FACULTY OF ECONOMICS AND
APPLIED ECONOMIC SCIENCES
CENTER FOR ECONOMIC STUDIES
ENERGY, TRANSPORT & ENVIRONMENT

KATHOLIEKE
UNIVERSITEIT
LEUVEN

WORKING PAPER SERIES
n°2001-21

Pricing a stock-constrained congestible facility

Edward Calthrop
(K.U.Leuven-CES-ETE)

September 2001

contact:
Isabelle Benoit
KULeuven-CES
Naamsestraat 69, B-3000 Leuven (Belgium)
tel: +32 (0) 16 32.66.33
fax: +32 (0) 16 32.69.10
e-mail: Isabelle.Benoit@econ.kuleuven.ac.be
http://www.kuleuven.ac.be/ete

Pricing a stock-constrained congestible facility*

Edward Calthrop

Catholic University Leuven

September 2001

Abstract

This paper develops pricing rules for stock-constrained congestible facilities, such as an urban parking lot, swimming pool or museum. Pricing schedules optimally comprise of two components. Firstly, a per-time unit fee induces consumers to spend an efficient length of time within a facility. Secondly, an arrival time fee induces consumers to spread arrivals efficiently across the peak-period. The pricing rules are illustrated with a small numerical example. JEL H42;R48.

*This paper is based on chapter 5 of my doctoral thesis (Calthrop, [3]). It has been greatly improved by the comments of the committee including, in particular, Stef Proost, Piet Rietveld and Erik Schokkaert. Usual disclaimers apply. *Author's mailing address*: CES, K.U.Leuven, Naamsestraat 69, B3000 Leuven, Belgium. Tel:00-32-16-32-66-53; Fax: 00-32-16-32-67-96; e-mail: *Edward.Calthrop@econ.kuleuven.ac.be*

1 Introduction

Economists often model congestion in a 'reduced-form' manner. For instance, in club theory, it is often assumed that the average user benefit from joining a congestion-prone club declines as the number of other users increases. Or, in transport economics, the average driving speed is assumed to fall as more people use the road.

In recent times, however, authors have modelled the congestion technology in an explicit manner. One well known model is the bottleneck model of road congestion, which explicitly models the flow constraint on the road. The width of the road is such that only s cars per hour can pass through the bottleneck.

This paper concentrates on a third formulation: a congestible facility which is subject to a stock constraint. For example, there are only S parking spots available in the central area of a city. Similarly, fire regulations may permit a maximum of S people inside a museum at any one time, or a swimming pool. Expected wait time to use the facility at time t depends on the time path of both the arrivals up until t and the length of time each of these users remains inside the facility. In full generality this a complex problem. Fortunately, a subset of the theory on queueing provides the mechanics to simplify the analysis. This paper derives an optimal pricing function for a congestible facility. The pricing schedule comprises of two components: a per-time-unit of stay fee induces users to remain in the facility for the optimal length of time; while a time-varying arrival fee induces drivers to schedule arrivals optimally over the peak period.

Section 3 uses a single-time period model to derive the optimal per unit time fee. Section 4 expands this model to a multi-period problem, in which users choose both the length of time to remain within the facility, and schedule a time to arrive at the facility.

Unfortunately, closed-form solutions for the optimal per time unit fee and arrival fee cannot be derived. Section 5 presents a numerical model that illustrates the theory. Section 6 concludes.

2 Literature review

Table 1 distinguishes various characteristics which define the generalised form of the standard bottleneck model (Type 1)[1]. A variable number of heterogenous consumers decide whether to use a facility or not and at what time to join the queue. The congestion takes the form of delays rather than service denial. The model is therefore not suitable to a concert in which tickets may sell out and further applications for service are denied. Access to the facility is on a First-In, First-Out basis (FIFO). There is no possibility for jumping ahead in the queue. As stressed by Arnott and Kraus, [2], the state variable determining the evolution of the queue to time $t$is based only on arrivals at time $t$plus the time path of arrivals during all previous time periods. This makes the model unsuitable to a network traffic model, where earlier departing commuters may be held up by late departing local residents. Finally, the congestion is caused by a flow constraint of the facility: only $s$users can pass through per unit time, and if arrivals are greater than s, deterministic queues develop. Optimal pricing is achieved via a continuously time-dependent fee, $F(t)$. Examples of facilities to which this model may be well suited include a single-link road (with no overtaking), or a golf course without a reservation scheme, or a large capacity no reservation concert.

[1] A detailed discussion of the generalisation of the bottleneck model (Type 1) is found in Arnott and Kraus [2].

	Type 1	Type 2	Type 3
authors	[9]; [1]; [2]; [6]	[8]; [5]; [6]	this paper
users	heterogenous	identical	identical
decisions	to visit or not when to use	 to schedule use	to visit or not to schedule use how long to stay
congestion	deterministic delay	stochastic delay	stochastic delay
queue	FIFO	FIFO	FIFO
constraint	flow	stock	stock
fee	$F(t)$	$\sum_t p_t^{s_i} F_t$	$\sum_t p_t^{s_i} F_t + ml\,(m)$
e.g.	single-link road no res golf no res concert	hub airports	parking swimming pools art galleries

Table 1: Characteristics of bottleneck models

Daniel [5] applies the bottleneck equilibrium concept to a stochastic queueing model of a hub airport. Sdeparture and landing lanes - i.e. a stock constraint - determines expected queues at discrete time intervals via a non-stationary stochastic queueing model $(M(t)/D/S/K^2)$, developed by Koopmans [8]. This is shown in the column marked 'Type 2' in Table 1. A fixed number of users (who may act collusively) decide when to schedule arrivals. Each user is subject to a

[2] In standard Kendall-Lee Notation for queueing systems, M denotes that interarrival times are independent, identically distributed random variables having an exponential distribution - the dependence on t refers to the time dependent evolution of the queue; D denotes that service times (the length of stay) are iid and deterministic; S denotes the number of parallel servers while K denotes the maximum number of customers in the system. Further description is given in section 4 below.

shock such that actual arrival time is given by a known distribution around scheduled arrival time. The model is solved for an equilibrium vector of scheduled arrival times such that no user has an incentive to shift scheduled arrival time. A comparison of empirical results for an airport using a standard peak-load pricing model, a deterministic bottleneck model and a stochastic bottleneck model is presented in Daniel and Pahwa [6].

This paper generalises Daniel's model and is shown as column 'Type 3' in Table 1. Firstly, each user must decide how long to remain within the facility. Secondly, variable demand is introduced such that an infinite number of potential users decide whether or not to use the facility. In equilibrium, the number of users is such that the consumer surplus from using the facility equals the sum of expected schedule delay and queueing costs. If the consumer surplus is greater than the expected cost of using the facility, additional users gain by using the facility. Likewise, if the cost of using the facility is greater than the consumer surplus, some users decide not to use the facility.

The welfare maximising fee includes a component related to actual arrival time and a component related to length of stay within the facility. The separability of these decisions implies that a two-part tariff pricing scheme is optimal. Before turning to the full model, however, it is useful to see the mechanics of a simpler queueing model in a single time period version of the model.

3 A single-period version: a M/D/1 model

I consider an economy with an infinite number of identical risk-neutral consumers. In a single time-period of time of length t, consumers decide whether or not to use a congestible facility. A consumer decides to use the facility as long as the net benefit from making the trip is non-negative. Each person's benefit from the trip varies in function of the time spent within the facility. The facility has a fixed capacity (set for convenience equal to 1spot) - and the visitor may have to wait to gain entrance into the facility.

In order to use the simplifying mathematics of queueing theory, I assume that the actual arrival time of any user differs from planned arrival time: each consumer is subject to a shock given by an independent and identically distributed random variable such that the number of actual arrivals to occur in a time interval of $t$follows a Poisson-distribution with parameter nt, where $n$denotes the number of facility users within the single time period. Each potential user is therefore unsure of his or her actual arrival position relative to other users within the time period. Without loss of generality, I set $t = 1$.

The length of time a user spends within the facility is a deterministic variable given by ℓ. The marginal benefit of a unit of time in the congestible facility is given by $1 - q$, where $q$gives the time spent within the facility. Hence the consumer surplus, CS, conditional on gaining access to the facility, from a stay of ℓunits of time, at price $m$per unit time ($0 \leq m < 1$), is given by:

$$CS[\ell] = \int_0^{\ell} (1 - q - m)dq \tag{1}$$

It is clear that an individual maximises consumer surplus by staying in the facility for $\ell(m) = 1 - m$units of time.

Given the above assumptions, and applying the standard Pollaczek-Khintchine formulas[3] (Winston, [10] pp.1147 equation 48) expected

[3] With a deterministic service rate, expected waiting time EW is given as:

$$EW[\lambda, \mu] = \frac{\lambda}{2(1-\rho)\mu^2}$$

where λ is the arrival rate, μ is the service rate (equal to one over the length of stay) and $\rho = \frac{\lambda}{\mu}$.

waiting time $EW$to use a single-server facility is given by[4]:

$$EW\left[n,m\right]=\frac{\ell\left[m\right]^{2}n}{2\left(1-n\ell\left[m\right]\right)} \tag{2}$$

This waiting time function is strictly convex in arrival rate and in user time, as long as the fee is such that users wish to park for a strictly positive length of time. This implies that the facility is subject to congestion: average wait time for existing users rises if one more consumer decides to use the facility. The expected queueing cost per person of using the facility is given by:

$$\kappa\left[n,m\right]=\alpha EW\left[n,m\right] \tag{3}$$

where $\alpha$is the value of time spent in the queue (henceforth set equal to $1$for ease of notation). The expected net benefit of visiting the facility, NB, can be written as a function of the number of users and the per time unit fee:

$$NB\left[n,m\right]=CS\left[m\right]-\kappa\left[n,m\right] \tag{4}$$

Each consumer decides whether to use the facility or not. The equilibrium number of consumers, n^{*}, is given by the solution to the following equation:

$$CS\left[m\right]-\kappa\left[m,n^{*}\right]=0 \tag{5}$$

Consumers decide to use the facility as long as the expected benefits from doing so outweigh the expected costs. In equilibrium, the

[4]Strictly speaking, this expression only holds if $\lambda\ell\leq 1$ - otherwise the queueing system 'explodes'. The condition holds in equilibrium as can be seen from expanding $n^{*}\left[m\right]\ell^{*}\left[m\right]$ via equations 6 and **??**, which is less than one.

expected queueing costs exactly match the consumer surplus gained from using the facility: no-one individual has the incentive to switch from using to not using the facility and vice-versa[5].

Equations 5, 3, 1 and 2 can be used to solve for an equilibrium mean number of facility users, n^*, as a function of m. This is given as:

$$n^*[m] = \frac{1}{2-m} \tag{6}$$

The impact of increasing $m$on the equilibrium arrival rate depends on two opposing effects. Increasing $m$reduces the amount of time spent in the facility, $\ell[m]$. Consumer surplus falls. For a fixed queueing cost function, therefore, the equilibrium number of arrivals falls. But as each user spends less time in the facility, expected waiting costs decline, thus decreasing the cost of making a trip. By taking the first derivative of the equilibrium arrival rate, it can be shown that the net effect of raising the per time unit fee is to increase the arrival rate: $n^{*'}[m] > 0$[6].

Figure 1 shows expected costs and benefits from a trip as a function of the number of facility users. The subscript L denotes a relatively low per time unit fee of 0.25, while the subscript H denotes a higher fee of 0.5 per time period.

Expected queueuing cost per user, κ, rises as a function of the number of users, while the consumer surplus from visiting the facility is independent of the number of other users. For a relatively

[5] One is reminded of the story of two people discussing a new restaurant. One says to the other: 'no one goes there anymore - it is too crowded'.

[6] This point is also made by Glazer and Niskanen [7] in the last section of their paper. They use this result to argue that raising per time unit meter fees may result in increased arrivals and hence increase congestion levels. However, they do not consider the feedback of higher congestion levels on the equilibrium number of arrivals. Nor do they discuss the optimal meter fee.

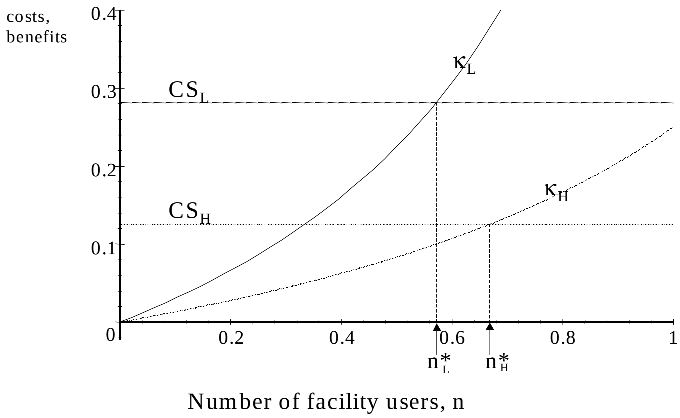

Figure 1: The equilibrium number of facility users

low per time unit fee, expected queueing costs entirely offset consumer surplus once n^*_L users make the trip. Increasing the fee reduces consumer surplus. For a given cost function, this would reduce the equilibrium number of facility users. But, by inducing people to use the facility for a shorter period of time, the cost function falls to κ_H. The reduction in expected queueing cost is sufficiently large to outweigh the fall in consumer surplus: the equilibrium number of users increases to n^*_H.

As a result of the endogenous arrival rate given in equation 5, social welfare, SW, is given by:

$$SW\left[m\right] = n^*\left[m\right] m\ell\left[m\right] = \frac{m(1-m)}{2-m} \tag{7}$$

Social welfare is equivalent to total revenue: it follows that a private operator that maximises profit sets a meter fee at the socially-optimal level. This is a striking result. It follows directly from the endogenous arrival rate. If no fee is charged, social welfare is zero.

Either individuals use the facility - in which case, in equilibrium, the expected queueing cost exactly outweighs the consumer surplus - or they do not. Either way, social welfare per person is zero. At any positive meter fee level, there is a wedge between the consumer surplus and the social surplus from using the facility. Expected queueing costs exactly offset the consumer surplus, leaving the difference between social and consumer surplus (i.e. the tax revenue) as a measure of welfare.

Maximising equation 7 with respect to the meter fee, gives the optimal per time unit fee as:

$$m^* = 2 - \sqrt{2} = 0.59 \tag{8}$$

A second point follows from the formulation of the social welfare function in expression 7. Any level of time restriction - a common alternative to a meter fee - results in zero social welfare[7]. For each successive tightening of the restriction, more new users will arrive at the facility such that social surplus is zero. This can be seen directly by setting $m = 0$in equation 7 above.

4 A multi-period version: a $M[t]/D/S/K$ model

The previous model is limited. It assumes a single time period, while in reality customers can also decide whether to use the facility during busier periods or not. This section applies non-stationary queueing theory to derive a bottleneck equilibrium: each consumer trades off expected queueing costs and schedule delay costs such that

[7] The relative inefficiency of non-decreasing non-linear pricing mechanisms as a means of rationing the urban on-street parking market is discussed in Calthrop and Proost [4]. In that paper, the additional consumer surplus resulting from a time restriction, in comparison with a meter fee, induces excessive search behaviour.

in the equilibrium, no individual has an incentive to alter scheduled arrival time. As in the previous section, once a consumer gains access to the facility, she decides how long to stay as a function of the per time unit fee. A notional planner is assumed to maximise social welfare with respect to two instruments: a fee dependent on time of arrival into the queue, plus a linear fee per time unit spent within the facility. Given this restriction on instruments, welfare is optimised with a time dependent fee spreading arrivals efficiently across the peak period, and a per-time unit linear fee inducing users to stay within the facility for an optimal period of time.

I assume that the evolution of the queue over time is given by an $M[t]/D/S/K$ queueing system. In standard Kendall-Lee Notation for queueing systems, Mdenotes that interarrival times are independent, identically distributed random variables having an exponential distribution - the dependence on $t$refers to the time dependent evolution of the queue; Ddenotes that service times (the length of stay) are iid and deterministic; Sdenotes the number of parallel servers while Kdenotes the maximum number of customers in the system. I assume that after a person exits from the facility another user can enter whom in turn exits after ℓunits of time. A sequence of points $t_1, t_2, ...$ marked on the time axis can be imagined, all spaced at distances ℓ apart representing the deterministic epochs of service completion. Following the discussion in Koopmans [8], I confine attention to the queue at these points: at each one, if the number in the queue had previously been positive, just Swill be removed. The number of arrivals in each service period is approximately Poisson distributed with time varying parameter, λ_t. A state vector, q_t, represents the state of the queueing system at each period t. It consists of a column of elements, $q_{t0}, q_{t1}, ..., q_{tK}$which give the probability the queue is of length $k$at the beginning of the period t. The state vector evolves according to the transition rule:

$$q_{t+1} = Q_t q_t \tag{9}$$

where $Q_t$is the state-transition matrix determined by the $M[t]/D/S/K$ queueing model and the arrival rates λ_t. Let $n$denote the number of drivers that choose to use the facility. I denote the column vector of scheduled arrival times by s(of dimension n). The probability of a customer i, scheduled to arrive at time s^i,actually arriving at time $t$is given by $p_t^{s^i}$. The average arrival rate is given by summation of expected arrivals at any time t :

$$\lambda_t = \sum_{i=1}^{n} p_t^{s_i} \tag{10}$$

Appendix A gives more details on the state-transition matrix $Q_t[\ell, \mathbf{s}, n]$. To aid comparability, the notation is chosen to follow as closely as possible the model in Daniel [5].

4.1 The expected cost of accessing the facility

The expected cost for consumer $i(i = 1, ..., n)$of gaining access to the facility is again given by $\kappa_i[\cdot]$which is the sum of expected queueing costs, schedule delay costs and an arrival fee, and is defined below in equation 12.

Define the expected waiting time for a user who arrives at time $t$with $k$other people in the queue as $w[k]$. All consumers wish to use the facility at the same time, t^*. Schedule delay costs are given by:

$$SDC = \begin{cases} \beta(t^* - t - w[k]) & \text{if} \quad t^* - t - w[k] \geq 0 \\ \gamma(t + w[k] - t^*) & \text{otherwise} \end{cases}$$

where $\beta$is the value of time when arriving earlier than desired at the facility and $\gamma$is the corresponding measure when arriving later than desired[8]. To ease notation, I introduce a dummy variable:

$$d = \begin{cases} 1 & \text{if} \quad t^* - t - w[k] \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The expected queueing and schedule delay costs of arriving at time $t$with queue $k$is given by:

$$c_{tk} = \alpha w[k] + d\beta(t^* - t - w[k]) + (1-d)\gamma(t + w[k] - t^*)$$

where, as in the previous section, αdenotes the value of time spent queueing. The vector of costs for different queue lengths $k = 0, .., K$is given by c_t. For any given number of customers, n, the expected cost of arriving at t, $C_t[\cdot]$is given by:

$$C_t[m, \mathbf{s}, n] = c_t q_t[m, \mathbf{s}, n] \tag{11}$$

The expected consumer cost of scheduling to arrive at s^i, κ_i, is given by:

$$\kappa_i[m, \mathbf{s}, n, \mathbf{F}] = \sum_t p_t^{s_i} \{C_t[m, \mathbf{s}, n] + F_t\} \tag{12}$$

where $p_t^{s_i}$, as defined earlier, gives the probability of a user scheduled to arrive at time s^iarriving at time t. The vector Fgives the arrival fee, F_t, for all possible arrival times.

$\kappa_i$gives the cost of gaining access to the facility for any individual i, for any number of users, n; for any vector of scheduled arrival times, s, and any vector of arrival fees, $F$and per time unit fee m.

[8]It is assumed that schedule delay costs take a linear form: this keeps the analysis significantly simpler.

4.2 Consumers' problem

Each consumer maximises the expected net benefit of a trip with respect to the scheduled time of arrival and length of stay. The consumer surplus from time spent within the facility is given as in the section above, and thus we can write $\ell[m]$. The net benefit from making a trip is therefore given by:

$$NB[\cdot] = CS[m] - \kappa_i [m, \mathbf{s}, n, \mathbf{F}]$$

Each consumer takes the cost $C_t[\cdot]$as parametric: thus ignoring his or her effect on expected waiting and schedule delay costs of other facility users. For any given number of facility users, the optimisation problem is given by:

$$\underset{s^i}{Max} NB[m, \mathbf{s}, n, \mathbf{F}] \tag{13}$$

The first-order condition with respect to scheduled arrival time is such that:

$$s^i : \sum_t \frac{\partial p_t^{s^i}}{\partial s^i} \{C_t[m, \mathbf{s}, n] + F_t\} = 0 \tag{14}$$

A consumer equilibrium (in scheduled arrival times) is given by the following conditions. For a given level of $\alpha, \beta,$and γ, the following equations must hold: 10, 9, 11, plus:

$$\kappa^i[m, \mathbf{s}, n, \mathbf{F}] = \sum_t p_t^{s^i} \{C_t[m, \mathbf{s}, n] + F_t\} \leq \sum_t p_t^{s^{i'}} \{C_t[m, \mathbf{s}, n] + F_t\} \text{ for all } s^{i'} \tag{15}$$

which implies that:

$$\kappa^i[m, \mathbf{s}, n, \mathbf{F}] = \kappa[m, n, \mathbf{F}] \text{ for all i}$$

In a bottleneck (Nash) equilibrium, no one user can adjust his or her scheduled arrival time to reduce expected queueing costs, scheduled delay costs and entrance fee. In equilibrium, therefore, private user cost $\kappa$is equal for all users.

As in the equation 5, it is straightforward to solve for the equilibrium number of arrivals. Given a bottleneck equilibrium in scheduling times and optimal length of use, the equilibrium number of arrivals, $n^*[m, \mathbf{F}]$, is given implicitly by:

$$\kappa[m, n^*, \mathbf{F}] = CS[m] \tag{16}$$

4.3 Social welfare

Using the equilibrium number of arrivals given in condition 16, (expected) social welfare is given by:

$$SW[m, \mathbf{F}] = n^*[m, \mathbf{F}]\, m(1-m) + \sum_{0}^{n^*[m,\mathbf{F}]} \sum_t p_t^{s_i} F_t \tag{17}$$

As in the single time-period version of the model, social welfare is given by total revenue - the difference between social and consumer surplus on this market.

The social planner can maximise this objective function with respect to the per time unit fee, $m$and the vector of arrival fees, F. Taking the first-order conditions with respect to the meter fee reveals:

$$n^*(1-2m) + n_m \left\{ m(1-m) + \sum_t p_t^{s_n} F_t \right\} + \sum_0^n \sum_t \frac{\partial p_t^{s_i}}{\partial s^i} \frac{\partial s^i}{\partial m} F_t = 0 \tag{18}$$

where $n_m$denotes the derivative of the equilibrium number of arrivals with respect to the per time unit fee. Marginally increasing the meter fee from a low level increase revenues collected from the original set of facility users - this is captured in the first term. The second term shows that account needs to be taken of the effect of increasing meter fees on the equilibrium number of arrivals. The sign of this term is not obvious a priori. By reducing mean length of stay, mean queue length decreases and for a given number of users and arrival fees, the value of κfalls. However, so does the consumer surplus from making a trip. Whether more or less drivers will use the facility in the new equilibrium depends on the net impact of these two opposing effects. The third effect gives the impact on total arrival fee revenues from the induced changes in scheduled arrival time that result from the change in mean length of usage.

The first-order condition for a particular arrival time fee $F_t$is given by:

$$n_{F_t}\left\{m(1-m)+\sum_t p_t^{s_n}F_t\right\}+\sum_0^n\sum_k \frac{\partial p_k^{s_i}}{\partial s^i}\frac{\partial s^i}{\partial F_t}F_k di=0 \qquad (19)$$

The marginal increase in an arrival fee at time $t$alters the equilibrium consumer surplus from drivers and thus the total number of arrivals. The first term captures the change in revenues from any change in the number of facility users. The second term captures the effect of altered scheduling on expected arrival fee revenues.

5 Numerical example

The system of first order conditions given by equations 18 and 19, do not lend themselves to a closed form solution solution for the optimal tax rates. In order to illustrate the theory, I present a small numerical example. The model is designed to illustrate the mechanisms at work.

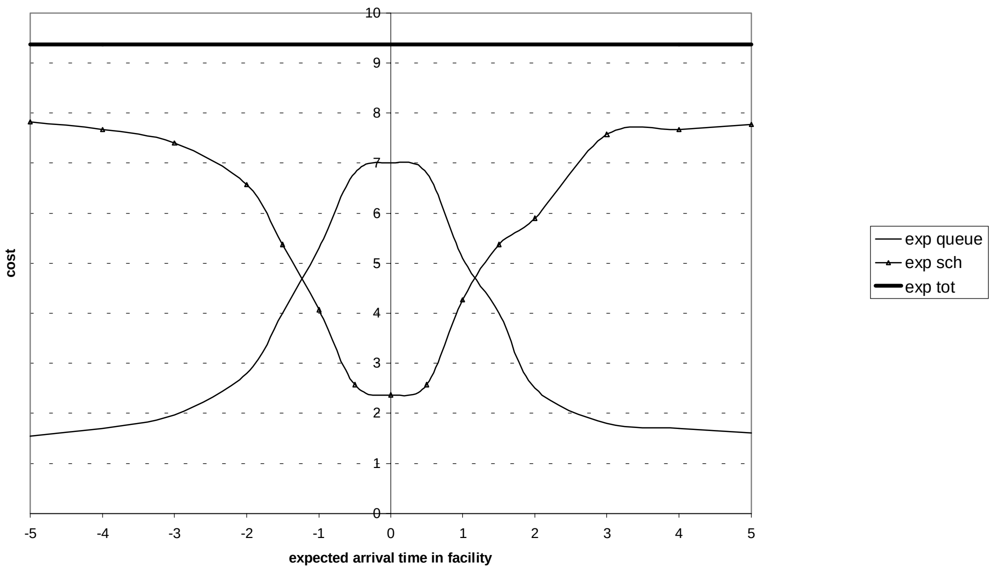

Figure 2: no fee bottleneck eqm

To simplify matters, I consider an $M[t]/D/1/20$queueing model, although the number of servers and queue limit can easily be extended. Desired time of usage of the facility, t^*, is set equal to 0.

5.1 The no-fee consumer bottleneck equilibrium

In the absence of a per time unit fee, each consumer uses the facility for $\ell[0]$time units, given by 1. Figure 2 gives a equilibrium pattern of scheduled arrivals for N consumers such that $CS[0] = \kappa_i[0, n[0,0], 0]$for all i. The expected combined queueing and schedule delay costs equals 9.09 for all scheduled arrival times. The consumer that arrives at the facility closest to the desired arrival time faces the lowest expected schedule delay costs. Hence in equilibrium, this consumer must face the highest expected queueing costs. Note that the first arrival at the facility occurs at around -5 and the last at around 5.

5.2 Optimal fee structure

The government sets a uniform meter fee, $m$and a vector of arrival fees, F, to maximise social welfare. The numerical model can be used to illustrate the effect of each tax separately. First, consider that demand remains fixed at $n\,[0,0]$and the government uses the vector of arrival tolls to minimise social costs of using the facility. Figure 3 shows the result. The upper curve shows the combined queueing and schedule delay costs for each individual. This cost is smallest for the consumer arriving in the facility at the desired arrival time. It is clear that without an arrival fee, this distribution of arrival times could not be in equilibrium. A consumer arriving at the edge of the peak would have the incentive to switch to arrive nearer the desired time of usage. The result of this process would be the distribution of arrivals given in the no-fee equilibrium in Figure 2 above.

Figure 4 shows the arrival fee necessary to decentralise the desired distribution of arrivals. Note that the fee is based on the time of arrival into the queue and not into the facility. With a first-come first served queue discipline, any fee schedule based on arrival time into the queue can be translated into a fee schedule based on arrival time in the facility.

The optimal scheduling of arrivals results in the average social cost of queueing and schedule delay costs falling from 9.09to 6.47. This occurs in part by spreading the arrivals over a longer period. Under optimal arrival-toll equilibrium, the first arrival occurs almost at -10, whilst the last occurs at 10. The peak-period is almost twice the length of the no-fee equilibrium peak. This demonstrates the importance of using a time-of-arrival dependent fee to spread arrivals away from the desired time of usage. This corresponds to the results derived in the context of airport congestion pricing by Daniel [5] and

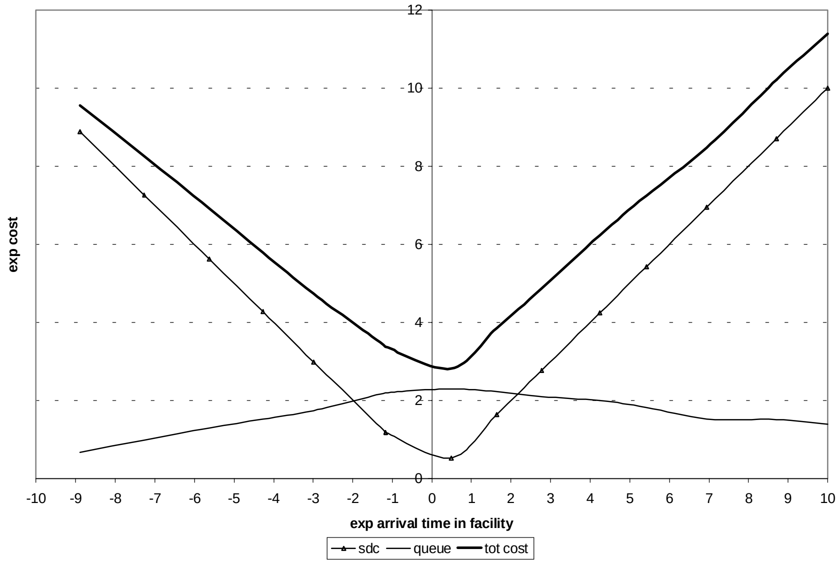

Figure 3: social optimal scheduling

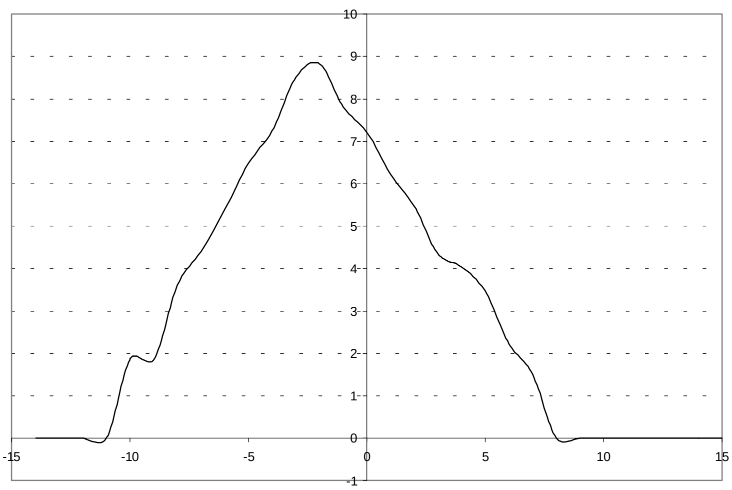

Figure 4: Time of arrival fee schedule

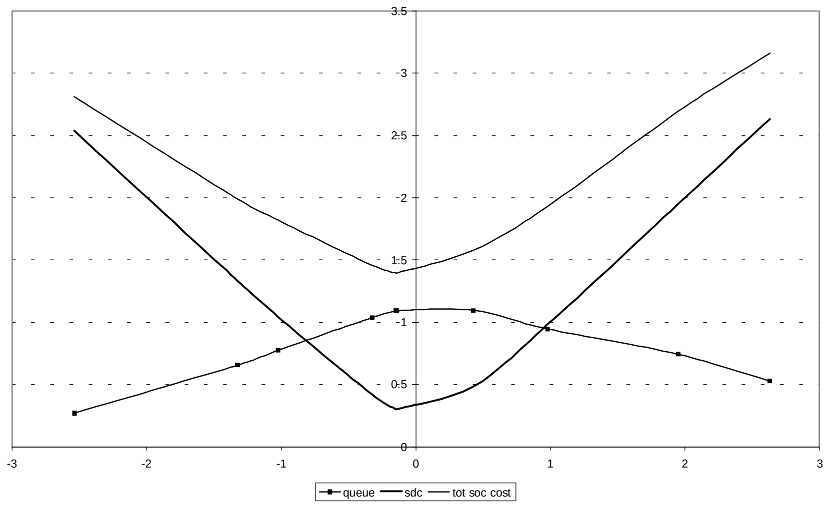

Figure 5: Social optimal scheduling

Daniel and Pahwa [6]. Secondly, the equilibrium average private cost of using the facility rises from the no-fee equilibrium of 9.09to 11.4with the optimal arrival fee schedule. However, the net consumer benefit from using the facility, $CS\,[0]$remains constant. Hence, by equation 16, the equilibrium number of users must fall such that average private cost equals $CS\,[0]$. Further computations show that the equilibrium number of users is approximately 80%of those in the no-arrival fee equilibrium, $n\,[0,0]$.

Figure 5 shows the equilibrium distribution of arrivals when the per time unit fee rises from zero to 0.4. Each consumer uses the facility for a shorter period of time, and hence the net consumer benefit from using the facility, $CS\,[0.4]$falls to 3.27. With an equilibrium number of users, this must also equal average equilibrium private user cost, $\kappa\,[0.4, n^*, \mathbf{F}]$. The model results indicate that the optimal number of users falls to approximately 73% of $n\,[0,0]$. The average social cost of using the facility is 2.05.

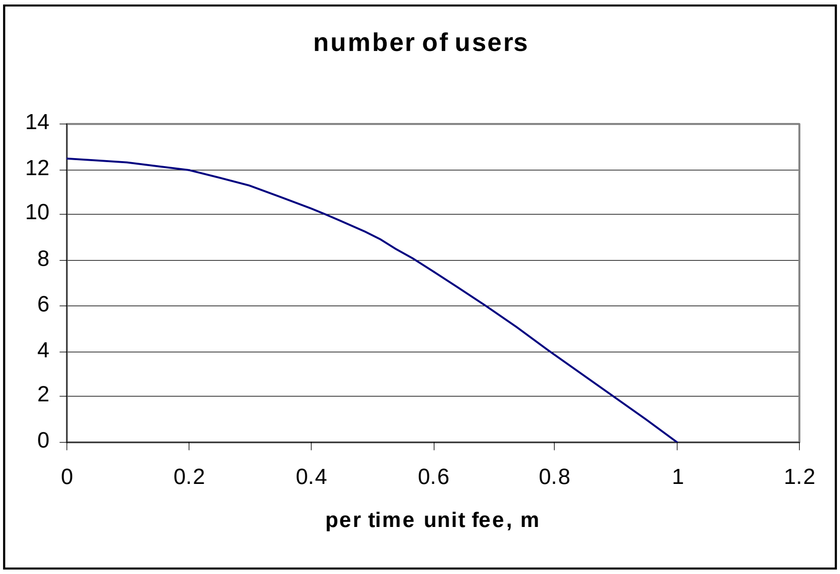

Figure 6: Equilibrium number of facility users

Figure 6 presents the model results on the relationship between the per time unit fee (with optimal arrival fee schedule) and the equilibrium number of users. Recall from the simplified model above (section 3) that for the $M/D/1$model, $n^{'}[m] > 0$.Raising per time unit fees leads to a reduction in consumer surplus (for a fixed number of users), but this is more than offset by the reduction in expected queueing costs from the reduction in average length of stay. Hence the equilibrium number of users rises. In this model, increasing meter fees also leads to a reduction in consumer surplus for a given number of users. With a more complicated queueing technology, however, the reduction in expected queueing costs and schedule delay costs from average shorter stay is insufficient to offset the loss in consumer surplus. Hence the equilibrium number of users falls as the per time unit fee rises.

Finally, Figure 7 shows the relationship between total revenues

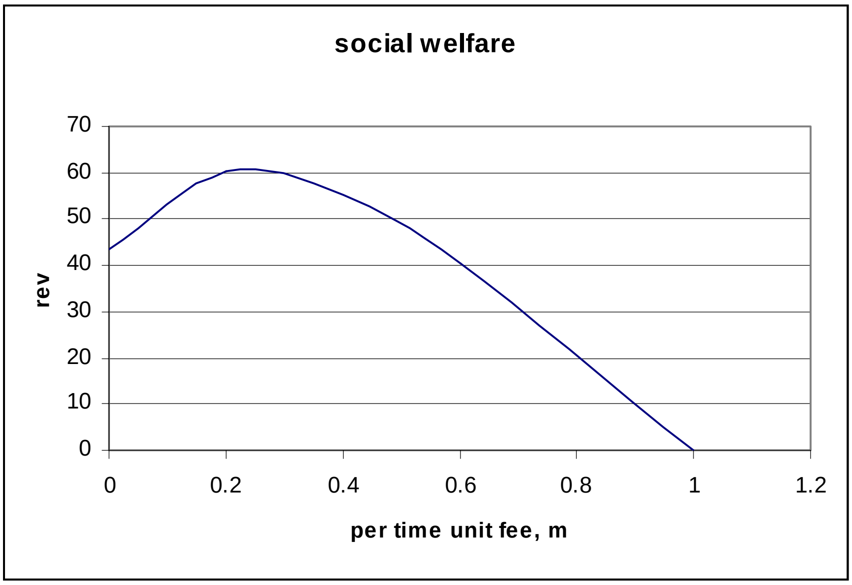

Figure 7: Social welfare

collected and the per time unit fee (with optimal arrival fee schedule and equilibrium number of users). Recall from equation 17 that social welfare can be measured by total revenues collected. Hence Figure 7 shows the optimal per time meter fee is approximately 0.25.

6 Conclusion

This paper has extended the stochastic bottleneck model to include an additional important margin of consumer behaviour: namely, how long to remain within a congestible facility, and for elastic number of users. The model is therefore suitable to a broader range of congestible facilities than previously considered. In setting optimal fees for urban parking spaces, public swimming pools or art galleries, it is clear that how long customers spend within the facility affects the probability of having to queue. In addition, the fee structure also needs to give sufficient incentive for customers to spread-out their planned arrivals around any desired time of usage. The optimal two-part tariff fee structure is captured in the solution to the two first order conditions given in 19 and 18. This seen more clearly in the numerical example. Customers would pay a per time unit fee equal to 0.25and an additional access fee dependent on time of entry into the queue.

Some caveats on the main findings are required. Firstly, the model assumes a single-preferred time of use for the facility. This is a simplification and further work should examine the implications of relaxing this assumption and allowing for a distribution of desired arrival times. More generally, the assumption of identical individuals needs to be relaxed.

Secondly, the model restricts government (or a private operator) to using a linear meter fee independent of time of arrival. The basic optimality result of a two-part tariff depends on this restriction. More general non-linear pricing schemes will be considered.

7 Appendix A

Define $a = \lambda_t \ell$[9]. The state-transition matrix of the queueing system is given by:

$$Q_t = \begin{bmatrix} e^{-a} & \ldots & e^{-a} & 0 & 0 & \ldots & 0 \\ ae^{-a} & \ldots & ae^{-a} & e^{-a} & 0 & \ldots & 0 \\ \frac{a^2}{2!}e^{-a} & \ldots & \frac{a^2}{2!}e^{-a} & ae^{-a} & e^{-a} & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{a^{K-1}}{(K-1)!}e^{-a} & \ldots & \frac{a^{K-1}}{(K-1)!}e^{-a} & \frac{a^{K-2}}{(K-2)!}e^{-a} & \frac{a^{K-3}}{(K-3)!}e^{-a} & \ldots & \frac{a^{S-1}}{(S-1)!}e^{-a} \\ u_K & \ldots & u_K & u_{K-1} & & \ldots & u_S \end{bmatrix}$$

The first three columns represent the first $S+1$ columns in the matrix Q_t.

[9] To ease notation, I do not indicate that a is a function of the expected number of arrivals in each time period λ_t and the length of time that each user stays for ℓ.

Element ij (where $i < K; j \leq S+1$) gives the probability (assuming a Poisson distribution of arrivals) of $i-1$ arrivals occuring during a period of length ℓ with an average arrival rate of λ_t. The remaining four columns represent the other $K-S$ columns of the matrix. Element ij (where $i < K; j > S+1$) gives the probability of $i-j+S$ arrivals.

Consider the probability of n people being in the queue at time $t+1$. This event can evolve from the period t in a finite number of ways only. For instance, 0 people in the queue at time t and n arrivals during the ℓ units of time between t and $t+1$. Equally, it can occur with 1 person in the queue at time t (who then enters the facility at t) and n arrivals. Or 2 people at time t and $n-1$ arrivals. And so forth.

The state-transition matrix gives the relevant probabilities of various numbers of arrivals during a period of length ℓ. Multiplying this by the vector of probable states of the queue in period t gives the vector of probable queue lengths in period $t+1$.

References

[1] Arnott, R., A.de Palma and R.Lindsey, 1993, A structural model of peak-period congestion: a traffic model with elastic demand, *American Economic Review*, 83(1), pp.161-179.

[2] Arnott, R. and M.Kraus, 1998, When are anonymous congestion charges consistent with marginal cost pricing?, *Journal of Public Economics*, 67, 45-64.

[3] Calthrop, E., 2001, *Essays in urban transport economics*, K.U.Leuven.

[4] Calthrop, E. and S.Proost, 2000, *Regulating urban parking*, Centre for Economic Studies Discussion Paper, Public Economics, DPS 00.21, K.U.Leuven (revised version September 2001).

[5] Daniel, J.I., 1995, Congestion pricing and capacity of large hub airports: a bottleneck model with stochastic queues, *Econometrica*, 63, 2, 327-370.

[6] Daniel, J.I and M.Pahwa, 2000, Comparison of Three Empirical Models of Airport Congestion Pricing, *Journal of Urban Economics*, 47, pp.1-38.

[7] Glazer, A. and E.Niskanen, 1992, Parking fees and congestion, *Regional Science and Urban Economics*, 22, pp.123-132.

[8] Koopmans, B.O., 1972, Air-Terminal Queues under Time-Dependent Conditions, *Operations Research*, 20, 1089-1114.

[9] Vickrey, W.S., 1969, Congestion theory and transport investment, *American Economic Review Proceedings*, 59, 251-260.

[10] Winston, 1990, *Operations Research: Applications and Algorithms*, 3^{rd} edition.

The Center for Economic Studies (CES) is the research division of the Department of Economics of the Katholieke Universiteit Leuven. The CES research department employs some 100 people. The division Energy, Transport & Environment (ETE) currently consists of about 15 full time researchers. The general aim of ETE is to apply state of the art economic theory to current policy issues at the Flemish, Belgian and European level. An important asset of ETE is its extensive portfolio of numerical partial and general equilibrium models for the assessment of transport, energy and environmental policies.

WORKING PAPER SERIES

n° 2001-21 Calthrop, E. (2001), Pricing a stock-constrained congestible facility

n° 2001-20 S. Proost, K. Van Dender, C. Courcelle, B. De Borger, J. Peirson, D. Sharp, R. Vickerman, E. Gibbons, M. O'Mahony, Q. Heaney, J. Van den Bergh, E. Verhoef (2001), How large is the gap between present and efficient transport prices in Europe?

n° 2001-19 Van Dender, K., and Proost, S. (2001), Optimal urban transport pricing with congestion and economies of density

n° 2001-18 Eyckmans, J., Van Regemorter, D., and van Steenberghe, V. (2001), Is Kyoto fatally flawed? An analysis with MacGEM

n° 2001-17 Van Dender, K. (2001), Transport taxes with multiple trip purposes

n° 2001-16 Proost, S., and Van Regemorter, D. (2001), Interaction between Local Air Pollution and Global Warming Policy and its Policy Implications

n° 2001-15 Franckx, L. (2001), Environmental enforcement with endogenous ambient monitoring

n° 2001-14 Mayeres, I. (2001), Equity and transport policy reform

n° 2001-13 Calthrop, E. (2001), On subsidising auto-commuting

n° 2001-12 Franckx, L. (2001), Ambient environmental inspections in repeated enforcement games

n° 2001-11 Pan, H. (2001), The economics of Kyoto flexible mechanisms: a survey

n° 2001-10 Calthrop, E. (2001), When consumers can decide not to pay a tax: enforicing and pricing urban on-street parking space

n° 2001-09 Franckx, L. (2001), Ambient environmental inspections eliminate the need for marginal deterrence

n° 2001-08 Franckx, L. (2001), Optimal fines for environmental noncompliance under a decentralized enforcement policy

n° 2001-07 Franckx, L. (2001), Reputation effects in regulatory enforcement

n° 2001-06 Franckx, L. (2001), Ambient inspections in environmental enforcement: an extension

n° 2001-05 Eyckmans, J., and Cornillie, J. (2001), Supplementarity in the European carbon emission market

n° 2001-04 Franckx, L. (2001), Ambient environmental inspections followed by sequential firm inspections

n° 2001-03 Eyckmans J. (2001), On the farsighted stability of the Kyoto Protocol