π is not imposed from the outset, but rather is obtained as a consequence of the bundle-dependent priority axiom in combination with the other axioms.

This result encompasses the two main approaches in the literature. Atkinson (2003) refers to these approaches as the 'social welfare approach' and the 'counting approach'. The social welfare approach deals exclusively with cardinal attributes and extends concepts of unidimensional social welfare and poverty measurement.[3] The counting approach deals exclusively with ordinal—usually binary—attributes and focuses on counting the number of dimensions in which an individual is deprived.[4] Our class of measures deals with cardinal and ordinal attributes jointly. Yet, it has much in common with the social welfare and counting approaches. For example, each member of the class satisfies uniform majorization (Kolm, 1977) and correlation increasing majorization (Atkinson and Bourguignon, 1982, and Tsui, 1999). Therefore, these two principles receive a new ethical underpinning using the bundle-dependent priority axiom.

We illustrate our approach with an empirical application. The application deals with the important case of a single cardinal attribute (income) augmented by several binary ordinal attributes. For this setting, we develop a unanimity criterion based on the characterized class of poverty measures. This unanimity criterion is applied in a cross-country

[3]See, e.g., Tsui (2002), Bourguignon and Chakravarty (2003), Duclos, Sahn, and Younger (2006), Chakravarty and Silber (2008), and Alkire and Foster (2011). Chakravarty (2009, Chapter 6) provides a survey. Related are studies dedicated to the assessment of poverty over time, e.g., Ligon and Schechter (2003) and Bossert, Chakravarty, and D'Ambrosio (2008). This framework is also exclusively cardinal because it deals with bundles of incomes, one income per period.

[4]See, e.g., Lasso de la Vega (2010), and Aaberge and Peluso (2011), and Bossert, Chakravarty, and D'Ambrosio (2013).

poverty comparison using European Union Statistics on Income and Living Conditions (EU-SILC) data on income, health, education, work, environment, physical security, and financial security.

The next section introduces the notation. Section 3 presents the identification criterion and discusses the axioms of representation, focus, and monotonicity. Section 4 develops the three priority axioms. Section 5 presents and discusses the main result. Section 6 concludes with the empirical illustration of the unanimity criterion using EU-SILC data.

2 Notation

A population is a finite set of individuals. Each individual is endowed with a bundle of attributes. An attribute bundle is a vector $x = (x_k)_{k \in K}$ of real numbers with K a finite set of at least three attributes and x_k the value of attribute k. The set K of attributes partitions as $C \cup O$ with C the set of cardinal and O the set of ordinal attributes. Let $B_C = \mathbb{R}^{|C|}$ and $B_O = \mathbb{R}^{|O|}$. Both sets allow for continuous or discrete variables, depending on the application. Our application in Section 6 combines one continuous cardinal attribute, income, with several binary ordinal attributes reflecting among others health and education. Each bundle x decomposes as (x_C, x_O) with $x_C = (x_k)_{k \in C}$ in B_C and $x_O = (x_k)_{k \in O}$ in B_O. The set $B = B_C \times B_O$ collects all possible bundles. The zero-bundles in B_C, B_O, and B are denoted by 0. For two bundles x and y in B, we write $x \geq y$ if $x_k \geq y_k$ for each k in K, and $x > y$ if $x \geq y$ and $x \neq y$. Let $x \circ y$ denote the attribute-wise product of two bundles x and y in B, i.e., $x \circ y = (x_k y_k)_{k \in K}$.

A set of individuals endowed with a bundle is said to be a distribution. A distribution is fully described by $X = (x^1, x^2, \ldots, x^n)$ with n the number of individuals in the population and x^i in B the bundle of individual $i = 1, 2, \ldots, n$.[5] The domain

$$D = \{(x^1, x^2, \ldots, x^n) \,|\, n \in \mathbb{N} \text{ and } x^i \in B \text{ for each } i = 1, 2, \ldots, n\}$$

collects all possible distributions. We do not make a distinction between one-person distributions and bundles, i.e., for each bundle x in B, we identify the distribution (x) with x.

A poverty ordering on D is a complete and transitive binary relation in D and is denoted by $\succsim$. We read $X \succsim Y$ as distribution X is at least as good as distribution Y, or equivalently, poverty in X is at most as high as in Y. The asymmetric and symmetric components of $\succsim$ are denoted by $\succ$ and $\sim$.

3 Identification and three axioms

The first step in assessing poverty consists of identifying the poor. In order to determine who is poor and who is not, individual attribute bundles are compared to the poverty thresholds (i.e., the minimally acceptable levels) for the different attributes. For each

[5]Below we require the poverty ranking to be anonymous. Therefore, we can use the same labels $1, 2, \ldots$ for individuals across different populations.

attribute k in K, let z_k denote the poverty threshold. The vector z in B lists the poverty thresholds for all attributes and is referred to as the poverty bundle. An individual with bundle x in B is said to be deprived in dimension k if $x_k < z_k$.

Typically there exist individuals who are deprived in some dimensions and non-deprived in others. Hence, the identification of the poor depends on the trade-off between the different dimensions. We require only that the trade-off is consistent with how the poverty ordering $\succsim$ on D ranks one-person distributions. That is, an individual with bundle x in B is said to be poor if $z \succ x$ and non-poor if $x \succsim z$. The set of poor bundles P and the set of non-poor bundles R are defined as

$$P \;=\; \{\, x \,|\, x \in B \text{ and } z \succ x \,\} \quad \text{and} \quad R \;=\; \{\, x \,|\, x \in B \text{ and } x \succsim z \,\}.$$

Identification implies that the poverty bundle z extends to a poverty frontier, i.e., the set of bundles that are equally good as the poverty bundle z (see Duclos, Sahn, and Younger, 2006). To sum up, the poverty bundle is exogenous, whereas the poverty frontier through the poverty bundle follows from the poverty ordering.

We now define the axioms of representation, focus, and monotonicity. The next section develops different versions of the priority axiom.

Representation requires that poverty in a distribution can be judged by its average individual poverty level.

Representation. There exists a continuous function $\pi : B \to \mathbb{R}$ such that, for all distributions $X = (x^1, x^2, \ldots, x^n)$ and $Y = (y^1, y^2, \ldots, y^m)$, we have

$$X \succsim Y \quad \text{if and only if} \quad \frac{1}{n}\sum_{i=1}^{n} \pi(x^i) \;\leq\; \frac{1}{m}\sum_{j=1}^{m} \pi(y^j). \tag{1}$$

The function π can be interpreted as a measure of poverty at the individual level. The axiom of representation combines four properties (Tsui, 2002): continuity (small changes in the attribute bundles do not cause large changes in the poverty ranking), anonymity (the names of the individuals do not matter), subgroup consistency (overall poverty increases if poverty increases in a subgroup of the population and remains the same in the complement of this subgroup)[6], and replication invariance (overall poverty does not change if the distribution is replicated).

Focus requires that two distributions are judged as equally poor if the bundles of the poor are the same. Equivalently, replacing a non-poor bundle by the poverty bundle z generates a distribution that is equally good. The focus axiom ensures that the 'focus' is solely on the poor individuals.

Focus. Let $X = (x^1, \ldots, x^{i-1}, x^i, x^{i+1}, \ldots, x^n)$ be a distribution. Let x^i be a non-poor bundle. Then we have $X \sim (x^1, \ldots, x^{i-1}, z, x^{i+1}, \ldots, x^n)$.

[6]Many authors consider the separability between individuals implied by subgroup consistency as essential to prioritarianism. The same goes for the monotonicity axiom presented below. See, e.g., Fleurbaey (2001), Tungodden (2003), and Esposito and Lambert (2011).

The focus axiom—as well as several of the axioms discussed next—makes the ordering $\succsim$ dependent on the poverty bundle z. In the notation we suppress this dependency and write $\succsim$ instead of $\succsim_z$. The imposition of representation and focus implies that we have $\pi(x) = \pi(z)$ for each non-poor bundle x.

Monotonicity demands that poverty decreases if a poor individual receives an additional amount of an attribute.

Monotonicity. Let $X = (x^1, \ldots, x^{i-1}, x^i, x^{i+1}, \ldots, x^n)$ be a distribution. Let $z \succ x^i$. Let $\varepsilon > 0$ be a bundle in B. Then we have $(x^1, \ldots, x^{i-1}, x^i + \varepsilon, x^{i+1}, \ldots, x^n) \succ X$.

An implication of monotonicity is that the poverty ordering is sensitive to both the deprived and the non-deprived attributes of a poor individual. Monotonicity, however, does not prevent giving more weight to changes in deprived attributes than to changes in non-deprived attributes.[7] The combination of representation, focus, and monotonicity implies that the map π, restricted to the set P of poor bundles, is strictly decreasing in each attribute.

4 Priority axioms

Consider a distribution with at least two poor individuals. Assume an indivisible non-negative bundle becomes available. Priority axioms answer the question to which poor individual this extra bundle should be allocated. We distinguish cardinal and ordinal priority axioms, depending on whether the extra bundle—denoted by ε—includes only cardinal or only ordinal attributes.

Let the bundle $\varepsilon = (\varepsilon_C, 0) > 0$ include only cardinal attributes. Let x and y be two bundles such that $x \leq y$. *Cardinal dominance priority* recommends to allocate the extra bundle ε to the individual endowed with bundle x.

Cardinal dominance priority. Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (\varepsilon_C, 0) > 0$ be a bundle in $B_C \times B_O$. If $x^i \geq x^j$, then

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Cardinal poverty priority relies on the ordering $\succsim$ (restricted to one-person distributions) and recommends to allocate the extra bundle ε to the poorer individual.

Cardinal poverty priority. Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (\varepsilon_C, 0) > 0$ be a bundle in $B_C \times B_O$. If $x^i \succsim x^j$, then

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

If the poverty ordering $\succsim$ satisfies monotonicity, then $x \geq y$ implies $x \succsim y$. As a consequence, monotonicity and cardinal poverty priority entail cardinal dominance priority.

[7] Tsui (2002) and Bourguignon and Chakravarty (2003) require poverty to be invariant under changes in non-deprived attributes. In contrast, monotonicity interprets an increase in a non-deprived attribute as a (possibly small) improvement.

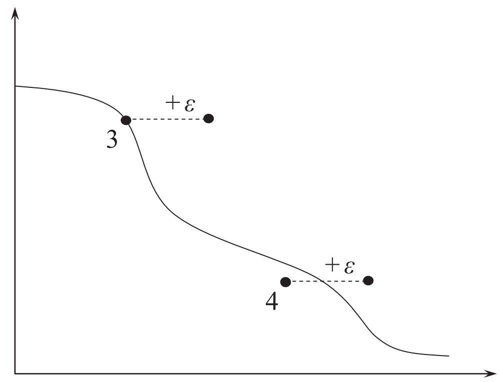

Figure 2. Who should receive the extra bundle?

Cardinal poverty priority is a rather demanding ethical requirement. Consider Figure 2, which repeats the example of the introduction. According to the poverty ordering (of which an isoline is depicted), individual 4 is poorer than individual 3. By consequence, cardinal poverty priority recommends giving the extra bundle ε to individual 4. However, individual 4 already has more than individual 3 of the attribute in ε. Bundle ε better complements the bundle of 3 than the bundle of 4. Therefore, bundle ε possibly entails greater benefits for individual 3. Cardinal poverty priority disregards these possibly greater benefits for the better off. This goes much beyond prioritarianism, which allows "benefits to the worse off [to] be morally outweighted by sufficiently great benefits to the better off" (Parfit, 1997, p. 213).

We formulate a third version of priority, *cardinal (bundle-dependent) priority*. It gives priority to bundle x over bundle y if, in addition to $x \succsim y$, bundle x contains at least as much as bundle y of each attribute for which ε is not zero, i.e., if the attribute-wise product $x^i \circ \varepsilon$ is at least as great as $x^j \circ \varepsilon$.

Cardinal priority. Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (\varepsilon_C, 0) > 0$ be a bundle in $B_C \times B_O$. If $x^i \succsim x^j$ and $x^i \circ \varepsilon \geq x^j \circ \varepsilon$, then

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Given monotonicity, this version is intermediate in strength between cardinal dominance priority and cardinal poverty priority.

Now, let the extra bundle $\varepsilon = (0, \varepsilon_O)$ include only ordinal attributes. The meaning of an increase in an ordinal attribute depends on the amount of the attribute already present. For example, it is not meaningful to say that an increase of an ordinal attribute from 2 to 3 is the same improvement as an increase from 4 to 5. But if two individuals both have an initial endowment of 2, then an increase to 3 does constitute the same improvement. We therefore impose the condition that the two individuals should have the same initial values of the ordinal attributes in bundle ε. We define three versions of ordinal priority.

Ordinal dominance priority. Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (0, \varepsilon_O) > 0$ be a bundle in $B_C \times B_O$. If $x^i \geq x^j$ and $x^i \circ \varepsilon = x^j \circ \varepsilon$, then

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Ordinal poverty priority. Let $X = (x^1, x^2, \ldots, x^n)$ be a distribution. Let $\varepsilon = (0, \varepsilon_O) > 0$ be a bundle in $B_C \times B_O$. If $x^i \succsim x^j$ and $x^i \circ \varepsilon = x^j \circ \varepsilon$, then

$$(x^1, \ldots, x^i, \ldots, x^j + \varepsilon, \ldots, x^n) \succsim (x^1, \ldots, x^i + \varepsilon, \ldots, x^j, \ldots, x^n).$$

Due to the restriction on the initial values, ordinal (bundle-dependent) priority coincides with ordinal poverty priority.

Ordinal priority. Coincides with ordinal poverty priority.

Table 1. Priority axioms

	Poverty	(Bundle-dependent)	Dominance
Cardinal	$x^i \succsim x^j$	$x^i \succsim x^j$	$x^i \geq x^j$
$\varepsilon = (\varepsilon_C, 0) > 0$		$x^i \circ \varepsilon \geq x^j \circ \varepsilon$	
Ordinal	$x^i \succsim x^j$	$x^i \succsim x^j$	$x^i \geq x^j$
$\varepsilon = (0, \varepsilon_O) > 0$	$x^i \circ \varepsilon = x^j \circ \varepsilon$	$x^i \circ \varepsilon = x^j \circ \varepsilon$	$x^i \circ \varepsilon = x^j \circ \varepsilon$

Table 1 summarizes the different priority axioms. In each of the six cases the corresponding priority axiom recommends to allocate the extra bundle to individual j. Because ordinal (bundle-dependent) priority and ordinal poverty priority coincide, the corresponding entries in the table also coincide.

We now combine the different priority axioms to obtain three final versions of priority. Each version deals with both cardinal and ordinal attributes, but differs in assigning priority.

Dominance priority. Cardinal dominance priority and ordinal dominance priority hold.

Poverty priority. Cardinal poverty priority and ordinal poverty priority hold.

Priority. Cardinal priority and ordinal priority hold.

Given monotonicity, the priority axioms are logically connected. A monotonic poverty ordering that satisfies poverty priority also satisfies priority. And a monotonic poverty ordering that satisfies priority also satisfies dominance priority.

5 Main result

Our main result characterizes poverty orderings that satisfy representation, focus, monotonicity, and priority. See the appendix for the proof.

Theorem. *A poverty ordering $\succsim$ on D with poverty bundle z in B satisfies representation, focus, monotonicity, and priority if and only if there exist*

- *strictly increasing and concave functions $g_k : \mathbb{R} \to \mathbb{R}$,*
- *strictly increasing functions $h_\ell : \mathbb{R} \to \mathbb{R}$,*
- *a decreasing and convex continuous function $f : \mathbb{R} \to \mathbb{R}$ with*
  - *$f(r) = 0$ for each $r \geq \zeta = \sum_C g_k(z_k) + \sum_O h_\ell(z_\ell)$, and*
  - *f strictly decreasing on the interval $(-\infty, \zeta]$,*

such that, for all distributions X and Y in D, we have $X \succsim Y$ if and only if

$$\frac{1}{n}\sum_{i=1}^{n} f\Big(\sum\nolimits_C g_k(x_k^i) + \sum\nolimits_O h_\ell(x_\ell^i)\Big) \;\leq\; \frac{1}{m}\sum_{j=1}^{m} f\Big(\sum\nolimits_C g_k(y_k^j) + \sum\nolimits_O h_\ell(y_\ell^j)\Big). \quad (2)$$

The four axioms together impose a strong structure on the poverty ordering. In particular, the measure of individual poverty $\pi = f(\sum g_k + \sum h_\ell)$ must be separable in each attribute. This property is common in the social welfare approach, which deals only with cardinal attributes (e.g., Bourguignon and Chakravarty, 2003, Tsui, 2003, and Duclos, Sahn, and Younger, 2006). In Section 6, we consider binary ordinal attributes and show that the sum $\sum h_\ell$ reduces to a weighted count. As such, the above poverty ordering accommodates both the social welfare and the counting approaches.

The sum $\sum g_k + \sum h_\ell$ is a measure of individual well-being, where the functions g_k and h_ℓ determine the trade-offs between the attributes. The decreasing function f maps individual well-being levels to individual poverty levels. The curvature of f determines the reduction of poverty for a given increase in individual well-being.[8] For a linear f, this reduction does not depend on the initial level of well-being of the poor individual. By increasing the curvature of f, the reduction of poverty increases as the initial level well-being decreases. 'Absolute priority' (lexicographically) to the worst off can be approached arbitrarily closely.[9]

The poverty bundle z and the sum $\sum g_k + \sum h_\ell$ specify the shape of the poverty frontier. Note that if z goes to infinity, then the focus axiom loses its power and the poverty ordering becomes a welfare ordering.

[8]Criteria to compare unidimensional poverty measures on the basis of distribution-sensitivity fully apply here (Zheng, 2000, Bosmans, 2012).

[9]This follows from Theorem 4.4 in Lambert (2001).

We now show that the above poverty ordering satisfies the prominent uniform majorization and correlation increasing majorization principles. Uniform majorization presupposes a setting where each individual has the same ordinal bundle. The principle demands that post-multiplying the distribution of cardinal bundles by a non-permutation bistochastic matrix does not increase poverty.[10] We decompose a distribution X as (X_C, X_O) with X_C the matrix $(x_C^1, x_C^2, \ldots, x_C^n)$ and $X_O = (x_O^1, x_O^2, \ldots, x_O^n)$.

Uniform majorization. Let $X = (X_C, X_O)$ be a distribution with $x_O^1 = x_O^2 = \cdots = x_O^n$. Let M be a non-permutation bistochastic matrix. Then, $(X_C M, X_O) \succsim (X_C, X_O)$.

The poverty ordering in the theorem satisfies uniform majorization. Tsui (2002, Proposition 3) shows that convexity of the function π (in the cardinal attributes) is a necessary and sufficient condition for uniform majorization. The concavity of the functions g_k and the decreasingness and convexity of f indeed imply $\pi(\alpha x_C + (1-\alpha) y_C, x_O) \leq \alpha\pi(x_C, x_O) + (1-\alpha)\pi(y_C, x_O)$ for all bundles x_C and x_C in B_C, each bundle x_O in B_O, and each α in the interval $[0, 1]$.

Correlation increasing majorization requires that switching attributes between two individuals until one individual has more of each attribute than the other does not decrease poverty. Correlation increasing majorization applies to both cardinal and ordinal attributes. Consider two bundles x and y. Let $x \wedge y$ be the bundle $(\min\{x_k, y_k\})_{k\in K}$ and let $x \vee y$ be $(\max\{x_k, y_k\})_{k\in K}$. Note that $x + y = (x \wedge y) + (x \vee y)$.

Correlation increasing majorization. Let $X = (x^1, \ldots, x^i, \ldots, x^j, \ldots, x^n)$ be a distribution. Then, $X \succsim (x^1, \ldots, x^i \vee x^j, \ldots, x^i \wedge x^j, \ldots, x^n)$.

Dominance priority, and thus also the stronger versions of priority, imply correlation increasing majorization. To see this, consider a distribution X and two individuals i and j. Construct a distribution Y from X such that $y^j = x^i \wedge x^j$ and $y^k = x^k$ for each individual $k \neq j$. We have that $y^i \geq y^j$. Define $\varepsilon = x^j - y^j = x^j - (x^i \wedge x^j) \geq 0$ and verify that $y^i \circ \varepsilon = y^j \circ \varepsilon$ holds. Dominance priority implies

$$(\ldots, y^i, \ldots, y^j + \varepsilon, \ldots) \succsim (\ldots, y^i + \varepsilon, \ldots, y^j, \ldots),$$

or equivalently,

$$(\ldots, x^i, \ldots, x^j, \ldots) \succsim (\ldots, x^i \vee x^j, \ldots, x^i \wedge x^j, \ldots),$$

as required.

We close this section by indicating how the main result changes if priority is weakened to dominance priority, or is strengthened to poverty priority. The combination of representation, focus, monotonicity, and dominance priority imposes that π has non-increasing increments, i.e.,

$$\pi(y) - \pi(y + \varepsilon) \geq \pi(x) - \pi(x + \varepsilon)$$

[10] A bistochastic matrix is a non-negative square matrix each row and each column of which sums to one. A permutation matrix is a bistochastic matrix that only contains zeros and ones.

for all bundles x and y in P with $y \leq x$ and for each bundle ε in B with $x_O \circ \varepsilon_O = y_O \circ \varepsilon_O$. To see this, start from expression (1) and apply dominance priority to the two-person distribution (x, y). Clearly, dominance priority imposes little structure on π.

Strengthening priority to poverty priority implies that the functions g_k in the theorem are linear. The individual poverty measure becomes

$$\pi(x) \;=\; f\Big(\sum\nolimits_C w_k\, x_k + \sum\nolimits_O h_\ell(x_\ell)\Big)$$

with $w_k > 0$ a weight for each attribute k in C. To see this, start from expression (2). Consider two bundles x and y in P such that $\pi(x) = \pi(y)$ and a cardinal attribute k. Let ε be a non-negative bundle in B with $\varepsilon_\ell = 0$ for each attribute $\ell \neq k$. Cardinal poverty priority imposes $\pi(x + \varepsilon) = \pi(y + \varepsilon)$. Hence, the gain from x to $x + \varepsilon$ is equal to the gain from y to $y + \varepsilon$, i.e.,

$$g_k(x_k + \varepsilon_k) - g_k(x_k) \;=\; g_k(y_k + \varepsilon_k) - g_k(y_k).$$

Because the slope $(g_k(x_k + \varepsilon_k) - g_k(x_k))/\varepsilon_k$ does not depend on x_k, linearity follows. This result emphasizes the demandingness of poverty priority. Poverty priority requires efficiency losses to be accepted as long as the worse off benefits. This turns out to be possible only by completely removing efficiency considerations, viz., by imposing perfect substitutability of the cardinal attributes.[11]

6 Empirical illustration

6.1 A dominance criterion

The empirical application involves data on income levels and ℓ non-material binary ordinal attributes for 26 European countries. In this context, an attribute bundle is a $(1+\ell)$-tuple (y, t) with y in $\mathbb{R}$ the income level (cardinal) and t in $T = \{0, 1\}^\ell$ the vector listing the values of the ℓ ordinal attributes. We will refer to T as the set of types. The individual poverty measure—obtained in expression (2) and restricted to the domain $\mathbb{R} \times T$—is

$$\pi(y, t) \;=\; f(g(y) + \alpha \cdot t)$$

with f and g as before, α in $\mathbb{R}^\ell_+$, and $\cdot$ the vector product. In the notation of expression (2), the value $h_i(1)$ coincides with $\alpha_i = \alpha \cdot (0, \ldots, 1, \ldots, 0)$. Let $\iota = (1, 1, \ldots, 1)$ denote the highest type and let (z_ι, ι) be the poverty bundle for this type. The poverty frontier collects the bundles (z_t, t), one for each type t in T, that are equally poor as (z_ι, ι). Hence, for each type t, the income level z_t satisfies $g(z_t) + \alpha \cdot t = g(z_\iota) + \alpha \cdot \iota = g(z_\iota) + \sum_{i \in T} \alpha_i$. Let $(p, F) = (p_t, F_t)_{t \in T}$ be the empirical distribution of incomes and types of a country. For each type t in T, the number p_t denotes the fraction of individuals of type t and F_t denotes the corresponding cumulative distribution of income. These income distributions

[11] An alternative would be to focus lexicographically on the worst off. But this solution is excluded by continuity.

have finite support: within each type the lowest and the highest income are finite numbers. Aggregate poverty in country (p, F) can then be rewritten as

$$\Pi_{f,g,\alpha,z_\iota}(p, F) \;=\; \sum_{t\in T} p_t \int\limits_{y=-\infty}^{z_t} f(g(y) + \alpha \cdot t)\, dF_t(y).$$

Poverty within type t corresponds to incomes below z_t and is measured by the integral. The weighted sum of these integrals measures poverty in the total population.

In order to compare two countries we develop the following robust, but incomplete, unanimity criterion.

Unanimity criterion. Country (p, F) dominates country (q, G) up to level z^* if

$$\Pi_{f,g,\alpha,z_\iota}(p, F) \;\leq\; \Pi_{f,g,\alpha,z_\iota}(q, G)$$

for each

- income poverty level $z_\iota \leq z^*$,
- vector $\alpha \geq 0$, and
- pair (f, g) of C^2-maps with $f' < 0$, $f'' \geq 0$, $g' > 0$, and $g'' \leq 0$.[12]

The next proposition provides an equivalent criterion based on the average (transformed) poverty gap. See the appendix for the proof.

Proposition. *Country (p, F) dominates country (q, G) up to level z^* if and only if*

$$\sum_{t\in T} \int\limits_{y=-\infty}^{s_t} (g(s_t) - g(y))\, \{p_t dF_t(y) - q_t dG_t(y)\} \;\leq\; 0 \qquad (3)$$

for each

- $s_t = g^{-1}(s - \alpha \cdot t)$ *with* $s \leq g(z_\iota) + \alpha \cdot \iota$ *and* $z_\iota \leq z^*$,
- *vector* $\alpha \geq 0$, *and*
- C^2*-map* g *with* $g' > 0$ *and* $g'' \leq 0$.

In comparing two countries we use the proposition to compute the maximal poverty line z^* for which one country dominates the other. We let z^* run from €0 to €3000 in steps of €100. For each z^*, we use a grid and test the null hypothesis (confidence level 0.95) of no dominance against dominance following Kaur, Prakasa Roa, and Singh (1994). The grid

[12] We strengthen continuity of the poverty measure to differentiability.

points $(s_t = g^{-1}(s - \alpha \cdot t))_{t \in T}$ are calculated for specific choices of α, g, and s. The vector α runs over the zero vector and 99 vectors with attribute weights that are independently and identically drawn from the uniform distribution over the interval $[0, 500]$. The map g runs over two linear maps ($x \mapsto ax$ with $a = 1$ and $a \to 0$), and a piece-wise linear combination of these two linear maps ($x \mapsto ax$, $a = 1$, if $x < 1000$, and $x \mapsto 1000 + ax$, $a \to 0$, if $x \geq 1000$). Income s is set equal to $g(z_\iota) + \alpha \cdot \iota$, and z_ι runs from 0 to z^* in steps of €100. In case α is the zero vector and g is the identity map, the dominance test is based only on the income distribution. As a consequence, multidimensional poverty dominance implies income poverty dominance. The maximal poverty line does not increase if ordinal attributes are also included. In case g is arbitrarily close to zero, the dominance test is based only on the ordinal attributes.

6.2 The data

The application involves the cardinal attribute income and six binary ordinal attributes. We use data from the 2007 European Union Statistics on Income and Living Conditions (EU-SILC).[13] EU-SILC covers 24 EU member states (all 2006 EU members except Malta) as well as Norway and Iceland.

Income is defined as the total disposable household income in 2006, expressed per month, divided by the modified OECD equivalence scale, and adjusted for price differences using the ICP (2008) price index.[14] The problem of contaminated data can be severe, especially for testing second-order dominance in the tails of the distribution (see Cowell and Victoria-Feser, 2002, 2006). We therefore apply bottom and top coding for incomes below $\max\{0, Q_1 - 1.5(Q_3 - Q_1)\}$ and above $Q_3 + 1.5(Q_3 - Q_1)$ with Q_1 and Q_3 the lower and upper quartile in the country.

We now focus on the ordinal attributes. The Stiglitz-Sen-Fitoussi (2009) report identifies the following key non-material dimensions: health, education, personal activities including work, political voice and governance, social connections and relationships, environment, and insecurity (of an economic and physical nature). Six of these dimensions are available in the EU-SILC data and we transform them into binary attributes:

- health = 0 if the individual reports strong limitations in daily activities because of health problems for at least the last six months, and health = 1 otherwise,
- education = 0 if a degree of secondary education is not obtained by the individual, and education = 1 otherwise,[15]
- job = 0 if the individual indicates that (s)he would like to work but either cannot find a job or cannot work due to disability, and job = 1 otherwise,

[13]Data are obtained under contract EU-SILC/2006/28 - project 4.

[14]Total disposable household income is the sum across all household members of gross personal income components *plus* gross income components at the household level *minus* taxes and social insurance contributions.

[15]We only select individuals aged 18 years and older.

Table 2. Mean statistics for the cardinal and ordinal attributes

	Income	Health	Education	Job	Environmt	Phys. sec.	Fin. sec.
Austria (AT)	1740.49	0.90	0.77	0.97	0.93	0.89	0.75
Belgium (BE)	1569.65	0.93	0.67	0.93	0.82	0.83	0.80
Cyprus (CY)	1737.90	0.91	0.61	0.96	0.75	0.87	0.56
Czech Republic (CZ)	898.10	0.94	0.84	0.92	0.85	0.88	0.63
Germany (DE)	1666.36	0.92	0.86	0.93	0.80	0.89	0.68
Denmark (DK)	1746.65	1.00	0.71	0.94	0.93	0.87	0.86
Estonia (EE)	703.11	0.90	0.75	0.93	0.76	0.83	0.78
Spain (ES)	1266.12	0.91	0.41	0.93	0.85	0.84	0.71
Finland (FI)	1615.83	0.92	0.74	0.91	0.87	0.88	0.75
France (FR)	1541.65	0.95	0.69	0.92	0.83	0.84	0.69
Greece (GR)	1189.21	0.92	0.47	0.94	0.84	0.92	0.69
Hungary (HU)	622.34	0.86	0.71	0.86	0.87	0.88	0.38
Ireland (IE)	1829.89	0.92	0.52	0.91	0.92	0.86	0.65
Iceland (IS)	1928.94	0.94	0.65	0.94	0.89	0.97	0.73
Italy (IT)	1475.41	0.92	0.47	0.95	0.81	0.87	0.71
Lithuania (LT)	654.77	0.89	0.74	0.92	0.85	0.93	0.58
Luxembourg (LU)	2632.28	0.94	0.59	0.92	0.85	0.89	0.75
Latvia (LV)	639.98	0.89	0.71	0.95	0.67	0.74	0.34
Netherlands (NL)	1804.77	0.93	0.70	0.95	0.86	0.83	0.83
Norway (NO)	2089.73	0.93	0.77	0.92	0.92	0.95	0.90
Poland (PL)	609.82	0.92	0.75	0.87	0.88	0.93	0.44
Portugal (PT)	949.40	0.85	0.22	0.94	0.80	0.89	0.81
Sweden (SE)	1532.21	0.93	0.80	0.93	0.93	0.87	0.82
Slovania (SI)	1329.21	0.91	0.73	0.96	0.80	0.90	0.58
Slovak Republic (SK)	725.68	0.90	0.85	0.93	0.82	0.92	0.57
United Kingdom (UK)	1886.83	0.90	0.75	0.94	0.87	0.74	0.78
Spearman (y,t)		0.10	0.18	0.15	0.01	-0.01	0.40

- environment = 0 if the (main) respondent reports pollution, grime or other environmental problems in the neighborhood, and environment = 1 otherwise,
- physical security = 0 if the (main) respondent reports crime, violence or vandalism in the neighborhood, and physical security = 1 otherwise,
- financial security = 0 if the (main) respondent reports problems to face unexpected expenses, and financial security = 1 otherwise.

Table 2 shows the summary statistics. The mean attribute values vary considerably across the different countries. The final row of the table presents the (rank) correlation between income (first column) and each of the non-material attributes (the other columns).

These correlations are significant at the 1% level. Individuals in higher income countries have better health, better education, more jobs, a better environment, and more financial security. Higher income individuals, however, report lower physical security.

6.3 Results

Table 3 presents two maximal poverty lines for each country pair. Above the horizontal separator it reports the maximal poverty line for the unidimensional case (only income included), below the separator for the multidimensional case (income and type included). For example, the United Kingdom dominates Sweden up to €700 in the unidimensional case, and does not dominate Sweden for any poverty line in the multidimensional case.[16] Recall that the unidimensional maximal poverty line is necessarily at least as great as the multidimensional maximal poverty line.

The results for the unidimensional and multidimensional cases differ considerably. Adding the ordinal attributes decreases the maximal poverty line in 46% of the pairwise country comparisons. For each of these comparisons, the maximal poverty line drops to zero, which reveals the strength of the unanimity criterion if the ordinal attributes are included.[17] Luxembourg is an interesting example. Table 2 shows that Luxembourg has the highest income, but performs moderately in terms of the ordinal attributes. With the exception of the Czech Republic, Luxembourg dominates all countries up to €3000 in terms of income poverty. But if the ordinal attributes are also included, then Luxembourg dominates only Greece, Ireland, Italy, Portugal, and Spain.[18]

We now move from the pairwise comparisons in Table 3 to a comparison of overall performance. For each country, we compute the *net* average maximal poverty line, defined as the average 'good' maximal poverty line (in the country's row in Table 3) minus the average 'bad' maximal poverty line (in the country's column). Figure 3 plots the net average maximal poverty lines for the unidimensional and multidimensional cases. Two groups of countries can be distinguished. First, the countries that perform above average in both the unidimensional and multidimensional cases: the Nordic countries (DK, FI, IS, NO, SE), several Western European countries (AT, FR, LU, NL), the Czech Republic, and Slovenia. Second, the countries that perform below average in both cases: most Southern European countries (ES, GR, IT), the Eastern and Central European countries (EE, HU, LT, LV, PL), Belgium, and the United Kingdom. The regression line in the figure allows a further distinction between countries. Countries above the trend line relatively gain by adding the ordinal attributes, whereas countries below the line relatively lose. Striking

[16] Sweden also dominates the United Kingdom up to €1400 in the unidimensional case. For low income poverty lines, the average income poverty gaps of Sweden and the United Kingdom are close to each other. Non-dominance is therefore not easily rejected in the unidimensional case.

[17] Choosing a lower upper bound on the ordinal attribute weights (and hence weakening the unanimity criterion) produces less extreme drops in the maximal poverty lines.

[18] The fact that Luxembourg does not dominate the Czech Republic in terms of income poverty highlights that poverty dominance is especially sensitive in the tails of the distribution. Compared to Luxembourg, the Czech Republic is poor, but has few incomes below €200. Therefore, it dominates Luxembourg up to €200, whereas Luxembourg does not dominate the Czech Republic for any poverty line.

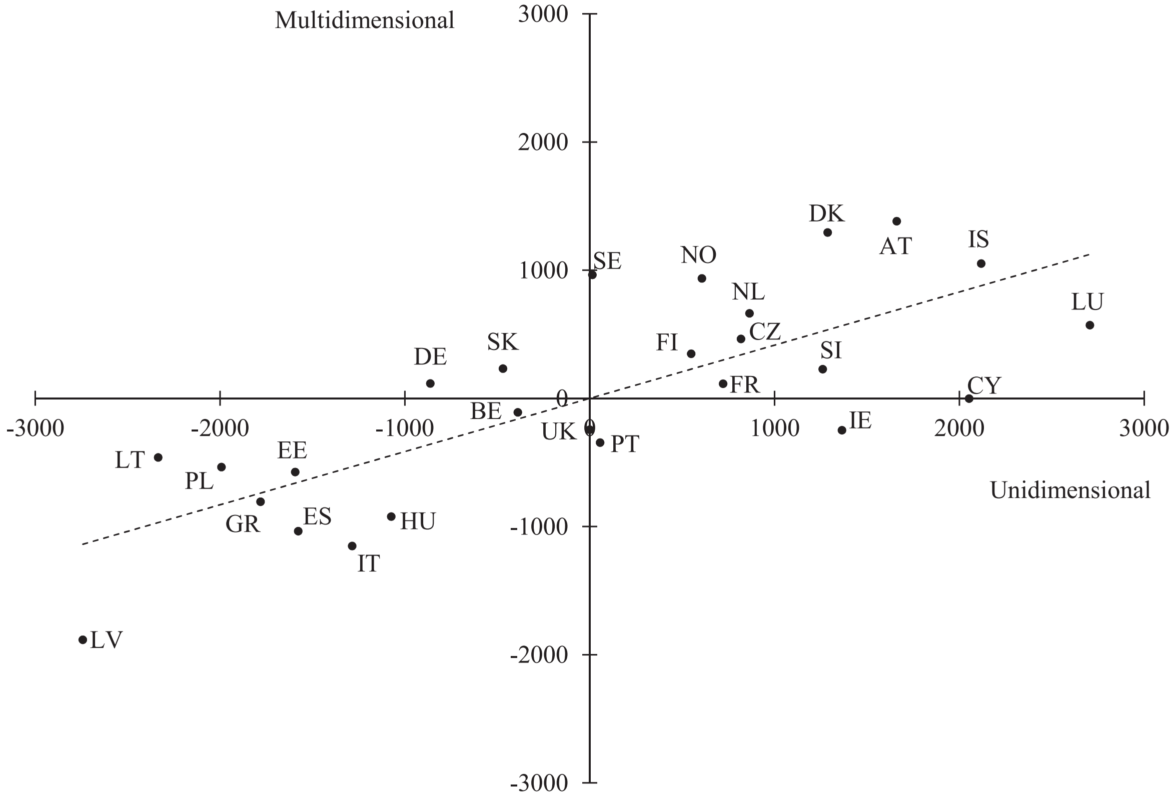

Figure 3. Net average maximal poverty lines

examples in the former category are Denmark, Norway, and Sweden, and in the latter category Cyprus, Ireland, and Luxembourg.

Table 3. Maximal poverty line in pairwise country comparisons
"row country dominates column country up to"

	AT	BE	CY	CZ	DE	DK	EE	ES	FI	FR	GR	HU	IE
AT		$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{600}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{500}{500}$
BE	$\frac{0}{0}$		$\frac{0}{0}$	$\frac{0}{0}$	$\frac{900}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
CY	$\frac{700}{0}$	$\frac{3000}{0}$		$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$
CZ	$\frac{300}{0}$	$\frac{400}{0}$	$\frac{200}{0}$		$\frac{500}{0}$	$\frac{400}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$	$\frac{400}{0}$	$\frac{300}{0}$	$\frac{800}{0}$	$\frac{3000}{3000}$	$\frac{300}{0}$
DE	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$		$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DK	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$		$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{100}{100}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$
EE	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$		$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
ES	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$		$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
FI	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{900}{0}$	$\frac{400}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$		$\frac{300}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{300}{0}$
FR	$\frac{0}{0}$	$\frac{1500}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{1000}{0}$	$\frac{500}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{1400}{0}$		$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{400}{0}$
GR	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{400}{0}$	$\frac{0}{0}$	$\frac{0}{0}$		$\frac{0}{0}$	$\frac{0}{0}$
HU	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{100}{0}$	$\frac{600}{0}$	$\frac{200}{0}$	$\frac{100}{0}$	$\frac{100}{0}$	$\frac{200}{0}$		$\frac{100}{0}$
IE	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	
IS	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{100}{100}$	$\frac{100}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$
IT	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
LT	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
LU	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{3000}{3000}$
LV	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
NL	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$
NO	$\frac{0}{0}$	$\frac{300}{300}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{100}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$
PL	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
PT	$\frac{100}{0}$	$\frac{200}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{300}{0}$	$\frac{100}{0}$	$\frac{100}{0}$	$\frac{300}{0}$	$\frac{3000}{0}$	$\frac{100}{0}$
SE	$\frac{0}{0}$	$\frac{2600}{2600}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{1600}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$
SI	$\frac{500}{0}$	$\frac{700}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{800}{0}$	$\frac{500}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$	$\frac{600}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{400}{0}$
SK	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{300}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{400}{0}$	$\frac{100}{0}$	$\frac{100}{0}$	$\frac{400}{0}$	$\frac{3000}{3000}$	$\frac{100}{0}$
UK	$\frac{0}{0}$	$\frac{400}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$

Table 3 (continued). Maximal poverty line in pairwise country comparisons "row country dominates column country up to"

	IS	IT	LT	LU	LV	NL	NO	PL	PT	SE	SI	SK	UK
AT	$\frac{500}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{200}{0}$	$\frac{3000}{3000}$	$\frac{600}{0}$	$\frac{1000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{100}{100}$	$\frac{3000}{0}$	$\frac{2300}{2300}$
BE	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{800}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{900}{0}$
CY	$\frac{600}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{500}{0}$	$\frac{3000}{0}$	$\frac{600}{0}$	$\frac{800}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{1400}{0}$
CZ	$\frac{400}{0}$	$\frac{600}{0}$	$\frac{3000}{3000}$	$\frac{200}{0}$	$\frac{3000}{3000}$	$\frac{400}{0}$	$\frac{500}{0}$	$\frac{3000}{3000}$	$\frac{1700}{0}$	$\frac{500}{0}$	$\frac{300}{0}$	$\frac{3000}{0}$	$\frac{500}{0}$
DE	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
DK	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{900}{800}$	$\frac{1200}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{2700}{2700}$
EE	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{100}{0}$	$\frac{100}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$
ES	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
FI	$\frac{400}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{400}{0}$	$\frac{700}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{800}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{900}{0}$
FR	$\frac{500}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{500}{0}$	$\frac{800}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{800}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{1000}{0}$
GR	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
HU	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{1000}{0}$	$\frac{0}{0}$	$\frac{1300}{0}$	$\frac{100}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{200}{0}$
IE	$\frac{700}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{100}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$	$\frac{900}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{1100}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{1200}{0}$
IS		$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{200}{200}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{1400}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{100}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$
IT	$\frac{0}{0}$		$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
LT	$\frac{0}{0}$	$\frac{100}{0}$		$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
LU	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$		$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$
LV	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$		$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$
NL	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$		$\frac{1300}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{0}$
NO	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{100}{0}$		$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$
PL	$\frac{0}{0}$	$\frac{100}{0}$	$\frac{200}{0}$	$\frac{0}{0}$	$\frac{1000}{1000}$	$\frac{100}{0}$	$\frac{100}{0}$		$\frac{0}{0}$	$\frac{100}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{100}{0}$
PT	$\frac{100}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$	$\frac{100}{0}$	$\frac{3000}{0}$	$\frac{200}{0}$	$\frac{200}{0}$	$\frac{3000}{0}$		$\frac{200}{0}$	$\frac{100}{0}$	$\frac{200}{0}$	$\frac{200}{0}$
SE	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{3000}{3000}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{200}{0}$	$\frac{800}{0}$	$\frac{3000}{3000}$	$\frac{0}{0}$		$\frac{0}{0}$	$\frac{0}{0}$	$\frac{1400}{1400}$
SI	$\frac{500}{0}$	$\frac{1500}{0}$	$\frac{3000}{0}$	$\frac{300}{0}$	$\frac{3000}{3000}$	$\frac{500}{0}$	$\frac{700}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{700}{0}$		$\frac{3000}{0}$	$\frac{800}{0}$
SK	$\frac{100}{0}$	$\frac{300}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{200}{0}$	$\frac{300}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{200}{0}$	$\frac{0}{0}$		$\frac{200}{0}$
UK	$\frac{0}{0}$	$\frac{3000}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{3000}{3000}$	$\frac{100}{0}$	$\frac{700}{0}$	$\frac{3000}{0}$	$\frac{0}{0}$	$\frac{700}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	

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Appendix: Proofs

Proof of the theorem. The representation defined in the theorem satisfies all axioms. We discuss the reverse implication. Recall that a poverty ordering $\succsim$ on D satisfies representation, focus, and monotonicity if and only if there exists a continuous individual poverty function $\pi : B \to \mathbb{R}$,

- with $\pi(x) = \pi(z)$ for each non-poor bundle x, and
- strictly decreasing on the set P of poor bundles,

such that expression (1) holds.

We add priority. Consider an attribute k in K and two equally poor bundles x and y with $x_k = y_k$. Let ε be a bundle in B with $\varepsilon_\ell = 0$ for each attribute $\ell \neq k$ and $\varepsilon_k > 0$. Since x and y are equally poor, priority implies that also the distributions $(x + \varepsilon, y)$ and $(x, y + \varepsilon)$ are equally poor. Representation implies that $\pi(x + \varepsilon) = \pi(y + \varepsilon)$.

We argue that, starting from the same assumptions, also the equality $\pi(x-\varepsilon) = \pi(y-\varepsilon)$ must hold. The argument is by contradiction and starts from the inequality $\pi(x - \varepsilon) < \pi(y - \varepsilon)$. Since π is strictly decreasing in P, we have $\pi(y) < \pi(x - \varepsilon) < \pi(y - \varepsilon)$. Because π is continuous, there must exist a scalar λ with $0 < \lambda < 1$ such that

$$\pi(x - \varepsilon) \; = \; \pi(y - \lambda\varepsilon).$$

The distribution $(x - \varepsilon, y - \lambda\varepsilon)$ satisfies $x_k - \varepsilon_k \leq y_k - \lambda\varepsilon_k$. Apply cardinal priority and obtain

$$(x - \varepsilon + \varepsilon, y - \lambda\varepsilon) \; \succsim \; (x - \varepsilon, y - \lambda\varepsilon + \varepsilon).$$

Consequently,

$$\pi(x) + \pi(y - \lambda\varepsilon) \; \leq \; \pi(x - \varepsilon) + \pi(y + (1 - \lambda)\varepsilon).$$

The equality $\pi(x - \varepsilon) = \pi(y - \lambda\varepsilon)$ implies the inequality $\pi(x) \leq \pi(y + (1 - \lambda)\varepsilon)$. Recall that x and y are equally poor ($\pi(x) = \pi(y)$) and $(1 - \lambda)\varepsilon > 0$. Hence, this inequality conflicts with monotonicity in P.

Both arguments together imply the following result. Let k in K be an attribute, and x and y two bundles in P with $x_k = y_k$. Obtain x' and y' in P from x and y by replacing $x_k = y_k$ by $x'_k = y'_k$. Then, $\pi(x) = \pi(y)$ implies $\pi(x') = \pi(y')$. Now, modify the assumption $\pi(x) = \pi(y)$ to $\pi(x) \geq \pi(y)$. Let $c = (\max\{y_k - x_k, 0\})_{k \in K}$ and note that $c_k = 0$. We have $\pi(x) \geq \pi(y) \geq \pi(x + c)$. Because π is continuous, there must exist a scalar λ with $0 \leq \lambda \leq 1$, such that $\pi(y) = \pi(x + \lambda c)$. As a consequence, $\pi(x' + \lambda c) = \pi(y')$ holds and, using monotonicity, we get $\pi(x') \geq \pi(y')$. To sum up, π is separable in attributes, i.e., for all bundles x, x', y, and y' in P and for each attribute k in K, if $x_k = y_k$ and $x'_k = y'_k$, and $x_\ell = x'_\ell$ and $y_\ell = y'_\ell$ for each attribute $\ell \neq k$, then $\pi(x) \geq \pi(y)$ implies $\pi(x') \geq \pi(y')$.

Continuity and separability of the poverty function π in P, and the assumption of at least three attributes, allows for an additive representation (Debreu, 1960). More precisely, there exist continuous maps $\bar{f}_k : \mathbb{R} \to \mathbb{R}$ for each k in K such that, for each x and y in P, we have

$$\pi(x) \leq \pi(y) \quad \text{if and only if} \quad \sum_K \bar{f}_k(x_k) \ \leq \ \sum_K \bar{f}_k(y_k).$$

As π is strictly decreasing in the set P of poor bundles, also the maps $\bar{f}_k$ are strictly decreasing. Let $f_k = \bar{f}_k(0) - \bar{f}_k$ for each k and obtain that $\pi(x) = f(\sum_K f_k(x_k))$ for each x in P with

- f_k continuous, strictly increasing, with $f_k(0) = 0$, and
- $f : \mathbb{R} \to \mathbb{R}$ continuous and strictly decreasing on the interval $(-\infty, \zeta]$ with $\zeta = \sum_K f_k(z_k)$.

We can normalize $f(\zeta) = 0$ without loss of generality.

We now argue that the map f is convex. Consider two poor bundles x and y such that $x \leq y$ and $x_k = y_k$. Let ε be a bundle that is zero in each dimension except for dimension k ($\varepsilon_k > 0$). Apply priority to the distribution (x, y) and obtain the inequality

$$f\Big(f_k(x_k{+}\varepsilon_k){+}\sum_{\ell \neq k} f_\ell(x_\ell)\Big) - f\Big(\sum_K f_k(x_k)\Big) \ \leq \ f\Big(f_k(y_k{+}\varepsilon_k){+}\sum_{\ell \neq k} f_\ell(y_\ell)\Big) - f\Big(\sum_K f_k(y_k)\Big).$$

Since $x_k = y_k$, the inequality can be rewritten as

$$f(\delta + a) - f(a) \ \leq \ f(\delta + b) - f(b)$$

with $a = \sum_K f_k(x_k) \leq b = \sum_K f_k(y_k)$ and $\delta = f_k(x_k + \varepsilon_k) - f_k(x_k) = f_k(y_k + \varepsilon_k) - g_k(y_k)$. Conclude that the map f is convex.

Finally we show that, for each cardinal attribute k in C, the map f_k is concave. Consider two equally poor bundles x and y that satisfy $x_k \leq y_k$. Again, let ε be a bundle that is zero in each dimension except for dimension k ($\varepsilon_k > 0$). Apply cardinal priority to the distribution (x, y) and obtain the inequality

$$f\Big(f_k(x_k + \varepsilon_k) + \sum_{\ell \neq k} f_\ell(x_\ell)\Big) + \pi(y) \ \leq \ f\Big(f_k(y_k + \varepsilon_k) + \sum_{\ell \neq k} f_\ell(y_\ell)\Big) + \pi(x).$$

Since x and y are assumed to be equally poor ($\pi(x) = \pi(y)$) and since f is strictly decreasing, it follows that

$$f_k(x_k + \varepsilon) - f_k(x_k) \ \geq \ f_k(y_k + \varepsilon) - f_k(y_k),$$

and hence the map f_k is concave for each k in C. Write g_k for f_k with k in C, and h_ℓ for f_ℓ with ℓ in O. Expression (2) follows. □

Proof of the proposition. First, we rewrite the aggregate poverty level $\Pi_{f,g,\alpha,z_\iota}(p,F)$. As $g' > 0$, the map g is everywhere increasing and the inverse map g^{-1} is well defined. We change the integration variable y into $x = g(y) + \alpha \cdot t$ and obtain

$$\Pi_{f,g,\alpha,z_\iota}(p,F) \;=\; \sum_{t\in T} p_t \int\limits_{x=-\infty}^{g(z_t)+\alpha\cdot t} f(x)\, dF_t(g^{-1}(x-\alpha\cdot t)).$$

Recall that $g(z_t) + \alpha \cdot t = g(z_\iota) + \alpha \cdot \iota$ for each type t. Hence,

$$\Pi_{f,g,\alpha,z_\iota}(p,F) \;=\; \int\limits_{x=-\infty}^{g(z_\iota)+\alpha\cdot\iota} f(x)\, d\{\sum_{t\in T} p_t F_t(g^{-1}(x-\alpha\cdot t))\}.$$

In order to compare (p,F) and (q,G), let $H(x) = \sum_{t\in T} p_t F_t(x) - q_t G_t(x)$ and obtain that country (p,F) dominates country (q,G) up to level z^* if and only if

$$\int\limits_{x=-\infty}^{g(z_\iota)+\alpha\cdot\iota} f(x)\, dH(g^{-1}(x-\alpha\cdot t)) \;\leq\; 0$$

for each z_ι, α, f, and g as specified before. As $f(g(z_\iota)+\alpha\cdot\iota) = 0$, $f' < 0$ and $f'' \geq 0$, the previous condition holds if and only if

$$\int\limits_{x=-\infty}^{s} H(g^{-1}(x-\alpha\cdot t))\, dx \;\leq\; 0$$

for each vector α in $\mathbb{R}^\ell_+$, each C^2-map g with $g' > 0$ and $g'' \leq 0$, each $s \leq g(z_\iota) + \alpha\cdot\iota$, and each $z_\iota \leq z^*$. The condition "for each f with $f' < 0$ and $f'' \geq 0$" can be eliminated using integration by parts twice (e.g., Lambert, 2001, p. 52). Use integration by parts and rewrite the integral as

$$\int\limits_{x=-\infty}^{s} H(g^{-1}(x-\alpha\cdot t))\, dx \;\;=\;\; \int\limits_{x=-\infty}^{s} (s-x)\, dH(g^{-1}(x-\alpha\cdot t)).$$

Finally, return to the variable $y = g^{-1}(x-\alpha\cdot t)$ and let $s_t = g^{-1}(s-\alpha\cdot t)$ for each t in T. The unanimity criterion becomes:

Country (p,F) dominates country (q,G) up to level z^* if and only if

$$\sum_{t\in T} \int\limits_{y=-\infty}^{s_t} (g(s_t) - g(y))\, \{p_t dF_t(y) - q_t G_t(y)\} \;\leq\; 0$$

for each

- $s_t = g^{-1}(s - \alpha \cdot t)$ with $s \le g(z_\iota) + \alpha \cdot \iota$ and $z_\iota \le z^*$,
- vector $\alpha \ge 0$, and
- C^2-map g with $g' > 0$ and $g'' \le 0$,

as required. $\square$