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Ranking opportunity sets, averaging versus preference for freedom

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Ranking opportunity sets, averaging versus preference for freedom*

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Abstract. We introduce and axiomatize two quasi-orderings that extend preferences on a set to its power set. First, a modified version of indirect utility takes into account the number of maximal elements in the opportunity set. This rule meets Puppe's axiom of preference for freedom. Second, an averaging rule takes into account the number of non-maximal elements in the opportunity set. Such a rule satisfies the Gärdenfors principle. Axioms that involve no more than two alternatives capture the differences between the two rules.

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1 Introduction

How should one extend a relation on a set towards a relation on the collection of subsets? The answer on this question depends upon the particular problem at hand. Barberà, Bossert and Pattanaik (2004) and Foster (2011) present overviews of the literature on ranking sets of objects. Let us discuss some of the applications.

First, there is the attempt to rank opportunity sets on the basis of a preference relation on the set of alternatives. Opportunity sets are subsets of the set of all outcomes from which an agent chooses or obtains one outcome. The ranking of opportunity sets may be based on the most preferred elements in each set, on the degree of freedom, or on

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the diversity or similarity of the different alternatives available in the opportunity set; e.g. Pattanaik and Xu (1990, 1998, 2000), Sen (1993), Klemish-Ahlert (1993), Bossert et al (1994). Next, in the aftermath of the Gibbard-Satterthwaite theorem on strategy-proofness, mechanisms were studied that generate—for each reported preference profile—a set of possible outcomes, over which a randomization device finally selects a particular alternative, e.g. Barberà (1977), Roth (1985). Furthermore, matching theory presupposes the agents to have preferences over sets. In the college admissions problem, colleges need to have preferences over sets of students. In a context of many-to-many matching, Echenique and Oviedo (2000) study the problem where workers can work at several firms and firms want to match with several workers. Each of these cases invites the researcher to think about ordering opportunity sets on the basis of a preference relation.

This note employs the axiomatic approach to the ordering of sets. Its contribution is twofold. *First*, we appeal to the idea of preference for freedom as modeled by Puppe (1996): each opportunity set contains at least one alternative such that its exclusion reduces an agent's well-being. Applied to a 'simple' setting with only two equally good options, preference for freedom recommends the pair to be ranked strictly above the singletons. We introduce a ranking rule in the spirit of the indirect utility rule[1] that meets this simple axiom. This new rule, that we label the indirect-utility-freedom rule, satisfies monotonicity with respect to set inclusion (i.e. additional options do not deteriorate the opportunity set) as well as Puppe's axiom of preference for freedom.

Second, we contribute to the literature that does not endorse the axiom of monotonicity on the basis of temptation, thinking cost, or limited attention; e.g. Gül and Pesendorfer (2001), Lleras et al (2010), Masatlioglu et al (2010a,b). Here, we impose Barberà's principle and require that a pair is ranked strictly in between its best and its worst element. We introduce a class of averaging rules that rank opportunity sets on the basis of their best element and the number of non-best elements. As a consequence, such a rule is not sensitive to each element in the opportunity set, this in contrast to the average Borda rule (Baigent and Xu, 2004) and the uniform expected utility rule (Gravel et al, 2012).

We provide an axiomatization of the indirect-utility-freedom rule, the averaging rules, and the indirect utility rule. We want to highlight that the differences between these three rules are captured by simple axioms, i.e. axioms that involve no more than two different alternatives. The difference between the indirect utility and the indirect-utility-freedom rule boils down to whether or not a pair of equally good options is ranked strictly above the singletons. The difference between the indirect utility rule and the averaging rules boils down to the axiom of simple monotonicity versus Barberà's principle. Simple axioms are attractive since the intuition about well-being is likely to be firmer in 'simple' cases than in more complex cases (e.g. Bossert et al, 1994). In this note, simple axioms provide a guide to one of the above three rules.

The outline of the note is as follows. The next section introduces notation, recalls the indirect utility rule, and defines two new ranking rules. Section 3 discusses nine axioms. We provide further insights in the incompatibility of the axioms of independence and

[1]The indirect utility rule ranks opportunity sets according to a most preferred element.

of Gärdenfors-averaging (Kannai and Peleg, 1984 and Fishburn, 1984). In particular, Barberà's principle in combination with a restricted version of independence implies the Gärdenfors principle. Table 1 at the end of this section lists the axioms a rule satisfies and *summarizes the main results* as it announces the characterizations of the three rules discussed. Section 4 axiomatizes our averaging rule. Section 5 axiomatizes the indirect utility rule and its refinement the indirect-utility-freedom rule.

2 Notation, one old and two new rules

The set X of alternatives is finite and R is a transitive and complete (hence, reflexive) binary relation on X. We write $a\,R\,b$ instead of $(a,b) \in R$. The asymmetric and the symmetric factors of R are denoted by P and I. A pair of alternatives that are equally good is said to be an I-pair. A nonempty subset of A is interpreted as a possible opportunity set that may be available to an individual equipped with the preference relation R. The collection of nonempty subsets of X is denoted by Ω. We abuse notation—recall that R is defined on X—and we write

$$A\,R\,x \quad \text{(resp., } x\,R\,A\text{)},$$

with x in X and A in Ω as a shorthand for

$$a\,R\,x \text{ for each } a \text{ in } A \quad \text{(resp., } x\,R\,a \text{ for each } a \text{ in } A\text{)}.$$

An option a in the opportunity set A is said to be R-maximal if $a\,R\,A$. For each A in Ω, let $\max A$ denote the set of R-maximal elements in A. Since X is finite, the set $\max A$ is nonempty. Let A^- denote the subset $A - \max A$. Hence, each opportunity set A decomposes into $(\max A) \cup A^-$. We now recall the indirect utility rule $\succsim_I$ and we introduce the indirect-utility-freedom rule, denoted by $\succsim_{IF}$. The relation $\succsim_{IF}$ keeps the strict ranking $\succ_I$ and refines the indifference relation $\sim_I$.

Definition 1. Let R be a complete and transitive relation on X. Let Ω be the collection of opportunity sets. The *indirect utility* rule, denoted by $\succsim_I$, is defined as

$$A \succsim_I B \qquad \text{if} \qquad |A \cap \max(A \cup B)| > 0,$$

for each pair A and B of opportunity sets. The *indirect-utility-freedom* rule, denoted by $\succsim_{IF}$, is defined by

$$A \succsim_{IF} B \qquad \text{if} \qquad |A \cap \max(A \cup B)| \geq |B \cap \max(A \cup B)|,$$

for each pair A and B of opportunity sets.

The indirect utility rule and the indirect-utility-freedom rule are both complete relations on the collection Ω. Although strongly in the spirit of the indirect utility rule, the indirect-utility-freedom rule meets Puppe's axiom of preference for freedom: dropping one of the best options in an opportunity set always decreases its ranking.

We now prepare the introduction of the averaging rule. Let a and b be two alternatives in X with aPb. Let $d(a,b)$ in $\mathbb{N}_0$ denote the length k of the longest sequence $b_1, b_2, \ldots, b_k = b$ of different alternatives in $X - \{a\}$ that satisfy

$$a\,P\,b_1\,R\,b_2\,R\,b_3\,R\cdots R\,b_k.$$

The number $d(a,b)$ measures the gap between the alternatives a and b. Averaging implies that opportunity set $\{a, b_1, b_2, b_3, \ldots, b_k\}$ has a lower rank than $\{a\}$ and a higher rank than $\{b\}$. In case $aPbPc$ with a, b, and c in X, then $d(a,c) = d(a,b) + d(b,c)$. We will write d_R instead of d in case a reference to the relation R is needed.

Definition 2. Let R be a complete and transitive relation on X. Let Ω be the collection of opportunity sets. Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be a map that satisfies superadditivity, i.e. $f(k+k') \geq f(k) + f(k')$ for each k, k' in $\mathbb{N}_0$. The φ-rule, denoted by $\succsim_\varphi$, is defined as follows.

- If $A \sim_I B$ and $|A^-| = |B^-|$, then $A \sim_\varphi B$,
- If $A \sim_I B$ and $|A^-| < |B^-|$, then $A \succ_\varphi B$,
- If $A \succ_I B$ and $|A^-| \leq |B^-| + \varphi(d(\max(A), \max(B)))$, then $A \succ_\varphi B$,
- If $A \succ_I B$ and $|A^-| > |B^-| + \varphi(d(\max(A), \max(B)))$, then $A \not\succ_\varphi B$.

If two opportunity sets are equally good according to the indirect utility rule, then the set with the smallest number of non-maximal elements is the better. On the other hand, the averaging rule and the indirect utility rule disagree in case the number of non-maximal elements in the set with the highest indirect utility is too high. The rule $\succsim_\varphi$ is not complete.

3 Nine axioms

We briefly discuss nine axioms. Throughout, $\succsim$ is a transitive and reflexive binary relation on Ω. The asymmetric and the symmetric factors of $\succsim$ are denoted by $\succ$ and $\sim$. We start with six axioms that involve the ranking of opportunity sets with no more than two distinct alternatives each. As indicated by Bossert et al (1994) and Foster (2011) the intuition on ranking small sets might be on a more sound basis than the ranking in more complex cases. The interpretations of this type of axioms are indeed straightforward.

Extension imposes that the ordering of the singleton opportunity sets reflects the ordering of the alternatives. This axiom is typical for rules that take indirect utility into account. The quantity approach, in contrast, considers singleton opportunity sets as equally good (cf. Pattanaik and Xu, 1990).

Extension. For each x and y in X, we have $\{x\} \succsim \{y\}$ if and only if $x\,R\,y$.

Simple monotonicity. For each x and y in X with $x \neq y$, we have $\{x, y\} \succsim \{x\}$.

I-pair preference. For each x and y in X with $x \neq y$, we have

$$x \, I \, y \implies \{x, y\} \succ \{x\}.$$

I-pair indifference. For each x and y in X with $x \neq y$, we have

$$x \, I \, y \implies \{x, y\} \sim \{x\}.$$

The axiom of I-pair indifference does not value the 'freedom' when a second equally good option becomes available and, hence, conflicts with Puppe's axiom of preference for freedom.

Barberà's principle. For each x and y in X, we have

$$x \, P \, y \implies \{x\} \succ \{x, y\} \succ \{y\}.$$

Barberà's principle conflicts with simple monotonicity. In case the set X of alternatives is a convex subset of the Euclidean space $\mathbb{R}^n$ and R is continuous, then Barberà's principle 'converges' to I-pair indifference when x converges to y. Hence, I-pair indifference is more in the spirit of Barberà's principle (the combination of I-pair preference and Barberà's principle is not considered). In this note, however, the set X is assumed to be finite. Combining Barberà's principle and I-pair indifference implies the axiom of extension. The next 'simple' axiom involves three different alternatives.

Simple indirect indifference. For each triple, x, y, and z, of different alternatives in X, we have

$$x \, P \, y \, R \, z \implies \{x, y\} \sim \{x, z\}.$$

Simple indirect indifference requires that the best elements are dominant in ranking pairs (see also Bossert et al, 1994). We now present three axioms that are more complex as they consider arbitrary opportunity sets. The axiom of independence requires that comparisons between opportunity sets are maintained when these opportunity sets expand or contract in the same way.

Independence. For each A and B in Ω and for each x in $X - (A \cup B)$, we have

$$[(\max A) \, R \, x \;\text{ and }\; (\max B) \, R \, x] \implies [A \succsim B \iff A \cup \{x\} \succsim B \cup \{x\}].$$

Consider two opportunity sets. The ranking between these two opportunity sets does not depend upon alternatives that are not preferred to the R-maximal alternatives of the two opportunity sets: the addition (or removal) of such an alternative to (or from) both opportunity sets does not change their ranking. This axiom is related to the axiom of weak independence of Bossert et al (1994). Let us illustrate the strength of this axiom.

Lemma 1. Let R be a complete and transitive relation on X. Let the transitive and reflexive relation $\succsim$ on Ω satisfy Independence and Simple monotonicity. Then, $\succsim$ satisfies monotonicity, i.e., for each A and B in Ω, we have $A \supseteq B$ implies $A \succsim B$.

Proof. Let $A \supseteq B$. If $A = B$, then the reflexivity of $\succsim$ implies that $A \succsim B$. If $A \supset B$, write the options in A in R-decreasing order

$$a_1 R\, a_2 R \cdots R\, a_k R\, b_1 R \cdots ,$$

with k in $\mathbb{N}$ and b_1 in $\max B$. Use simple monotonicity to obtain $\{a_1, a_2\} \succsim \{a_2\}$. Use independence and obtain the following sequence of weak inequalities:

$$\begin{aligned} \{a_1, a_2, a_3\} \succsim \{a_2, a_3\} \succsim \{a_3\}, \\ \{a_1, a_2, a_3, \cdots, a_k\} \succsim \{a_2, a_3, \cdots, a_k\} \succsim \cdots \succsim \{a_k\}, \end{aligned}$$

In each line, the final inequality follows from simple monotonicity. The transitivity of $\succsim$ implies $\{a_1, a_2, a_3, \cdots, a_k\} \succsim \{a_k\}$. Again, use independence to add b_1 and simple monotonicity to obtain

$$\{a_1, a_2, a_3, \cdots, a_k, b_1\} \succsim \{a_2, a_3, \cdots, a_k, b_1\} \succsim \{a_k, b_1\} \succsim \{b_1\}.$$

The next options to be added are not R-better than b_1. Let c be an option in $A - B$ such that $b_1 R c$. From independence and simple monotonicity it follows

$$\{a_1, a_2, a_3, \cdots, a_k, b_1, c\} \succsim \{a_k, b_1, c\} \succsim \{b_1, c\} \succsim \{b_1\}.$$

Proceed by adding all further options in $A - B$. Conclude that $(A - B) \cup \{b_1\} \succsim \{b_1\}$. Conclude the proof by adding the options in $B - \{b_1\}$. □

Independence lifts simple monotonicity to monotonicity for arbitrary sets. In a similar way, the axiom of independence lifts Barberà's principle to the Gärdenfors principle according to which the addition of a better (resp. worse) option improves (resp. deteriorates) the opportunity set. The next lemma provides a formal statement.

Lemma 2. Let R be a complete and transitive relation on X. Let the transitive and reflexive relation $\succsim$ on Ω satisfy Independence and Barberà's principle. Then, for each opportunity set A and for each option x and y in X, we have

$$xPA\,Py \qquad \text{implies} \qquad A \cup \{x\} \succ A \succ A \cup \{y\}.$$

Proof. Let $a \in \max A$. Then, according to Barberà's principle, $\{x, a\} \succ \{a\}$. Use independence to add the further elements of A and obtain $A \cup \{x\} \succ A$. Similarly, Barberà's principle implies $\{a\} \succ \{a, y\}$. From independency it follows that $A \succ A \cup \{y\}$. □

Lemma 2 provides a further comment on the Kannai-Peleg (1984) impossibility result: when independence is unconditional (i.e., the conditions $\max A\, R\, x$ and $\max B\, R\, x$ are dropped), then an incompatibility with the Gärdenfors principle occurs. See also Fishburn (1984) and Barberà and Pattanaik (1984). Lemma 2 underpins the Gärdenfors principle by the combination of independence and Barberà's principle.

Two further axioms close this section.

Composition. For each A and B in Ω and for each z in $X - (A \cup B)$, we have

$$A \succsim B \text{ and } (\max B) P z \quad \Longrightarrow \quad A \succsim B \cup \{z\}.$$

The ranking of two opportunity sets does not change when an option dominated by the worst-off set is added to this worst-off set. Apply this axiom to $A = B = \{x\}$ and obtain that "$\{x\} \succsim \{x, z\}$ as soon $x\, P\, z$" which is close to 'one' part of Barberà's principle ($\{x\} \succ \{x, z\}$ as soon $x\, P\, z$). Furthermore, this axiom is related to the robustness axiom of Bossert et al (1994) and ensures that a lack of indirect utility cannot be compensated by adding worse alternatives.

Finally, consider two different sets (X', R') and (X, R) of alternatives equipped with a preference relation and a map $f : X' \to X$. Then, f is said to be d-preserving if for each a and b in X' with $aP'b$ we have $f(a)Pf(b)$ and $d_{R'}(a, b) = d_R(f(a), f(b))$. Let Ω' and Ω be the corresponding collections of opportunity sets. It seems natural that a d-preserving map f lifts the ordering of alternatives towards the ordering of opportunity sets: for each pair A and B in Ω' we have

$$A \succsim' B \qquad \text{if and only if} \qquad f(A) \succsim f(B).$$

Neutrality restricts this idea to the case where X' is a subset of X and R' is the restriction of R to A. For example, if for one particular I-pair $\{a, b\}$ we have $\{a, b\} \sim \{a\}$, then neutrality implies that simple indirect indifference holds. Obviously, neutrality is a very weak axiom.

Neutrality. Let X' be a subset of X and let $f : X' \to X$ be a d-preserving injection. Then, for each pair A and B of subsets of X' we have

$$A \succsim B \qquad \text{if and only if} \qquad f(A) \succsim f(B).$$

Barberà et al (2004) discuss related versions of this axiom of neutrality. The next table indicates which axioms are satisfied by the rules discussed in this note. A '+' or a '$*$' means that the rule satisfies the axiom. A '$-$' means that the rule violates the axiom. Each column, when restricted to the axioms indicated by a $*$, presents an axiomatization. The proofs are discussed in the next sections. Each of the three ranking rules satisfies extension, simple indirect indifference, independence, composition, and neutrality.

	$\succsim_I$	$\succsim_{IF}$	$\succsim_\varphi$
1. Extension	$*$	$*$	$+$
2. Simple monotonicity	$*$	$*$	$-$
3. I-pair preference	$-$	$*$	$-$
4. I-pair indifference	$*$	$-$	$*$
5. Barberà's principle	$-$	$-$	$*$
6. Simple indirect indifference	$+$	$+$	$*$
7. Independence	$*$	$*$	$*$
8. Composition	$*$	$*$	$+$
9. Neutrality	$+$	$+$	$*$

Table 1. Three rules and nine axioms.

The differences between the three rules boil down to the simple axioms in rows 2-5. Start from indirect utility, drop I-pair indifference, impose I-pair preference, and obtain the indirect-utility-freedom rule. Start from indirect utility, drop simple monotonicity, impose Barberà's principle, and obtain an averaging rule.

4 Barberà's principle and averaging

The averaging rules $\succsim_\varphi$ satisfy the axioms as listed in Table 1. Here, we start from these axioms and we build up the averaging rule. The main idea is to reduce an opportunity set to its best element and the number of worse elements.

Lemma 3. Let R be a complete and transitive relation on X. Let the transitive and reflexive relation $\succsim$ on Ω satisfy Extension, I-pair indifference, Simple indirect indifference, and Independence. Consider two opportunity sets A and B. Then,

- if a belongs to $\max A$, then $A \sim \{a\} \cup A^-$;
- if $A \succsim_I B$ and $|A^-| = |B^-|$, then $A \succsim B$.

Proof. Write the options in A and B in R-decreasing order: $a_1 I a_2 I \cdots I a_k P c_1 R c_2 R \cdots R c_\ell$ for A and $b_1 I b_2 I \cdots I b_n P d_1 R d_2 R \cdots R d_\ell$ for B, with $k, n \geq 1$. Apply I-pair indifference and independence to obtain that $\max(A) \sim \{a_1\}$. Use independence and add the options in A^-. The first item follows. Applied to B, we have $\max(B) \sim \{b_1\}$. From $(\max A) R (\max B)$ it follows that $a_1 R b_1$. Extension implies that $\{a_1\} \succsim \{b_1\}$. Hence, $(\max A) \sim \{a_1\} \succsim \{b_1\} \sim (\max B)$. Simple indirect indifference implies $\{a_1, c_1\} \sim \{a_1, c_m\} \sim \{a_1, d_n\}$, for each $1 \leq m, n \leq \ell$. Repeated use of independence implies

$$\{a_1, c_1, c_2\} \sim \{a_1, c_m, d_{m'}\} \sim \{a_1, d_n, d_{n'}\},$$

for each $1 \leq m, m', n, n' \leq \ell$; and that $\{a_1\} \cup A^- \succsim \{b_1\} \cup B^-$. The second item follows. □

Hence, $\{a\} \cup B \sim \{a\} \cup C$ provided $a\,P\,(B \cup C)$ and $|B| = |C|$. Furthermore, if $a\,P\,(B \cup C)$ and $|B| < |C|$, then $\{a\} \cup B \succ \{a\} \cup C$. Indeed, assume that $B \subset C$ and that the options in $C - B$ are R-bottom ranked in C. Next, use the Gärdenfors principle: the addition of bottom ranked alternatives lowers the ranking of the opportunity set.

We now strengthen the axiom of extension to the combination of Barberà's principle and I-pair indifference.

Lemma 4. Let R be a complete and transitive relation on X. Let the transitive and reflexive relation $\succsim$ on Ω satisfy Barberà's principle, I-pair indifference, Simple indirect indifference, and Independence. Let A be an opportunity set for which A^- is nonempty. Then, $\{a\} \succ A \succ \{b\}$, with a an R-maximal and b an R-minimal option in A.

Proof. Write the options in A in R-decreasing order: $aIa_2I \cdots Ia_kPb_1Rb_2R \cdots Rb_\ell Rb$. Lemma 2 implies that $\{a\} \sim \max A$. Barberà's principle implies $\{a\} \succ \{a, b_1\} \succ \{b_1\}$. Use independence and add b_2,

$$\{a\} \succ \{a, b_2\} \succ \{a, b_1, b_2\} \succ \{b_1, b_2\} \succ \{b_2\},$$

where the first and the final inequality follow from Barberà's principle. Continue by adding $b_3, b_4, \ldots$. Finally, use $A \sim \{a\} \cup A^-$. $\square$

Let $\succsim$ satisfy simple indirect indifference, independence, and Barberà's principle. For each pair a and b of alternatives with $a\,P\,b$ we have

$$\{a\} \succ \{a, b\} \sim \{a, z_1\} \succ \{b\},$$

for each alternative z_1 with $a\,P\,z_1$. Now, add different 'bad' alternatives $z_1, z_2, \ldots$ to the singleton $\{a\}$ and obtain a decreasing sequence

$$\{a\} \succ \{a, z_1\} \succ \{a, z_1, z_2\} \succ \cdots \succ \{a, z_1, z_2, \ldots, z_k\}.$$

Let k be the largest number for which $\{a, z_1, z_2, \ldots, z_k\} \succ \{b\}$, hence

$$\{a, z_1, z_2, \ldots, z_k, z_{k+1}\} \not\succ \{b\}.$$

Obviously, k depends upon the particular alternatives a and b, and we write $k = k(a, b)$. Also, $k(a, b) \geq d(a, b)$. Indeed, if there exists a decreasing sequence

$$a\,P\,b_1\,R\,b_2\,R\,b_3\,R \cdots R\,b_{\ell-1}\,R\,b_\ell = b,$$

with $\ell = d(a, b)$, then

$$\{a, z_1, z_2, \ldots, z_\ell\} \sim \{a, b_1, b_2, b_3, \ldots, b_{\ell-1}, b_\ell\} \succ \{b\}.$$

Now, add neutrality. It follows that $k(a, b) = k(a', b')$ for each pair a' and b' with $a'\,P\,b'$ and $d(a', b') = d(a, b)$. Hence, k only depends on $d(a, b)$ and we write $k(a, b) = \varphi(d(a, b))$.

Finally, the map φ is superadditive: $\varphi(k+k') \geq \varphi(k) + \varphi(k')$. Indeed, consider three alternatives a, b, and c that satisfy $a\,P\,b\,P\,c$. Then, $d(a,c) = d(a,b) + d(b,c)$. Denote $k = d(a,b)$ and $k' = d(b,c)$. Then,

$$\{a, z_1, \ldots, z_{\varphi(k)}\} \succ \{b\} \text{ while } \{a, z_1, \ldots, z_{\varphi(k)}, z_{\varphi(k)+1}\} \not\succ \{b\}, \text{ and}$$
$$\{b, z'_1, \ldots, z'_{\varphi(k')}\} \succ \{c\} \text{ while } \{b, z'_1, \ldots, z'_{\varphi(k')}, z'_{\varphi(k')+1}\} \not\succ \{c\}.$$

Independence implies that $\varphi(d(a,c)) \geq \varphi(k) + \varphi(k')$. The next proposition summarizes.

Proposition 1. Let R be a complete and transitive relation on the set X. Let the transitive and reflexive relation $\succsim$ on Ω satisfy Barberà's principle, I-pair indifference, Simple indirect indifference, Independence, and Neutrality. Then, there exists a superadditive map $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\succsim$ extends $\succsim_\varphi$.

Remark 1 (*Independence of the five axioms*)**.** For each axiom that occurs in the previous proposition, we present a transitive and reflexive relation on Ω that violates this axiom and satisfies the other four axioms.

- The trivial ordering considers each opportunity set equally good and violates Barberà's principle.
- Replace I-pair indifference with I-pair preference and extension, obtain a rule that refines $\succsim_\varphi$. This rule violates I-pair indifference.
- Let $X = \{a, b, c\}$. Define R by $aPbPc$. Impose extension, Barberà's principle, and independence. Let $\{a, b\} \succ \{a, c\}$ and $\{b\} \sim X \succ \{a, c\}$. As R is a linear order, this rule satisfies I-pair indifference. This rule violates simple indirect indifference.
- Let $\varphi : X^2 \to \mathbb{N}_0$ be map for which $\varphi(a, b) \neq \varphi(a', b')$ while $d(a, b) = d(a', b')$ for at least one quadruple a, b, a', b' of options in X. The rule $\succsim_\varphi$ violates neutrality.
- Let the cardinality of X be at least four. Pairs and singletons are compared according to Barberà's principle, I-pair indifference, and simple indirect indifference. Opportunity sets of cardinality three or more are considered equally good. This rule violates independence.

5 Indirect utility and preference for freedom

We provide an axiomatization of the indirect utility rule $\succsim_I$ and the indirect-utility-freedom rule $\succsim_{IF}$. The difference between these two rules is captured by 'I-pair indifference' and 'I-pair preference'. We close this section with three remarks: we return to a rule proposed by Pattanaik and Xu (1998), we show the independency of the axioms used to characterize the indirect-utility-freedom rule, and we provide a further refinement towards the leximax rule of Bossert et al (1994).

The next lemma is the starting point: the imposition of simple monotonicity, extension, composition, and independence partitions the collection Ω. Each partition class collects the opportunity sets with the same indirect utility and the same number of maximal elements. These partition classes filter out the information—indirect utility and the number of maximal elements—that might be relevant for the ranking of opportunity sets.

Lemma 5. Let R be a complete and transitive relation on X. Let the transitive and reflexive relation $\succsim$ on Ω satisfy Simple monotonicity, Composition, Independence, and Extension. Then, for each A and B in Ω, we have

$$\max A\, I\, \max B \;\text{ and }\; |\max A| = |\max B| \quad \Longrightarrow \quad A \sim B.$$

Proof. Lemma 1 implies that $\succsim$ satisfies monotonicity. Let A and B be two opportunity sets that satisfy the premise of the statement in the lemma. We proceed in three steps.

Step 1. Monotonicity, extension, composition, and independence imply that $A \sim \max A$.

Proof. Since $A \supseteq \max A$, monotonicity implies $A \succsim \max A$. We now focus on the reverse inequality, $\max A \succsim A$. If $\max A$ happens to coincide with A, this inequality follows from the reflexivity of $\succsim$. Otherwise, let $x \in A - \max A$. Hence, for each a in $\max A$ we have $a\, P\, x$ and, by extension, $\{a\} \succ \{x\}$. Monotonicity and transitivity imply $\max A \succ \{x\}$. Since $\max A \succsim \max A$, composition implies $\max A \succsim \max A \cup \{x\}$. In case $A - \max A$ contains an option y different from x, then $\max A \succsim \max A \cup \{y\}$. Furthermore, independence implies

$$\max A \cup \{y\} \;\succsim\; \max A \cup \{x\} \cup \{y\}.$$

From transitivity it follows that $\max A \succsim \max A \cup \{x, y\}$. Repeat this argument for each z in $A - \max A$ and obtain $\max A \succsim A$.

Step 2. Let $a_1 I a_2 I \ldots I a_n$. Opportunity sets with k of these options are equally good.

Proof. If $k = 1$, then extension implies that $\{a_i\} \sim \{a_1\}$ for each $i = 2, 3, \ldots, n$. Next, we consider $k = 2$. Apply independence to $\{a_1\} \sim \{a_2\}$ and obtain that $\{a_1, a_i\} \sim \{a_2, a_i\}$ for each $i \notin \{1, 2\}$. Apply independence to $\{a_1\} \sim \{a_i\}$ with $i \neq 1$ and obtain that $\{a_1, a_j\} \sim \{a_i, a_j\}$ with $j \notin \{1, i\}$. Conclude that each pair $\{a_i, a_j\}$ of options $(i \neq j)$ is equally good as $\{a_1, a_2\}$. Continue this argument to obtain the statement.

Step 3. Conclude that $A \sim B$ and that $\succsim$ satisfies indifference.

Proof. From Step 1 we learn that $A \sim \max A$ and $B \sim \max B$. Furthermore, all options in $\max A \cup \max B$ are equally good and $|\max A| = |\max B|$. Step 2 implies that $\max A \sim \max B$. Use transitivity of $\succsim$ to conclude that $A \sim B$. □

Proposition 2. Let R be a complete and transitive relation on the set X. There is only one transitive and reflexive relation $\succsim$ on Ω that satisfies Simple monotonicity, Composition, Extension, Independence, and I-pair preference. It is the indirect-utility-freedom rule.

Proof. It is easy to check that $\succsim_{IF}$ satisfies the axioms. Next, let $\succsim$ satisfy the axioms. We have to prove that $\succsim$ coincides with $\succsim_{IF}$. We proceed in different steps.

Step 1. Simple monotonicity, Composition, Independence, and Extension imply

$$\max A\, P\, \max B \quad \Longrightarrow \quad A \succ B.$$

Proof. Let $a \in \max A$. For each b in B, we have $a\, P\, b$ and $\{a\} \succ \{b\}$ (use extension). In case $B = \{b\}$, we obtain $\{a\} \succ B$. Otherwise, let b and b' be two different options in B. Use independence to obtain $\{a, b'\} \succ \{b, b'\}$. In combination with $\{a\} \sim \{a, b'\}$ (use Lemma 2) it follows that $\{a\} \succ \{b, b'\}$. Repeat this argument for each option in B different from b and b' and, again, conclude that $\{a\} \succ B$. Monotonicity implies $A \succsim \{a\}$. From the transitivity of $\succsim$ it follows that $A \succ B$.

Step 2. Add I-pair preference to obtain

$$\max A\, I\, \max B \;\text{ and }\; |\max A| > |\max B| \quad \Longrightarrow \quad A \succ B.$$

Proof. Let A' be a strict subset of $\max A$ with cardinality equal to $|\max B|$. Lemma 2 (*ii*) implies that $B \sim A'$. Hence, it is sufficient to show that $A \succ A'$. Now, according to the I-pair preference principle we have $\{x, y\} \succ \{x\}$ with x in A' and y in $A - A'$. Independence, simple monotonicity, monotonicity (Lemma 2), and transitivity imply that $A \succ A'$.

Step 3. Conclusion.
Lemma 2 partitions the collection Ω into classes of opportunity sets that are equally good according to $\succsim_{IF}$. In order to complete the proof, it suffices to check whether these different partition classes are ordered in the 'right' way. This can be done by checking the following exhaustive list:

- $\max A\, P\, \max B$,
- $\max A\, I\, \max B$ and $|\max A| > |\max B|$,
- $\max A\, I\, \max B$ and $|\max A| = |\max B|$.

Step 1 (Step 2, Lemma 2) tackles the first (second, third) item. Conclude that the relation $\succsim$ coincides with $\succsim_{IF}$. □

Let us now compare the *indirect-utility-freedom* rule and the *indirect utility* method. Lemma 6 shows that the difference between I-pair *preference* and I-pair *indifference* is able to capture the difference between these two rules. Indeed, replace in Proposition 2 the axiom of I-pair preference by I-pair indifference. The following characterization shows up.

Lemma 6. Let R be a complete and transitive relation on X. There is only one transitive and reflexive relation on Ω that satisfies Simple monotonicity, Composition, Extension, Independence, and I-pair indifference. It is the indirect utility rule.

Proof. The indirect utility rule satisfies the five axioms. To show the converse, we first prove the following statement: for each A in Ω, we have

$$a_1 \in \max A \quad \Longrightarrow \quad A \sim \{a_1\}.$$

Let $A = \{a_1, a_2, a_3, \ldots, a_r\}$ with $a_1 R\, a_i$ for each $i = 2, 3, \ldots, r$. If $r = 1$, the statement is immediate. In case $r > 1$, then the following iteration holds. First, $\{a_1\} \sim \{a_1, a_i\}$ for each $i = 2, \ldots, r$. Indeed, we have either $a_1 I a_i$ or $a_1 P a_i$. The indifference between $\{a_1\}$ and $\{a_1, a_i\}$ follows from either I-pair indifference, or extension, composition, and simple monotonicity. Next, apply independence (add a_j) to obtain

$$\{a_1, a_i, a_j\} \sim \{a_1, a_j\}, \quad \text{with } i, j = 2, 3, \ldots, r \text{ and } i \neq j.$$

From the transitivity of $\succsim$ it follows that $\{a_1\}$ is indifferent to each triple subset $\{a_1, a_i, a_j\}$ of A. By repeated application of independence we obtain that $\{a_1\}$ and A are equally good. Now, use extension and conclude the lemma. □

Lemma 6 is in line with Foster's (2011) axiomatization of the indirect utility approach. In the above formulation of Lemma 6, the axiom of independence can be weakened to semi-independence:

Semi-independence. For each A and B in Ω and for each x in $X - (A \cup B)$, we have

$$[\max A\, R\, x] \implies [A \succsim B \implies A \cup \{x\} \succsim B \cup \{x\}].^2$$

Nevertheless, the above formulation of Lemma 6 is, for obvious reasons, intentional. Our characterizations of the indirect utility rule and the indirect-utility-freedom rule have four axioms in common: simple monotonicity, composition, extension, and independence. Let us impose these four axioms. The way in which a pair of equally good options is ranked against one of these options distinguishes the indirect utility rule from the indirect-utility-freedom rules. In case the level of well-being attached to a pair of equally good options is considered higher than the level attached to one of these options, the indirect-utility-freedom rule comes forward. In case such a pair is considered equally good as a singleton opportunity set, the indirect utility rule comes forward.

Remark 2. Pattanaik and Xu (1998) define the 'maximal set' of an opportunity set A as the collection of options in A that are best in terms of some preference on X that a 'reasonable' person may have. They propose to rank opportunity sets on the basis of the cardinalities of their 'maximal sets'. This rule can be retrieved as follows. Denote by $\mathcal{P}$ the collection of the reasonable preferences and define an approval utility function $u : X \to \{0, 1\}$ by $u(x) = 1$ if option x is R-maximal for at least one ordering R in $\mathcal{P}$ and $u(x) = 0$ otherwise. Then, the indirect-utility-freedom rule based upon this 'approval' preference relation (induced by u) coincides with the rule of Pattanaik and Xu.

[2] Starting from the axiom of independence, the condition $\max BRx$ is dropped and the 'if and only if' in the conclusion is weakened to 'if, then'.

Remark 3 (*Independence of the five axioms*)**.** Let us list again our five axioms: simple monotonicity, composition, extension, independence, and I-pair preference. For each axiom in this list we present a transitive and reflexive relation on Ω that violates this axiom and satisfies the other four axioms.

- Refine $\succsim_{IF}$ as follows:

$$A \succsim B \quad \text{if} \quad \begin{cases} A \succ_{IF} B \text{ or,} \\ A \sim_{IF} B \text{ and } |A^-| \leq |B^-|. \end{cases}$$

  According to $\succsim$, non-maximal elements attach a negative effect to the opportunity set. The ranking rule $\succsim$ violates simple monotonicity: if xPy, then $\{x\} \succ \{x, y\}$.

- The leximax rule of Bossert et al (1994)—see remark 4—ranks the pair $\{x, y\}$ with xPy above $\{x\}$ and, hence, violates composition.

- Let a and b be two options. Let R be a preference relation on X such that each option in $X - \{a, b\}$ is preferred to a and to b. Define a rule $\succsim$ on Ω as follows. The singletons $\{a\}$ and $\{b\}$ are not comparable. The remaining singletons are ordered according to R. Sets with two or more options are ordered according to $\succsim_{IF}$. This rule violates extension.

- Let $|X| \geq 3$. Restrict $\succsim_{IF}$ to the collection of singletons and pairs (hence, extension, simple monotonicity, and I-pair preference are satisfied). Let $\{a, b\}$ be an $\succsim_{IF}$-maximal pair. Consider all the other opportunity sets equally good as $\{a, b\}$. This rule violates independence.

- The indirect utility rule violates I-pair preference.

Remark 4 (*First best, second best, ..., leximax*)**.** We close this section by introducing a further refinement of the indirect-utility-freedom rule.

Let us recall the leximax rule (Bossert et al, 1994). Let X be the finite set of options and let R be a complete and transitive relation on X. Let $n = |X|$ be the number of different options in X. Let $u : X \to \mathbb{R}_{++}$ be a representation of the preference relation R on X, each option obtains a positive utility. For each opportunity set $A = \{a_1, a_2, \ldots, a_r\}$ in Ω, we define an n-vector

$$u(A) \;=\; (u(a_1), u(a_2), \ldots, u(a_r), 0, \ldots, 0),$$

that collects the utilities of the r options in A, extended with $n - r$ zeros. The ordering $\succsim_L$ on Ω is based upon the leximax ordering $\geq_L$ on $\mathbb{R}^n$: for each A and B in Ω,

$$A \succsim_L B \qquad \text{if} \qquad u(A) \geq_L u(B).$$

The ordering $\succsim_L$ does not depend upon the particular representation provided $u(x) > 0$ for each x in X.

The link between $\succsim_L$ and $\succsim_{IF}$ is as follows. The indirect-utility-freedom rule first cuts away from the opportunity set the alternatives that are not R-maximal sets and then compares the remaining sets by means of $\succsim_L$. More formally, $A \succsim_{IF} B$ if $(\max A) \succsim_L (\max B)$.

The rule $\succsim_{IF}$ can be further refined. The set $\max(A^-)$ collects the second best options in A. Let $\max_2(A) = \max(A) \cup \max(A^-)$ collect the first best and the second best options in the opportunity set A. Define the rule $\succsim_2$ as follows. For each pair A and B of opportunity sets, we have

$$A \succsim_2 B \qquad \text{if} \qquad \max_2(A) \succsim_L \max_2(B).$$

In the same spirit, one can define the $\max_k$-rule that takes the first, second, third, ..., and k^{th}-best options into account. The $\max_1$-rule corresponds to the indirect-utility-freedom rule; the $\max_n$-rule to the leximax rule.

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